Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 1 (2014), pp. 90-107

DOI: 10.4208/ata.2014.v30.n1.6

All Meromorphic Solutions of Some Algebraic Differential Equations

Wenjun Yuan[1,2], Zifeng Huang[1], Jinchun Lai[1] and Jianming Qi[3,*]

[1] *School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China*
[2] *Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, Guangzhou 510006, China*
[3] *Department of Mathematics and Physics, Shanghai Dianji University, Shanghai 201306, China*

Received 30 November 2013; Accepted (in revised version) 2 March 2014

Available online 31 March 2014

Abstract. In this article, we introduce some results with respect to the integrality and exact solutions of some 2nd order algebraic DEs. We obtain the sufficient and necessary conditions of integrable and the general meromorphic solutions of these equations by the complex method, which improves the corresponding results obtained by many authors. Our results show that the complex method provides a powerful mathematical tool for solving a large number of nonlinear partial differential equations in mathematical physics.

Key Words: Differential equation, general solution, meromorphic function, elliptic function.

AMS Subject Classifications: 30D35, 34A05

1 Introduction

Nonlinear partial differential equations (NLPDEs) are widely used as models to describe many important dynamical systems in various fields of sciences, particularly in fluid mechanics, solid state physics, plasma physics and nonlinear optics Exact solutions of NLPDEs of mathematical physics have attracted significant interest in the literature Over the last years, much work has been done on the construction of exact solitary wave solutions and periodic wave solutions of nonlinear physical equations. Many methods have been developed by mathematicians and physicists to find special solutions of NLPDEs, such as the inverse scattering method [1], Darboux transformation method [2], Hirota

*Corresponding author. *Email addresses:* `wjyuan1957@126.com` (W. J. Yuan), `847071088@qq.com` (Z. F. Huang), `190534202@qq.com` (J. C. Lai), `qijianmingsdju@163.com` (J. M. Qi)

bilinear method [3], Lie group method [4], bifurcation method of dynamic systems [5–7], sine-cosine method [8], tanh-function method [9, 10], Fan-expansion method [11], and homogenous balance method [12]. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations. Recently, Kudryashov et al. [13–16] find exact meromorphic solutions for some nonlinear ordinary differential equations by using Laurent series and gave some basic results. Follow their work the complex method was introduced by Yuan et al. [17–19]. It is shown that the complex method provide a powerful mathematical tool for solving great many nonlinear partial differential equations in mathematical physics.

2 The second order algebraic differential equations with degree two

In 2013, Yuan et al. [17] derived all traveling wave exact solutions by using the complex method for a type of ordinary differential equations (ODEqs)

$$Aw''+Bw+Cw^2+D=0, \tag{2.1}$$

where A, B, C and D are arbitrary constants.

In order to state these results, we need some concepts and notations.

A meromorphic function $w(z)$ means that $w(z)$ is holomorphic in the complex plane $\mathbb{C}$ except for poles. α, b, c, c_i and c_{ij} are constants, which may be different from each other in different place. We say that a meromorphic function f belongs to the class W if f is an elliptic function, or a rational function of $e^{\alpha z}$, $\alpha\in\mathbb{C}$, or a rational function of z.

Theorem 2.1. *Suppose that $AC\neq 0$, then all meromorphic solutions w of an Eq.* (2.1) *belong to the class W. Furthermore, Eq.* (2.1) *has the following three forms of solutions:*

(I) The elliptic general solutions

$$w_{1d}(z)=-6\frac{A}{C}\Big\{-\wp(z)+\frac{1}{4}\Big[\frac{\wp'(z)+F}{\wp(z)-E}\Big]^2\Big\}+6\frac{AE}{C}-\frac{B}{2C}.$$

Here, $4DC=-12A^2g_2+B^2$, $F^2=4E^3-g_2E-g_3$, g_3 and E are arbitrary.

(II) The simply periodic solutions

$$w_{1s}(z)=-6\frac{A}{C}\alpha^2\coth^2\frac{\alpha}{2}(z-z_0)-\frac{A}{2C}\alpha^2-\frac{B}{2C},$$

where $4DC=-A^2\alpha^4+B^2$, $z_0\in\mathbb{C}$.

(III) The rational function solutions

$$w_{1r}(z)=-\frac{6\frac{A}{C}}{(z-z_0)^2}-\frac{B}{2C},$$

where $4CD=B^2$, $z_0\in\mathbb{C}$.

The Eq. (2.1) is a type of an important auxiliary equation, because many nonlinear evolution equations can be converted to the Eq. (2.1) using the travelling wave reduction. For instance, the classical KdV equation, Boussinesq equation, $(3+1)$-dimensional Jimbo-Miwa equation and Benjamin-Bona-Mahony equation can be converted to the Eq. (2.1) [17].

The KdV equation

The KdV equation has the form as

$$u_t+uu_x+\beta u_{xxx}=0, \tag{2.2}$$

where β is constant.

