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MIXED FINITE ELEMENT ANALYSIS OF THERMALLY COUPLED QUASI-NEWTONIAN FLOWS

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Dedicated to Professor Walter Allegretto on the occasion of his 70th birthday

Abstract. A mixed finite element method combined with a fixed point algorithm is proposed for solving the thermally coupled quasi-Newtonian flow problem. The existence and uniqueness of the mixed variational solution are established. A more general uniqueness result for the original system problem is presented. The convergence of the approximate solution is analyzed and the corresponding error estimates are given.

Key words. Quasi-Newtonian flow, viscous heating, existence, uniqueness, nonlinear mixed method, finite element approximations, error estimates

1. Introduction

In modeling quasi-Newtonian flows with thermal effects, see for instance [4, 6, 7, 14, 17, 18, 19], we encounter a coupled system involving a quasi-Newtonian flow with a temperature dependent viscosity and a thermal balance with viscous heating. A mathematical model for this problem in two dimensions can be written as:

$$(1.1)\quad \begin{cases} \text{(a)} & -\boldsymbol{\nabla}\cdot(k(\theta)|D(\boldsymbol{u})|^{r-2}D(\boldsymbol{u}))+\boldsymbol{\nabla}p = \boldsymbol{f} & \text{in } \Omega \\ \text{(b)} & \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega \\ \text{(c)} & -\Delta\theta = k(\theta)|D(\boldsymbol{u})|^{r} & \text{in } \Omega \\ \text{(d)} & \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma \\ \text{(e)} & \theta = 0 & \text{on } \Gamma \end{cases}$$

where $\boldsymbol{u} : \Omega \to I\!R^2$ is the velocity, $p : \Omega \to I\!R$ is the pressure, $\theta : \Omega \to I\!R$ is the temperature, Ω is a bounded open subset of $I\!R^2$, Γ its boundary. The viscosity k is a function of θ, $k = k(\theta)$. $D(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T)$ is the strain rate tensor, and $1 < r < \infty$.

Professor Walter Allegretto and his former student Dr. Hong Xie did the pioneer works [1, 2, 3, 28] on the thermistor problem, which is a special scalar model with $r = 2$ of the problem considered in this paper. Other works for the thermistor problem can be found in [9, 10, 11, 12, 13, 16, 21, 22], etc. In [23, 31], the complete mathematical and numerical studies such as existence, uniqueness, regularity, finite element approximations based on an iterative algorithm, convergence analysis and

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numerical implementations were presented, and then extended to the Stokes flows with viscous heating in [32]. A nonlinear finite element approximation and a mixed discontinuous Galerkin approximation were studied respectively in [8] and [34].

For the case of $r \neq 2$, the existence study for the thermally coupled nonlinear Darcy flows or Hele-Shaw flows can be found in, e.g. [7, 17, 18, 19], and [4, 29] for non-Newtonian flows with viscous heating. The existence, uniqueness, regularity, finite element approximations and convergence analysis based on the standard variational formulation for the thermally coupled nonlinear Darcy flows were studied by the second author of this paper in [30] and extended to the thermally coupled quasi-Newtonian flows in [33]. A nonlinear mixed variational formulation and finite element approximations to the thermally coupled nonlinear Darcy flows were studied recently by the authors in [35].

In this paper, we will continue the works in [33, 35] and study the nonlinear mixed variational formulation introduced in [24, 15], possessing local conservations of the momentum and the mass, for problem (1.1). We first establish the existence and uniqueness in Section 2. Because of the restriction of mathematical technique applied for nonlinear analysis as pointed out in [35], the uniqueness obtained here is for the case of $r \geq 2$ which is different from in [33], thus we get a more general result on uniqueness (see Theorem 2.3), which is another objective to study the mixed method for the nonlinear coupled problem, besides the usual one which is to get more precise numerical solution for the deviatoric stress tensor $\boldsymbol{\sigma}$. We propose a fixed point algorithm to decouple the problem in Sections 3 and its nonlinear mixed finite element approximation in Section 4. Also in Section 4, we present convergence analysis with an error estimate between continuous solution and its iterative finite element approximation.

2. Nonlinear mixed variational formulation

Let $W^{m,s}(\Omega)$ denote the Sobolev space with its norm $\|\cdot\|_{W^{m,s}}$, for $m \geq 0$ and $1 \leq s \leq \infty$. We write $H^m(\Omega) = W^{m,2}(\Omega)$ when $s = 2$, with the norm $\|\cdot\|_{H^m}$, and $L^s(\Omega) = W^{0,s}(\Omega)$ when $m = 0$, with the norm $\|\cdot\|_{L^s}$. $W_0^{m,s}(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ for the norm $\|\cdot\|_{W^{m,s}}$. Vector variables are, in general, denoted with bold face. We denote also $\boldsymbol{W}^{m,s}(\Omega) = [W^{m,s}(\Omega)]^2$, $\boldsymbol{W}_0^{m,s}(\Omega) = [W_0^{m,s}(\Omega)]^2$, $\boldsymbol{H}^m(\Omega) = [H^m(\Omega)]^2$, $\boldsymbol{H}_0^m(\Omega) = [H_0^m(\Omega)]^2$, and $\boldsymbol{L}^s(\Omega) = [L^s(\Omega)]^2$.

Throughout this work, we assume that, the coupling function μ is bounded, i.e., there exist constants $k^* \geq k_* > 0$ such that, for all $s \in I\!R$,

$$k_* \leq k(s) \leq k^* \ , \tag{2.1}$$

and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, which implies that $\boldsymbol{f} \in \boldsymbol{W}^{-1,r'}(\Omega)$, where $1/r' + 1/r = 1$, then the standard variational formulation of problem (1.1) can be defined as:

(2.2)

$$\begin{cases} \textit{Find } (\boldsymbol{u}, p, \theta) \in \boldsymbol{W}_0^{1,r}(\Omega) \times L_0^{r'}(\Omega) \times H_0^1(\Omega) \textit{ such that} \\ \text{(a)} \quad (k(\theta)|D(\boldsymbol{u})|^{r-2}D(\boldsymbol{u}), D(\boldsymbol{v})) - (p, \nabla\cdot\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}_0^{1,r}(\Omega) \\ \text{(b)} \quad (q, \nabla\cdot\boldsymbol{u}) = 0, \quad \forall\, q \in L_0^{r'}(\Omega) \\ \text{(c)} \quad (\nabla\theta, \nabla\eta) = (k(\theta)|D(\boldsymbol{u})|^r, \eta), \quad \forall\, \eta \in H_0^1(\Omega) \end{cases}$$

where $(\cdot,\cdot)$ denotes the dualities. $L_0^{r'}(\Omega) = \{q \in L^{r'}(\Omega) \ \big|\ \int_\Omega q = 0\}$.

Introduce the space $\boldsymbol{V}_{div} = \left\{ \boldsymbol{v} \in \boldsymbol{V} = \boldsymbol{W}_0^{1,r}(\Omega) \;\middle|\; \nabla\cdot \boldsymbol{v} = 0 \right\}$, and then problem (2.2) can be written equivalently by:

$$(2.3) \qquad \left\{ \begin{array}{lll} \multicolumn{3}{l}{\textit{Find } (\boldsymbol{u}, \theta) \in \boldsymbol{V}_{div} \times H_0^1(\Omega) \textit{ such that}} \\ \text{(a)} & (k(\theta)|D(\boldsymbol{u})|^{r-2} D(\boldsymbol{u}), D(\boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}), & \forall\, \boldsymbol{v} \in \boldsymbol{V}_{div} \\ \text{(b)} & (\nabla\theta, \nabla\eta) = (k(\theta)|D(\boldsymbol{u})|^r, \eta), & \forall\, \eta \in H_0^1(\Omega). \end{array} \right.$$

Lemma 2.1. (cf. [29]). *For any given θ, if $\boldsymbol{u} \in \boldsymbol{V}_{div}$ satisfies* (2.2a)*, then there exist $\delta > 0$ and a constant $C > 0$ depending only on Ω, k_* and k^* such that $\boldsymbol{u} \in \boldsymbol{W}_0^{1,r(1+\delta)}(\Omega)$ and the following estimate holds*

$$\|D(\boldsymbol{u})\|_{L^{r(1+\delta)}} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{\frac{1}{r-1}}.$$