Substituting

$$u(x,t)=w(z), \quad z=k(x-ct),$$

into Eq. (2.2), and integrating it yields

$$\beta k^2 w''+\frac{1}{2}w^2-cw-b=0.$$

It is converted to AOD equation (2.1), where

$$A=\beta k^2, \quad B=-c, \quad C=\frac{1}{2}, \quad D=-b.$$

The Boussinesq equation

The Boussinesq equation is the form as

$$u_{tt}-c_0^2u_{xx}-\alpha u_{xxxx}-\beta(u^2)_{xx}=0, \tag{2.3}$$

where c_0, α, β are constants.

Substituting

$$u(x,t)=w(z), \quad z=kx+\omega t,$$

into Eq. (2.3) and integrating it gives

$$\alpha k^4 w''+\beta w^2+(c_0^2k^2-\omega^2)w+D=0.$$

It is converted to AOD equation (2.1), where

$$A=\alpha k^4, \quad B=c_0^2k^2-\omega^2, \quad C=\beta.$$

The $(3+1)$-dimensional Jimbo-Miwa equation

The $(3+1)$-dimensional Jimbo-Miwa equation equation is considered as

$$u_{xxxy}+3u_yu_{xx}+3u_xu_{xy}+2u_{yt}-3u_{xz}=0. \tag{2.4}$$

Set k, m, r, ω are constants. Substituting

$$u(x,y,z,t)=w(z),\quad z=kx+my+rz+\omega t,$$

into Eq. (2.4), and integrating it deduces

$$k^3mw''+3k^2mw^2+(2m\omega-3kr)w+D=0.$$

It is converted to AOD equation (2.1), where

$$A=k^3m,\quad B=2m\omega-3kr,\quad C=3k^2m.$$

The Benjamin-Bona-Mahony equation

The Benjamin-Bona-Mahony equation has the form

$$u_t-u_{xxt}+u_x+\left(\frac{u^2}{2}\right)_x=0. \tag{2.5}$$

Let k, ω are constants. Substituting

$$u(x,t)=w(z),\quad z=kx-\omega t,$$

into Eq. (2.5) and integrating it gives

$$\omega k^2w''+\frac{k}{2}w^2+(k-\omega)w+D=0.$$

It is converted to AOD equation (2.1), where

$$A=\omega k^2,\quad B=k-\omega,\quad C=\frac{k}{2}.$$

Yuan et al. [20] employ the complex method to obtain all meromorphic solutions of anther Eq. (2.6) below

$$Aw''+Bw'+Cw+Dw^2+E=0, \tag{2.6}$$

where A, B, C, D, E are arbitrary constants.

Theorem 2.2. *Suppose that $AD\neq 0$, then the Eq.* (2.6) *is integrable if and only if*

$$B=0,\quad \pm\frac{5}{\sqrt{6}}\sqrt{-2AD\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}},\quad \pm\frac{5i}{\sqrt{6}}\sqrt{-2AD\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}.$$

Furthermore, the general solutions of the Eq. (2.1) *are of the form:*

(I) If $B=0$, then the elliptic general solutions of the Eq. (2.6)

$$w_{2d}(z)=-6\frac{A}{D}\Big\{-\wp(z)+\frac{1}{4}\Big[\frac{\wp'(z)+M}{\wp(z)-N}\Big]^2\Big\}+6\frac{AN}{D}-\frac{C}{2D}.$$

Here, $12A^2g_2=C^2$, $M^2=4N^3-g_2N-g_3$, g_3 and N are arbitrary.

In particular, which degenerates the simply periodic solutions

$$w_{2s}(z)=-6\frac{A}{D}\alpha^2\coth^2\frac{\alpha}{2}(z-z_0)-\frac{A}{2D}\alpha^2-\frac{C}{2D},$$

where $A^2\alpha^4=C^2$, $z_0\in\mathbb{C}$.

And the rational function solutions

$$w_{2r}(z)=-\frac{6\frac{A}{D}}{(z-z_0)^2}-\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}-\frac{C}{2D},$$

where $C^2=4DE$, $z_0\in\mathbb{C}$.

(II) If

$$B=\pm\frac{5}{\sqrt{6}}\sqrt{-2AD\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}},$$

the general solutions of the Eq. (2.6)

$$\begin{aligned}w_{g2}(z)=&\exp\Big\{\mp\frac{2}{\sqrt{6}}\sqrt{-\frac{2D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}z}\Big\}\wp\Big(\sqrt{-\frac{D}{A}}\exp\Big\{\mp\frac{1}{\sqrt{6}}\sqrt{-\frac{D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}z}\Big\}\\&-s_0;0,g_3\Big)-\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}-\frac{C}{2D},\end{aligned}$$

where $\sqrt{C^2/4D^2}=-C/2D$, both s_0 and g_3 are arbitrary constants.

In particular, which degenerates the one parameter family of solutions

$$w_{f2}(z)=2\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}\frac{1}{\Big\{1-\exp\Big\{\pm\frac{(z-z_0)}{\sqrt{6}}\sqrt{-\frac{2D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}\Big\}\Big\}^2}-\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}-\frac{C}{2D},$$

where $\sqrt{C^2/4D^2}=-C/2D$, $\quad z_0\in\mathbb{C}$.

(III) If

$$B=\pm\frac{5i}{\sqrt{6}}\sqrt{-2AD\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}},$$

then the general solutions of the Eq. (2.6)

$$w_{g2,i}(z)=\exp\Big\{\mp\frac{2i}{\sqrt{6}}\sqrt{-\frac{2D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}z\Big\}\wp\Big(\sqrt{\frac{D}{A}}\exp\Big\{\mp\frac{i}{\sqrt{6}}\sqrt{-\frac{D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}z\Big\}$$
$$-s_0;0,g_3\Big)-\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}-\frac{3C}{2D},$$

where $\sqrt{C^2/4D^2}=-C/2D$, *both* s_0 *and* g_3 *are arbitrary constants.*

In particular, which degenerates the one parameter family of solutions

$$w_{f2,i}(z)=-2\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}\frac{1}{\Big\{1-\exp\Big\{\pm\frac{i(z-z_0)}{\sqrt{6}}\sqrt{-\frac{2D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}\Big\}\Big\}^2}-\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}-\frac{3C}{2D},$$

where $\sqrt{C^2/4D^2}=-C/2D$, $z_0\in\mathbb{C}$.

The Fisher equation with degree two

Consider the Fisher equation

$$u_t=vu_{xx}+su(1-u),$$

which is a nonlinear diffusion equation as a model for the propagation of a mutant gene with an advantageous selection intensity s. It was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population in 1936.

Set $t'=st$ and $x'=(s/v)^{x/2}$ and drop the primes, above equation becomes

$$u_t=u_{xx}+u(1-u). \tag{2.7}$$

By substituting

$$u(x,t)=w(z),\quad z=x-ct,$$

into the Eq. (2.7) and integrating it, we obtain

$$w''+cw'+w(1-w)=0.$$

It is converted to the Eq. (2.6), where

$$A=1,\quad B=c,\quad C=1,\quad D=-1,\quad E=0.$$

Three nonlinear pseudoparabolic physical models

The one-dimensional Oskolkov equation, the Benjamin-Bona-Mahony-Peregrine-Burgers equation and the Oskolkov-Benjamin-Bona-Mahony-Burgers equation are the specially cases of our Eq. (2.6).

The one-dimensional Oskolkov equation is the form

$$u_t-\lambda u_{xxt}-\alpha u_{xx}+uu_x=0, \tag{2.8}$$

where $\lambda\neq 0$, $\alpha\in\mathbb{R}$.

Substituting

$$u(x,t)=w(z),\quad z=x-ct,$$

into the Eq. (2.8) and integrating the equation, we have

$$\lambda w''-\alpha w'-cw+\frac{1}{2}w^2=0.$$

It is converted to the Eq. (2.6), where

$$A=\lambda,\quad B=-\alpha,\quad C=-c,\quad D=\frac{1}{2},\quad E=0.$$

The Benjamin-Bona-Mahony-Peregrine-Burgers equation is the form as

$$u_t-u_{xxt}-\alpha u_{xx}++\gamma u_x+\theta uu_x+\beta u_{xxx}=0, \tag{2.9}$$

where α is a positive constant, θ and β are nonzero real numbers.

Substituting

$$u(x,t)=w(z),\quad z=x-ct,$$

into the Eq. (2.9), and then we get

$$(c+\beta)w''-\alpha w'+(\gamma-c)w+\frac{\theta}{2}w^2=0.$$

It is converted to the Eq. (2.6), where

$$A=c+\beta,\quad B=-\alpha,\quad C=\gamma-c,\quad D=\frac{\theta}{2},\quad E=0.$$

The Oskolkov-Benjamin-Bona-Mahony-Burgers equation is the form as

$$u_t-u_{xxt}-\alpha u_{xx}++\gamma u_x+\theta uu_x=0, \tag{2.10}$$

where α is a positive constant, θ is a nonzero real number.

Substituting

$$u(x,t)=w(z),\quad z=x-ct,$$

into the Eq. (2.10), we deduce

$$cw''-\alpha w'+(\gamma-c)w+\frac{\theta}{2}w^2=0.$$

It is converted to the Eq. (2.6), where

$$A=c,\quad B=-\alpha,\quad C=\gamma-c,\quad D=\frac{\theta}{2},\quad E=0.$$

The KdV-Burgers equation

The KdV-Burgers equation is of form

$$u_t+uu_x+u_{xxx}-\alpha u_{xx}=0, \tag{2.11}$$

where α is a constant.

Substituting the traveling wave transformation

$$u(x,t)=w(z),\quad z=x+Ct,$$

into the Eq. (2.11), and integrating it yields the auxiliary ordinary differential equation

$$w''-\alpha w'+\frac{1}{2}w^2+Cw+E=0,$$

where E is an integral constant. It is converted to Eq. (2.6), where

$$A=1,\quad B=-\alpha,\quad C=C,\quad D=\frac{1}{2},\quad E=E.$$

3 The second order algebraic differential equations with degree three

In 2012, Yuan et al. [21] employ the complex method to obtain all meromorphic solutions of the auxiliary ordinary differential equations [AOD equation (3.1)] below

$$Aw''+Bw+Cw^3+D=0, \tag{3.1}$$

where A, B, C and D are arbitrary constants. And then find all meromorphic exact solutions of the modified ZK equation, modified KdV equation, nonlinear Klein-Gordon equation and modified BBM equation.