To present the mixed variational formulation, let us introduce the nonlinear deviatoric stress tensor

$$\boldsymbol{\sigma} = k(\theta)|D(\boldsymbol{u})|^{r-2} D(\boldsymbol{u})$$

then

$$D(\boldsymbol{u}) = [k(\theta)]^{1-r'} |\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} = \mu(\theta)|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}.$$

where $\mu(\theta) = [k(\theta)]^{1-r'}$. Hence, the system (1.1) can be written in the form

$$(2.4) \qquad \left\{ \begin{array}{lrl} \text{(a)} & \mu(\theta)|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - D(\boldsymbol{u}) = 0 & \text{in } \Omega \\ \text{(b)} & \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma} - p\boldsymbol{I}) + \boldsymbol{f} = 0 & \text{in } \Omega \\ \text{(c)} & \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega \\ \text{(d)} & \Delta\theta + \mu(\theta)|\boldsymbol{\sigma}|^{r'} = 0 & \text{in } \Omega \\ \text{(e)} & \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma \\ \text{(f)} & \theta = 0 & \text{on } \Gamma \end{array} \right.$$

where $\boldsymbol{I}$ is the identity tensor.

As in [24, 15], we define the spaces

$$\begin{aligned} \boldsymbol{\Sigma} &= \{(\boldsymbol{\tau}, q) \in [\boldsymbol{L}^{r'}(\Omega)]^2 \times L_0^{r'}(\Omega) \;\big|\; \boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - qI) \in \boldsymbol{L}^{r'}(\Omega) \big| \; \boldsymbol{\nabla}\cdot\boldsymbol{\tau} \in \boldsymbol{L}^{r'}(\Omega)\} \\ \boldsymbol{M} &= \{(\boldsymbol{v}, \boldsymbol{\chi}) \in \boldsymbol{L}^r(\Omega) \times [\boldsymbol{L}^r(\Omega)]^2 \;\big|\; \boldsymbol{\chi} + \boldsymbol{\chi}^T = 0\} \\ \Theta &= H_0^1(\Omega). \end{aligned}$$

equipped with the norms

$$\|(\boldsymbol{\tau}, q)\|_\Sigma = (\|\boldsymbol{\tau}\|_{0,r',\Omega}^{r'} + \|q\|_{0,r',\Omega}^{r'} + \|\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - q\boldsymbol{I})\|_{0,r',\Omega}^{r'})^{\frac{1}{r'}},$$

$$\|(\boldsymbol{v}, \boldsymbol{\chi})\|_M = (\|\boldsymbol{v}\|_{0,r,\Omega}^r + \|\boldsymbol{\chi}\|_{0,r,\Omega}^r)^{\frac{1}{r}}.$$

For all $(\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma}$ such that $\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - qI) \in \boldsymbol{L}^{r'}(\Omega)$, as $\nabla\cdot\boldsymbol{u} = 0$, one has

$$\begin{aligned} (\mu(\theta)|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}, \boldsymbol{\tau}) &= (D(\boldsymbol{u}), \boldsymbol{\tau}) = (D(\boldsymbol{u}), \boldsymbol{\tau} - q\boldsymbol{I}) = (\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\omega}, \boldsymbol{\tau} - q\boldsymbol{I}) \\ &= (\boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\tau} - q\boldsymbol{I}) - (\boldsymbol{\omega}, \boldsymbol{\tau} - q\boldsymbol{I}) = -(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - q\boldsymbol{I}), \boldsymbol{u}) - (\boldsymbol{\omega}, \boldsymbol{\tau}), \end{aligned}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\boldsymbol{u}^T)$ is the vorticity tensor. Then, the mixed variational formulation of problem (2.4) can be defined as:

(2.5)

$$\left\{ \begin{array}{lrl} \multicolumn{3}{l}{\textit{Find } ((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega}); \theta) \in \boldsymbol{\Sigma} \times \boldsymbol{M} \times \Theta \textit{ such that}} \\ \text{(a)} & (\mu(\theta)|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - q\boldsymbol{I}), \boldsymbol{u}) + (\boldsymbol{\tau}, \boldsymbol{\omega}) = 0, & \forall\, (\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma} \\ \text{(b)} & (\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma} - p\boldsymbol{I}), \boldsymbol{v}) + (\boldsymbol{\sigma}, \boldsymbol{\chi}) + (\boldsymbol{f}, \boldsymbol{v}) = 0, & \forall\, (\boldsymbol{v}, \boldsymbol{\chi}) \in \boldsymbol{M} \\ \text{(c)} & (\nabla\theta, \nabla\eta) - (\mu(\theta)|\boldsymbol{\sigma}|^{r'}, \eta) = 0, & \forall\, \eta \in \Theta. \end{array} \right.$$

As in [15], a *Lagrange multiplier* is introduced in (2.5b) to relax the symmetry of $\boldsymbol{\sigma}$.

Now, for simplification, we assume that $\mu(\theta) = [k(\theta)]^{1-r'}$ satisfies the following conditions:

$$\text{(2.6)} \qquad \begin{array}{ll} \text{(a)} & \mu(s) \in C(I\!R), \quad \mu_* \leq \mu(s) \leq \mu^*, \quad \forall s \in I\!R, \\ \text{(b)} & \|\mu'\|_{L^\infty} \leq L \end{array}$$

where $\mu_* = [k^*]^{1-r'}$ and $\mu^* = [k_*]^{1-r'}$.

For any given θ, it is easily seen that

$$(\mu(\theta)|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}, \boldsymbol{\sigma}) \geq \mu_* \|\boldsymbol{\sigma}\|^{r'}_{L^{r'}}, \quad \forall \boldsymbol{\sigma} \in [\boldsymbol{L}^{r'}(\Omega)]^2.$$

Moreover, due to Proposition 2.2 and Proposition 2.3 in [15], we have

Lemma 2.2. *There exists a positive constant β such that*

$$\text{(2.7)} \qquad \inf_{(\boldsymbol{v},\boldsymbol{\chi})\in \boldsymbol{M}} \sup_{(\boldsymbol{\tau},q)\in \boldsymbol{\Sigma}} \frac{(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - q\boldsymbol{I}), \boldsymbol{v}) + (\boldsymbol{\tau}, \boldsymbol{\chi})}{\|(\boldsymbol{v},\boldsymbol{\chi})\|_{\boldsymbol{M}} \|(\boldsymbol{\tau}, q)\|_{\Sigma}} \geq \beta.$$

Lemma 2.3. *There exists a positive constant C such that*

$$\|q\|_{L^{r'}} \leq C\|\boldsymbol{\tau}\|_{\boldsymbol{L}^{r'}}, \quad \forall (\boldsymbol{\tau}, q) \in \boldsymbol{X},$$

where

$$\boldsymbol{X} = \{(\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma} \;\Big|\; (\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - q\boldsymbol{I}), \boldsymbol{v}) + (\boldsymbol{\tau}, \boldsymbol{\chi}) = 0, \quad \forall\ (\boldsymbol{v}, \boldsymbol{\chi}) \in \boldsymbol{M}\}.$$

Hence, due to Theorem 2.4 in [15], we have

Lemma 2.4. *For any given θ,* (2.5a,b) *has a unique solution $((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega})) \in \boldsymbol{\Sigma} \times \boldsymbol{M}$. Moreover $\boldsymbol{\sigma} = k(\theta)|D(\boldsymbol{u})|^{r-2}D(\boldsymbol{u})$, and $\boldsymbol{u}$ satisfies* (1.1a).

By Lemmas 2.1-2.4, we have

Lemma 2.5. *For any given θ, there exist $\delta > 0$ and a constant $C > 0$ depending only on Ω, μ_* and μ^* such that the solution $((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega})) \in \boldsymbol{L}^{r'(1+\delta)}(\Omega) \times L^{r'(1+\delta)}(\Omega) \times \boldsymbol{W}^{1,r(1+\delta)}(\Omega) \times \boldsymbol{L}^{r(1+\delta)}(\Omega)$ of* (2.5a,b) *and the following estimates hold*

$$\begin{array}{l} \|D(\boldsymbol{u})\|_{L^{r(1+\delta)}} + \|\boldsymbol{\omega}\|_{\boldsymbol{L}^{r(1+\delta)}} \leq C\|\boldsymbol{f}\|^{1/(r-1)}_{\boldsymbol{L}^2} \\ \|\boldsymbol{\sigma}\|_{\boldsymbol{L}^{r'(1+\delta)}} + \|p\|_{L^{r'(1+\delta)}} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}. \end{array}$$

Lemma 2.6. *If $((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega}); \theta)$ solves problem* (2.5)*, then $(\boldsymbol{u}, \theta)$ satisfies* (2.3)*. Conversely, if $(\boldsymbol{u}, \theta)$ is a solution to* (2.3)*, then $((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega}); \theta)$ solves problem* (2.5).