Theorem 3.1. *Suppose that $AC\neq 0$, then all meromorphic solutions w of an AOD equation* (2.1) *belong to the class W. Furthermore, AOD equation* (3.1) *has the following three forms of solutions:*

(I) The elliptic function solutions

$$w_{3d}(z)=\pm\frac{1}{2}\sqrt{-\frac{2A}{C}}\frac{(-\wp+c)(4\wp c^2+4\wp^2c+2\wp' d-\wp g_2-cg_2)}{((12c^2-g_2)\wp+4c^3-3cg_2)\wp'+(4\wp^3+12c\wp^2-3g_2\wp-cg_2)d}.$$

Here, $g_3=0$, $d^2=4c^3-g_2c$, g_2 and c are arbitrary.

(II) The simply periodic solutions

$$w_{3s,1}(z)=\alpha\sqrt{-\frac{A}{2C}}\Big(\coth\frac{\alpha}{2}(z-z_0)-\coth\frac{\alpha}{2}(z-z_0-z_1)-\coth\frac{\alpha}{2}z_1\Big),$$
$$w_{3s,2}(z)=\alpha\sqrt{-\frac{A}{2C}}\tanh\frac{\alpha}{2}(z-z_0),$$

where

$$z_0\in\mathbb{C},\quad B=A\alpha^2\Big(\frac{1}{2}+\frac{3}{2\sinh^2\frac{\alpha}{2}z_1}\Big),\quad D=\sqrt{-\frac{A}{2C}}\frac{\tanh\frac{\alpha}{2}z_1}{\sinh^2\frac{\alpha}{2}z_1},\quad z_1\neq 0,$$

in the former formula, or $B=A\alpha^2/2$, $D=0$.

(III) The rational function solutions

$$w_{3r,1}(z)=\pm\sqrt{-\frac{2A}{C}}\frac{1}{z-z_0},$$
$$w_{3r,2}(z)=\pm\sqrt{-\frac{2A}{Cz_1^2}}\Big(\frac{z_1}{z-z_0}-\frac{z_1}{z-z_0-z_1}-1\Big),$$

where $z_0\in\mathbb{C}$, $B=0$, $D=0$ *in the former case, or given* $z_1\neq 0$, $B=/z_1^2$, $D=\mp 2C(-2A/Cz_1^2)^{3/2}$.

The modified ZK equation, modified KdV equation, nonlinear Klein-Gordon equation and modified BBM equation are considered again and the exact solutions are derived with the aid of the AOD equation (3.1).

The Modified ZK equation

The Modified ZK equation is expressed as

$$u_t+\beta u^2u_x+u_{xxx}+u_{xyy}=0, \tag{3.2}$$

where β is a constant.

Substituting

$$u(x,t)=w(z),\quad z=k(x+ly-\omega t),$$

into Eq. (3.2) and integrating it yields

$$k^2(1+l^2)w''+\frac{\beta}{3}w^3-\omega w+b=0.$$

It is converted to AOD equation (3.1), where

$$A=k^2(1+l^2),\quad B=-\omega,\quad C=\frac{\beta}{3},\quad D=b.$$

The Modified KdV equation

The Modified KdV equation has the form

$$u_t+\tau u^2u_x+\beta u_{xxx}=0, \tag{3.3}$$

where τ, β are constant.

Substituting

$$u(x,t)=w(z), \quad z=kx-\omega t,$$

into Eq. (3.3) and integrating it yields

$$\beta k^3 w''+\frac{k\tau}{3}w^3-\omega w+d=0.$$

It is converted to AOD equation (3.1), where

$$A=\beta k^3, \quad B=-\omega, \quad C=\frac{k\tau}{3}, \quad D=d.$$

The nonlinear Klein-Gordon equation

The nonlinear Klein-Gordon equation is of the form

$$u_{tt}-c^2u_{xx}+\tau u-\beta u^3=0, \tag{3.4}$$

where c, τ, β are constants.

Substituting

$$u(x,t)=w(z), \quad z=kx-\omega t,$$

into Eq. (2.3) gives

$$\tau w(z)-\beta w^3(z)+(\omega^2-c^2k^2)w''(z)=0.$$

It is converted to AOD equation (3.1), where

$$A=\omega^2-c^2k^2, \quad B=\tau, \quad C=-\beta, \quad D=0.$$

The modified BBM equation

The modified BBM equation is considered as

$$u_t+u_x+u^2u_x+\beta u_{xxt}=0, \tag{3.5}$$

where β is constant. Substituting

$$u(x,t)=w(z), \quad z=k(x-\lambda t),$$

into Eq. (3.5), and integrating it deduces

$$\beta\lambda k^2w''-\frac{1}{3}w^3-(1-\lambda)w-b=0.$$

It is converted to AOD equation (3.1), where

$$A=\beta\lambda k^2,\quad B=-(1-\lambda),\quad C=-\frac{1}{3},\quad D=-b.$$

Recently, Yuan et al. [22] consider Eq. (3.6) below

$$Aw''+Bw+Cw^2+w^3+D=0, \tag{3.6}$$

where A, B, C and D are arbitrary constants. They obtained the following result and gave its two applications.

Theorem 3.2. *Suppose that $A\neq 0$, then all meromorphic solutions w of an Eq.* (3.6) *belong to the class W. Furthermore, the Eq.* (3.6) *has the following three forms of solutions:*
(I) All elliptic function solutions

$$\begin{aligned}w_{4d}(z)=&-\frac{C}{3}\pm\sqrt{-\frac{A}{2}}\times\\&\frac{(-\wp+E)(4\wp E^2+4\wp^2E+2\wp' F-\wp g_2-Eg_2)}{((12E^2-g_2)\wp+4E^3-3Eg_2)\wp'+4F\wp^3+12FE\wp^2-3Fg_2\wp-FEg_2},\end{aligned}$$

where $A(C^2-9B)=12C\sqrt{-A/2}$, $27D=C^3$, $g_3=0$, $F^2=4E^3-g_2E$, g_2 and E are arbitrary constants.
(II) All simply periodic solutions

$$\begin{aligned}&w_{4s,1}(z)=\pm\sqrt{-\frac{A}{2}}\alpha\coth\frac{\alpha}{2}(z-z_0)-\frac{C}{3},\\&w_{4s,2}(z)=\pm\sqrt{-\frac{A}{2}}\alpha\Big(\coth\frac{\alpha}{2}(z-z_0)-\coth\frac{\alpha}{2}\Big)-\frac{C}{3}\mp\sqrt{-\frac{A}{2}}\alpha\coth\frac{\alpha}{2}z_1,\end{aligned}$$

where $z_0\in\mathbb{C}$, $A(2C^2+9A\alpha^2-18B)=24C\sqrt{-A/2}$, $27D-C^3=27\alpha^2\sqrt{-A/2}$ in the former case, or $z_1\neq 0$, $8C\sqrt{-A/2}+6AB=3A^2\alpha^2(3/\sinh^2\frac{\alpha}{2}z_1+1)$,

$$162D\sqrt{-\frac{A}{2}}=\Big(2C\sqrt{-\frac{A}{2}}\mp 3A\alpha\coth\frac{\alpha}{2}z_1\Big)\times\Big(\frac{108A\alpha^2}{\sinh^2\frac{\alpha}{2}z_1}+3C^2\mp 9C\alpha\sqrt{-\frac{A}{2}}\coth\frac{\alpha}{2}z_1\Big).$$

(III) All rational function solutions

$$\begin{aligned}&w_{4r,1}(z)=\pm\frac{2\sqrt{-\frac{A}{2}}}{z-z_0}-\frac{C}{3},\\&w_{4r,2}(z)=\pm\frac{2\sqrt{-\frac{A}{2}}}{z-z_0}\mp\frac{2\sqrt{-\frac{A}{2}}}{z-z_0-z_1}\mp\frac{2\sqrt{-\frac{A}{2}}}{z_1}-\frac{C}{3},\end{aligned}$$

where $z_0\in\mathbb{C}$, $A(C^2-9B)=12C\sqrt{-A/2}$, $27D=C^3$ in the former case, or

$$A\Big(\frac{54A}{z_1^2}+C^2-9B\Big)=12C\sqrt{-\frac{A}{2}},\quad \frac{4A^2}{z_1^3}=\Big(\frac{C^3}{27}+\frac{2C}{z_1^2}-D\Big)\sqrt{-\frac{A}{2}},\quad z_1\neq 0.$$

All exact solutions of the Eq. (3.7) and Eq. (3.8) can be converted to the Eq. (3.6) making use of the traveling wave reduction.

The variant Boussinesq equations

The variant Boussinesq equations are expressed as

$$\begin{cases} u_t+(uw)_x+w_{xxx}=0, \\ w_t+u_x+ww_x=0. \end{cases} \tag{3.7}$$

As a model for water waves, w is the velocity and u the total depth, and the subscripts denote partial derivatives.

Substituting the traveling wave transformation

$$u(x,t)=u(z), \quad w(x,t)=w(z), \quad z=kx+\lambda t,$$

into the Eqs. (3.7), and integrating it yields

$$\begin{cases} \lambda u+kuw+k^3w''+C_1=0, \\ \lambda w+ku+\dfrac{k}{2}w^2+C_2=0, \end{cases}$$

where C_1 and C_2 are constants.

Solving the system, we get the relation

$$u=-\frac{1}{2}w^2-\frac{\lambda}{k}w-C_2,$$

and the auxiliary ordinary differential equation

$$k^3w''-\frac{\lambda^2+C_2}{k}w-\frac{3\lambda}{2}w^2-\frac{k}{2}w^3+C_3=0,$$

where $C_3=C_1-C_2\lambda/k$.