Using a similar technique as in [33], we can show that the following result holds.

Theorem 2.1. (Existence) *Problem* (2.5) *has a solution $((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega}); \theta)$, and for $\delta > 0$, defined in Lemma* 2.1*, there exists a constant $C > 0$ depending only on Ω, μ_*, μ^*, δ, r and r' such that*

$$\text{(2.8)} \qquad \|D(\boldsymbol{u})\|_{L^{r(1+\delta)}} + \|\boldsymbol{\omega}\|_{\boldsymbol{L}^{r(1+\delta)}} \leq C\|\boldsymbol{f}\|^{1/(r-1)}_{\boldsymbol{L}^2}$$

$$\text{(2.9)} \qquad \|\boldsymbol{\sigma}\|_{\boldsymbol{L}^{r'(1+\delta)}} + \|p\|_{L^{r'(1+\delta)}} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}$$

$$\text{(2.10)} \qquad \|\nabla\theta\|_{L^{\tilde{r}}} \leq C\|\boldsymbol{f}\|^{r'}_{\boldsymbol{L}^2}$$

where

$$\tilde{r} = \begin{cases} \frac{2(1+\delta)}{1-\delta}, & \text{if} \quad \delta < 1 \\ \text{any number in } (2,\infty), & \text{if} \quad \delta = 1 \\ \infty, & \text{if} \quad \delta > 1. \end{cases}$$

To study the uniqueness of problem (2.5), we will use the following technical lemma (see [5, 20]):

Lemma 2.7. *For all $r' > 1$, there exist two positive constants C_1 and C_2 such that for all X, $Y \in R^d$,*

$$(|X|^{r'-2}X - |Y|^{r'-2}Y, X - Y) \geq C_1|X-Y|^2(|X|+|Y|)^{r'-2} \tag{2.11}$$

$$||X|^{r'-2}X - |Y|^{r'-2}Y| \leq C_2|X-Y|(|X|+|Y|)^{r'-2}. \tag{2.12}$$

To analyze the uniqueness of problem (2.5), we restrict ourselves (from now on) in the case $1 < r' \leq 2$ or $r \geq 2$. If the problem (2.5) has two solutions $((\boldsymbol{\sigma}_1, p_1); (\boldsymbol{u}_1, \boldsymbol{\omega}_1); \theta_1)$ and $((\boldsymbol{\sigma}_2, p_2); (\boldsymbol{u}_2, \boldsymbol{\omega})_2; \theta_2)$, and let $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$, $\bar{p} = p_1 - p_2$, $\bar{\boldsymbol{u}} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\bar{\boldsymbol{\omega}} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2$ and $\bar{\theta} = \theta_1 - \theta_2$. Then, by (2.5), we have

$$\begin{array}{ll} \text{(a)} & (\mu(\theta_1)|\boldsymbol{\sigma}_1|^{r'-2}\boldsymbol{\sigma}_1 - \mu(\theta_2)|\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2, \boldsymbol{\tau}) \\ & \quad +(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - q\boldsymbol{I}), \bar{\boldsymbol{u}}) + (\boldsymbol{\tau}, \bar{\boldsymbol{\omega}}) = 0, \forall\, (\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma} \\ \text{(b)} & (\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{\sigma}} - \bar{p}\boldsymbol{I}), \boldsymbol{v}) + (\bar{\boldsymbol{\sigma}}, \boldsymbol{\chi}) = 0, \quad \forall\, (\boldsymbol{v}, \boldsymbol{\chi}) \in \boldsymbol{M} \\ \text{(c)} & (\nabla\bar{\theta}, \nabla\eta) - (\mu(\theta_1)|\boldsymbol{\sigma}_1|^{r'} - \mu(\theta_2)|\boldsymbol{\sigma}_2|^{r'}, \eta) = 0, \quad \forall\, \eta \in \Theta. \end{array} \tag{2.13}$$