It is converted to Eq. (3.6), where

$$A=-2k^2, \quad B=\frac{2\lambda+2C_2}{k^2}, \quad C=\frac{3\lambda}{k}, \quad D=-\frac{2C_3}{k}.$$

The combined KdV-mKdV equation

The combined KdV-mKdV equation is of form

$$u_t+auu_x+bu^2u_x+\delta u_{xxx}=0, \tag{3.8}$$

where a, b and δ are constants. The Eq. (3.8) is a real physical model concerning many branches in physics. The Eq. (3.8) may describe the wave propagation of bounded particle with a harmonic force in one-dimensional nonlinear lattice. Particularly, it describes the propagation of ion acoustic waves of small amplitude without Landau damping in plasma physics, and it is also used to explain the propagation of thermal pulse through single crystal of sodium fluoride in solid physics.

Substituting the traveling wave transformation

$$u(x,t)=w(z),\quad z=kx+\lambda t,$$

into the Eq. (3.8) and integrating it yields the auxiliary ordinary differential equation

$$\delta k^3 w''+\lambda w+\frac{ak}{2}w^2+\frac{bk}{3}w^3+d=0,$$

where d is an integral constant.

It is converted to Eq. (3.6), where

$$A=\frac{3\delta k^3}{b},\quad B=\frac{3\lambda}{bk},\quad C=\frac{3a}{2b},\quad D=\frac{3d}{bk}.$$

Very recently, Huang et al. [23] study the differential equation below.

$$Aw''+Bw'+Cw+Dw^3=0, \tag{3.9}$$

where A, B, C, D are arbitrary constants. They got the following theorem.

Theorem 3.3. *Suppose that $AD\neq 0$, then the Eq.* (3.9) *is integrable if and only if*

$$B=0,\quad \pm\frac{3}{\sqrt{2}}\sqrt{AC}.$$

Furthermore, the general solutions of the Eq. (3.9) *are of the form:*

(I) (see [21]*) When $B=0$, the elliptic general solutions of the Eq.* (3.9)

$$w_{5d,1}(z)=\pm\sqrt{-\frac{2A}{D}}\frac{\wp'(z-z_0:g_2,0)}{\wp(z-z_0:g_2,0)},$$

where, z_0 and g_2 are arbitrary. In particular, it degenerates the simply periodic solutions and rational solutions

$$w_{5s,1}(z)=\alpha\sqrt{-\frac{A}{2D}}\tanh\frac{\alpha}{2}(z-z_0),\quad w_{5r}(z)=\pm\sqrt{-\frac{2A}{D}}\frac{1}{z-z_0},$$

where $C=A\alpha^2/2$ and $z_0\in\mathbb{C}$.

(II) When

$$B=\pm\frac{3}{\sqrt{2}}\sqrt{AC},$$

the general solutions of the Eq. (3.9)

$$w_{5g,1}(z)=\pm\frac{1}{2}\exp\left\{\mp\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A}}z\right\}\frac{\wp'\left(\sqrt{-\frac{D}{C}}\exp\left\{\mp\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A}}z\right\}-s_0;g_2,0\right)}{\wp\left(\sqrt{-\frac{D}{C}}\exp\left\{\mp\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A}}z\right\}-s_0;g_2,0\right)},$$

where $\wp(s:g_2,0)$ is the Weierstrass elliptic function, both s_0 and g_2 are arbitrary constants. In particular, $w_{5g,1}(z)$ degenerates the one parameter family of solutions

$$w_{5f,1}(z)=\pm\sqrt{-\frac{C}{D}}\frac{1}{1-\exp\left\{\mp\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A}}(z-z_0)\right\}},$$

where $z_0\in\mathbb{C}$.

All exact solutions of the Eq. (3.10), nonlinear Scrödinger Eq. (3.11) and Eq. (3.12) can be converted to the Eq. (3.9) making use of the traveling wave reduction.

The Newell-Whitehead equation

The Newell-Whitehead equation is the form as

$$u_{xx}-u_t-ru^3+su=0, \tag{3.10}$$

where r, s are constants.

Substituting

$$u(x,t)=w(z), \quad z=x+\omega t,$$

into Eq. (3.10), and it gives

$$w''-\omega w'+sw-rw^3=0.$$

It is converted to Eq. (3.9), where

$$A=1, \quad B=-\omega, \quad C=1, \quad D=-1.$$

The NLS equation

The NLS equation is the form as

$$iu_t+\alpha u_{xx}+\beta|u|^2u=0, \tag{3.11}$$

where α, β are nonzero constants.

Substituting

$$u(x,t)=w(z)e^{kx-\omega t}, \quad z=x+ct,$$

into Eq. (3.11), and it gives

$$\alpha w''+i(2\alpha k-c)w'+(\omega-\alpha k^2)w+\beta w^3=0.$$

It is converted to Eq. (3.9), where

$$A=\alpha, \quad B=i(2\alpha k-c), \quad C=\omega-\alpha k^2, \quad D=\beta.$$

The Fisher equation with degree three

The Fisher equation with degree three is the form as

$$u_t = u_{xx} + u(1-u^2). \tag{3.12}$$

Substituting

$$u(x,t) = w(z), \quad z = x - ct,$$

into Eq. (3.12), and it gives

$$w'' + cw' + w(1-w^2) = 0.$$

It is converted to Eq. (3.9), where

$$A = 1, \quad B = c, \quad C = 1, \quad D = -1.$$

4 Some results and the complex method

In order to state our complex method, we need some notations and results.