Thus, taking $(\boldsymbol{\tau}, q) = (\bar{\boldsymbol{\sigma}}, \bar{p})$, $(\boldsymbol{v}, \boldsymbol{\chi}) = (\bar{\boldsymbol{u}}, \bar{\boldsymbol{\omega}})$ in (2.13a,b), and using (2.6) and (2.9), we have

$$\begin{aligned} & \mu_*|(|\boldsymbol{\sigma}_1|^{r'-2}\boldsymbol{\sigma}_1 - |\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2, \bar{\boldsymbol{\sigma}})| \\ \leq\ & |(\mu(\theta_1)[|\boldsymbol{\sigma}_1|^{r'-2}\boldsymbol{\sigma}_1 - |\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2], \bar{\boldsymbol{\sigma}})| \\ =\ & |([\mu(\theta_1) - \mu(\theta_2)]|\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2, \bar{\boldsymbol{\sigma}}) + (\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{\sigma}} - \bar{p}\boldsymbol{I}), \bar{\boldsymbol{u}}) + (\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\omega}})| \\ =\ & |([\mu(\theta_1) - \mu(\theta_2)]|\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2, \bar{\boldsymbol{\sigma}}) \\ \leq\ & L\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{L}^{r'}}\|\boldsymbol{\sigma}_2\|^{r'-1}_{\boldsymbol{L}^{r'(1+\delta)}}\|\bar{\theta}\|_{L^{r(1+\delta)/\delta}} \\ \leq\ & C\|\boldsymbol{f}\|^{r'-1}_{\boldsymbol{L}^2}\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{L}^{r'}}\|\nabla\bar{\theta}\|_{L^2}. \end{aligned} \tag{2.14}$$

Utilizing (2.11), we know that

$$\mu_*|(|\boldsymbol{\sigma}_1|^{r'-2}\boldsymbol{\sigma}_1 - |\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2, \bar{\boldsymbol{\sigma}})| \geq \mu_* C_1\|\bar{\boldsymbol{\sigma}}\|^2_{\boldsymbol{L}^{r'}}(\|\boldsymbol{\sigma}_1\|_{\boldsymbol{L}^{r'}} + \|\boldsymbol{\sigma}_2\|_{\boldsymbol{L}^{r'}})^{r'-2}. \tag{2.15}$$

Hence, combining (2.14) with (2.15), we get

$$\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{L}^{r'}} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}\|\nabla\bar{\theta}\|_{L^2}. \tag{2.16}$$

On the other hand, by (2.13c), we have

$$\begin{aligned} \|\nabla\bar{\theta}\|^2_{L^2} &= (\mu(\theta_1)|\boldsymbol{\sigma}_1|^{r'} - \mu(\theta_2)|\boldsymbol{\sigma}_2|^{r'}, \bar{\theta}) \\ &= ([\mu(\theta_1) - \mu(\theta_2)]|\boldsymbol{\sigma}_1|^{r'}, \bar{\theta}) + (\mu(\theta_2)[|\boldsymbol{\sigma}_1|^{r'} - |\boldsymbol{\sigma}_2|^{r'}], \bar{\theta}) \\ &= D_1 + D_2. \end{aligned} \tag{2.17}$$

It is easily seen that

$$\begin{aligned} D_1 &\leq L\|\boldsymbol{\sigma}_1\|^{r'}_{\boldsymbol{L}^{r'(1+\delta)}}\|\bar{\theta}\|^2_{L^{2(1+\delta)/\delta}} \\ &\leq C\|\boldsymbol{f}\|^{r'}_{\boldsymbol{L}^2}\|\nabla\bar{\theta}\|^2_{L^2}. \end{aligned} \tag{2.18}$$

Since

$$|\boldsymbol{\sigma}_1|^{r'} - |\boldsymbol{\sigma}_2|^{r'} = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot(|\boldsymbol{\sigma}_1|^{r'-2}\boldsymbol{\sigma}_1 - |\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2) + \frac{1}{2}\bar{\boldsymbol{\sigma}}\cdot(|\boldsymbol{\sigma}_1|^{r'-2}\boldsymbol{\sigma}_1 + |\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2)$$

then, using (2.12) and (2.16), we have

$$
\begin{aligned}
D_2 &= (\frac{\mu(\theta_2)}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(|\boldsymbol{\sigma}_1|^{r'-2}\boldsymbol{\sigma}_1-|\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2),\bar{\theta}) \\
&\quad +(\frac{\mu(\theta_2)}{2}\bar{\boldsymbol{\sigma}}\cdot(|\boldsymbol{\sigma}_1|^{r'-2}\boldsymbol{\sigma}_1+|\boldsymbol{\sigma}_2|^{r'-2}\boldsymbol{\sigma}_2),\bar{\theta}) \\
&\le C(\|\boldsymbol{\sigma}_1\|^{r'-1}_{\boldsymbol{L}^{r'(1+\delta)}}+\|\boldsymbol{\sigma}_2\|^{r'-1}_{\boldsymbol{L}^{r'(1+\delta)}})\|\bar{\boldsymbol{\sigma}}\|_{\boldsymbol{L}^{r'}}\|\bar{\theta}\|_{L^{r(1+\delta)/\delta}} \\
&\le C\|\boldsymbol{f}\|^{r'}_{\boldsymbol{L}^2}\|\nabla\bar{\theta}\|^2_{L^2}.
\end{aligned} \tag{2.19}
$$

Combining (2.17), (2.18) and (2.19), we have

$$
\|\nabla\bar{\theta}\|^2_{L^2} \le \bar{C}\|\boldsymbol{f}\|^{r'}_{\boldsymbol{L}^2}\|\nabla\bar{\theta}\|^2_{L^2}, \tag{2.20}
$$

where $\bar{C}$ is a constant dependent of Ω, C_1, C_2, μ_*, μ^*, L, r' and δ.

Therefore, if

$$
\bar{C}\|\boldsymbol{f}\|^{r'}_{\boldsymbol{L}^2} < 1, \tag{2.21}
$$

then it holds that $\bar{\theta}=0$, which implies that $\bar{\boldsymbol{\sigma}}=\mathbf{0}$ (by (2.16)) and $\bar{\boldsymbol{u}}=\mathbf{0}$, $\bar{\boldsymbol{\omega}}=\mathbf{0}$ and $\bar{p}=0$ (by Lemmas 2.2 and 2.3).

Theorem 2.2. (Uniqueness) *Under the assumption* (2.6)*, if condition* (2.21) *holds, then, problem* (2.5) *has a unique solution.*

By Theorem 2.2 and Lemma 2.6, we get the uniqueness of problem (2.3) if $r \ge 2$. On the other hand, this uniqueness has been shown in [33] for the case of $1 < r \le 2$. So, we have, for $1 < r < \infty$

Theorem 2.3. (General uniqueness) *Assume that $\boldsymbol{f}$ is sufficiently small such that both* (3.34) *in* [33] (*same form as* (2.21) *but with different constant* $\bar{C}$) *and* (2.21) *hold, then problem* (2.3) *has a unique solution.*

3. An iterative method

In this section, we introduce an iterative method to solve problem (2.5). For an arbitrary $\theta^0 \in \Theta$, and $n = 1, 2, \cdots$, we can calculate $((\boldsymbol{\sigma}^n, p^n); (\boldsymbol{u}^n, \boldsymbol{\omega}^n); \theta^n)$ by:

$$
\left\{
\begin{array}{lr}
\multicolumn{2}{l}{\text{Find } ((\boldsymbol{\sigma}^n,p^n);(\boldsymbol{u}^n,\boldsymbol{\omega}^n);\theta^n) \in \boldsymbol{\Sigma}\times\boldsymbol{M}\times\Theta \text{ such that}} \\
\text{(a)} & (\mu(\theta^{n-1})|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n,\boldsymbol{\tau})+(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}-q\boldsymbol{I}),\boldsymbol{u}^n)+(\boldsymbol{\tau},\boldsymbol{\omega}^n)=0, \quad \forall\,(\boldsymbol{\tau},q)\in\boldsymbol{\Sigma} \\
\text{(b)} & (\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}^n-p^n\boldsymbol{I}),\boldsymbol{v})+(\boldsymbol{\sigma}^n,\boldsymbol{\chi})+(\boldsymbol{f},\boldsymbol{v})=0, \quad \forall\,(\boldsymbol{v},\boldsymbol{\chi})\in\boldsymbol{M} \\
\text{(c)} & (\nabla\theta^n,\nabla\eta)-(\mu(\theta^{n-1})|\boldsymbol{\sigma}^n|^{r'},\eta)=0, \quad \forall\,\eta\in\Theta.
\end{array}
\right. \tag{3.1}
$$