Set $m \in \mathbb{N} := \{1,2,3,\cdots,\}$, $r_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $r = (r_0, r_1, \cdots, r_m)$, $j = 0,1,\cdots,m$. We define a differential monomial denoted by

$$M_r[w](z) := [w(z)]^{r_0}[w'(z)]^{r_1}[w''(z)]^{r_2} \cdots [w^{(m)}(z)]^{r_m}.$$

$p(r) := r_0 + r_1 + \cdots + r_m$ is called the degree of $M_r[w]$. A differential polynomial is defined by

$$P(w, w', \cdots, w^{(m)}) := \sum_{r \in I} a_r M_r[w],$$

where a_r are constants, and I is a finite index set. The total degree is defined by $\deg P(w, w', \cdots, w^{(m)}) := \max_{r \in I}\{p(r)\}$.

We will consider the following complex ordinary differential equations

$$P(w, w', \cdots, w^{(m)}) = bw^n + c, \tag{4.1}$$

where $b \neq 0$, c are constants, $n \in \mathbb{N}$.

Let p, $q \in \mathbb{N}$. Suppose that the Eq. (4.1) has a meromorphic solution w with at least one pole, we say that the Eq. (4.1) satisfies weak $\langle p,q \rangle$ condition if substituting Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k, \quad q > 0, \quad c_{-q} \neq 0, \tag{4.2}$$

into the Eq. (4.1) we can determinant p distinct Laurent singular parts below

$$\sum_{k=-q}^{-1} c_k z^k.$$

In order to give the representations of elliptic solutions, we need some notations and results concerning elliptic function [24].

Let ω_1, ω_2 be two given complex numbers such that $\mathrm{Im}\omega_1/\omega_2>0$, $L=L[2\omega_1,2\omega_2]$ be discrete subset $L[2\omega_1,2\omega_2]=\{\omega|\omega=2n\omega_1+2m\omega_2,\ n,m\in\mathbb{Z}\}$, which is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. The discriminant $\Delta=\Delta(c_1,c_2):=c_1^3-27c_2^2$ and

$$s_n=s_n(L):=\sum_{\omega\in L\setminus\{0\}}\frac{1}{\omega^n}.$$

Weierstrass elliptic function $\wp(z):=\wp(z,g_2,g_3)$ is a meromorphic function with double periods $2\omega_1$, $2\omega_2$ and satisfying the equation

$$(\wp'(z))^2=4\wp(z)^3-g_2\wp(z)-g_3, \tag{4.3}$$

where $g_2=60s_4$, $g_3=140s_6$ and $\Delta(g_2,g_3)\neq 0$.

Theorem 4.1 (see [24,25]). *Weierstrass elliptic functions $\wp(z):=\wp(z,g_2,g_3)$ have two successive degeneracies and addition formula:*
(i) Degeneracy to simply periodic functions (i.e., rational functions of one exponential e^{kz}) according to

$$\wp(z,3d^2,-d^3)=2d-\frac{3d}{2}\coth^2\sqrt{\frac{3d}{2}}z, \tag{4.4}$$

if one root e_j is double ($\Delta(g_2,g_3)=0$).
(ii) Degeneracy to rational functions of z according to

$$\wp(z,0,0)=\frac{1}{z^2},$$

if one root e_j is triple ($g_2=g_3=0$).
(iii) Addition formula

$$\wp(z-z_0)=-\wp(z)-\wp(z_0)+\frac{1}{4}\Big[\frac{\wp'(z)+\wp'(z_0)}{\wp(z)-\wp(z_0)}\Big]^2. \tag{4.5}$$

By above notations and results, we can give a method below, say complex method, to find exact solutions of some PDEs.

Step 1 Substituting the transform $T:u(x,t)\to w(z)$, $(x,t)\to z$ into a given PDE gives a non-linear ordinary differential equations (4.1).

Step 2 Substitute (4.2) into the Eq. (4.1) to determine that weak $\langle p,q\rangle$ condition holds, and pass the Painlevé test for the Eq. (4.1).

Step 3 Find the meromorphic solutions $w(z)$ of the Eq. (4.1) with pole at $z=0$, which have $m-1$ integral constants.

Step 4 By the addition formula of Theorem 4.1 we obtain all meromorphic solutions $w(z-z_0)$.

Step 5 Substituting the inverse transform T^{-1} into these meromorphic solutions $w(z-z_0)$, then we get all exact solutions $u(x,t)$ of the original given PDE.

Acknowledgments

This work was supported by the Visiting Scholar Program of Chern Institute of Mathematics at Nankai University when the authors worked as visiting scholars. The authors would like to express their hearty thanks to Chern Institute of Mathematics provided very comfortable research environments to them. The authors finally wish to thank Professor Robert Conte for supplying his useful reprints and suggestions. This work was completed with the support with the NSF of China (No. 11271090, 11326083) and NSF of Guangdong Province (S2012010010121), Shanghai university young teacher training program (ZZSDJ12020), projects 10XKJ01, 12C401 and 12C104 from the Leading Academic Discipline Project of Shanghai Dianji University.

References

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Vol. 149 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, UK, 1991.

[2] V. B. Matveev and M. A. Salle, Darboux Transformations and Solitons, Springer Series in Nonlinear Dynamics, Springer, Berlin, Germany, 1991.

[3] R. Hirota and J. Satsuma, Soliton solutions of a coupled KdV equation, Phys. Lett. A, 85(8-9) (1981), 407–408.

[4] P. J. Olver, Applications of Lie Groups to Differential Equations, Vol. 107 of Graduate Texts in Mathematics, Springer, New York, NY, USA, 2nd edition,.

[5] J. B. Li and Z. Liu, Travelling wave solutions for a class of nonlinear dispersive equations, Chinese Annals of Mathematics, 3B(3) (2002), 397–418.

[6] S. Tang and W. Huang, Bifurcations of travelling wave solutions for the generalized double sinh-Gordon equation, Appl. Math. Comput., 189(2) (2007), 1774–1781.

[7] D. Feng, T. He and J. Lü, Bifurcations of travellingwave solutions for $(2+1)$-dimensional Boussinesqtype equation, Appl. Math. Comput., 185(1) (2007), 402–414.

[8] S. Tang, Y. Xiao and Z. Wang, Travelling wave solutions for a class of nonlinear fourth order variant of a generalized Camassa-Holm equation, Appl. Math. Comput., 210(1) (2009), 39–47.

[9] S. Tang, J. Zheng and W. Huang, Travelling wave solutions for a class of generalized KdV equation, Appl. Math. Comput., 215(7) (2009), 2768–2774.

[10] W. Malfliet and W. Hereman, The tanh method: I Exact solutions of nonlinear evolution and wave equations, Phys. Scripta., 54(6) (1996), 563–568.

[11] E. Fan, Uniformly constructing a series of explicit exact solutions to nonlinear equations in mathematical physics, Chaos, Solitons and Fractals, 16(5) (2003), 819–839.

[12] M. L. Wang, Solitary wave solutions for variant Boussinesq equations, Phys. Lett. A, 199 (1995), 169–172.

[13] N. A. Kudryashov, Meromorphic solutions of nonlinear ordinary differential equations, Commun. Nonlinear Sci. Numer. Simul., 15 (2010), 2778–2790.

[14] M. V. Demina and N. A. Kudryashov, From Laurent series to exact meromorphic solutions: the Kawahara equation, Phys. Lett. A, 374(39) (2010), 4023–4029.

[15] M. V. Demina and N. A. Kudryashov, Explicit expressions for meromorphic solutions of autonomous nonlinear ordinary differential equations, Commun. Nonlinear Sci. Numer. Simul., 16(3) (2011), 1127–1134.

[16] N. A. Kudryashov, D. I. Sinelshchikov and M. V. Demina, Exact solutions of the generalized Bretherton equation, Phys. Lett. A, 375(7) (2011), 1074–1079.

[17] W. J. Yuan, Y. Z. Li and J. M. Lin, Meromorphic solutions of an auxiliary ordinary differential equation using complex method, Math. Methods Appl. Sci., 36(13) (2013), 1776–1782. DOI: 10.1002/mma.2723.

[18] W. J. Yuan, Y. Huang and Y. D. Shang, All travelling wave exact solutions of two nonlinear physical models, Appl. Math. Comput., 219(11) (2013), 6212–6223. DOI: 10.1016/j.amc.2012.12.023.

[19] W. J. Yuan, Y. D. Shang, Y. Huang and H. Wang, The representation of meromorphic solutions of certain ordinary differential equations and its applications, Scientia Sinica Mathematica, 43(6) (2013), 563–575. DOI: 10.1360/012012-159.

[20] W. J. Yuan, Z. F. Huang, J. C. Lai and J. M. Qi, The general meromorphic solutions of an auxiliary ordinary differential equation using complex method and its applications, Studia Mathematica, 2013, to appear.

[21] W. J. Yuan, W. L. Xiong, J. M. Lin and Y. H. Wu, All meromorphic solutions of an auxiliary ordinary differential equation and its applications, Acta Mathematica Scientia, 2012, to appear.

[22] W. J. Yuan, W. L. Xiong, J. M. Lin and Y. H. Wu, All meromorphic solutions of a kind of second order ordinary differential equation and its applications, Complex Vaviable and Elliptic Equations, 2013, to appear.

[23] Z. F. Huang, M. Z. Fu, J. C. Lai and W. J. Yuan, The general solutions of an auxiliary ordinary differential equation using complex method and its applications, Advance in Difference Equations, 2013, to appear.

[24] S. Lang, Elliptic Functions, Springer Verlag, 2nd Ed. New York, 1987.

[25] R. Conte and M. Musette, Elliptic general analytic solutions, Stud. Appl. Math., 123(1) (2009), 63–81.