(3.1a) implies that

$$
\boldsymbol{\sigma}^n = k(\theta^{n-1})|D(\boldsymbol{u}^n)|^{r-2}D(\boldsymbol{u}^n). \tag{3.2}
$$

Similarly to Theorems 2.1 , we have

Theorem 3.1. *The solution* $((\boldsymbol{\sigma}^n, p^n); (\boldsymbol{u}^n, \boldsymbol{\omega}^n); \theta^n)$ *to* (3.1) *satisfies:*

$$
\|D(\boldsymbol{u}^n)\|_{L^{r(1+\delta)}} + \|\boldsymbol{\omega}^n\|_{\boldsymbol{L}^{r(1+\delta)}} \le C\|\boldsymbol{f}\|^{1/(r-1)}_{\boldsymbol{L}^2}, \qquad \forall\, n \ge 1, \tag{3.3}
$$

$$
\|\boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'(1+\delta)}} + \|p^n\|_{L^{r'(1+\delta)}} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}, \qquad \forall\, n \ge 1, \tag{3.4}
$$

$$
\|\nabla\theta^n\|_{L^{\tilde{r}}} \le C\|\boldsymbol{f}\|^{r'}_{\boldsymbol{L}^2}, \qquad \forall\, n \ge 1 \tag{3.5}
$$

where δ and $\tilde{r}$ are same as in Theorem 2.1.

Subtracting (3.1) from (2.5), we have, for all $(\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma}, (\boldsymbol{v}, \boldsymbol{\chi}) \in \boldsymbol{M}$

$$
\begin{array}{ll}
\text{(a)} & (\mu(\theta)|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - \mu(\theta^{n-1})|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n, \boldsymbol{\tau}) \\
& = -(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - q\boldsymbol{I}), \boldsymbol{u} - \boldsymbol{u}^n) - (\boldsymbol{\tau}, \boldsymbol{\omega} - \boldsymbol{\omega}^n) \\
\text{(b)} & (\boldsymbol{\nabla}\cdot[\boldsymbol{\sigma} - \boldsymbol{\sigma}^n - (p - p^n)\boldsymbol{I}], \boldsymbol{v}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^n, \boldsymbol{\chi}) = 0 \\
\text{(c)} & (\nabla(\theta - \theta^n), \nabla\eta) = (\mu(\theta)|\boldsymbol{\sigma}|^{r'} - \mu(\theta^{n-1})|\boldsymbol{\sigma}^n|^{r'}, \eta).
\end{array} \tag{3.6}
$$

Similarly to (2.16) and (2.20), we can deduce that

$$
\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}} &\le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}\|\nabla(\theta - \theta^{n-1})\|_{L^2}, \\
\|\nabla(\theta - \theta^n)\|_{L^2} &\le \bar{C}\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'}\|\nabla(\theta - \theta^{n-1})\|_{L^2}
\end{aligned} \tag{3.7}
$$

where $\bar{C}$ is same as in (2.20). Using Lemma 2.3, we have

$$
\|p - p^n\|_{L^{r'}} \le C\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}\|\nabla(\theta - \theta^{n-1})\|_{L^2}. \tag{3.8}
$$

Now, we estimate the bound of $\|\boldsymbol{u} - \boldsymbol{u}^n\|_{\boldsymbol{L}^r}$.

By Lemma 2.2 and (3.6a), we have

(3.9)

$$
\begin{aligned}
\beta\|(\boldsymbol{u} - \boldsymbol{u}^n, \boldsymbol{\omega} - \boldsymbol{\omega}^n)\|_{\boldsymbol{M}} &\le \sup_{(\boldsymbol{\tau}, q)\in\boldsymbol{\Sigma}} \frac{(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau} - q\boldsymbol{I}), \boldsymbol{u} - \boldsymbol{u}^n) + (\boldsymbol{\tau}, \boldsymbol{\omega} - \boldsymbol{\omega}^n)}{\|(\boldsymbol{\tau}, q)\|_{\boldsymbol{\Sigma}}} \\
&= \sup_{(\boldsymbol{\tau}, q)\in\boldsymbol{\Sigma}} \frac{-(\mu(\theta)|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - \mu(\theta^{n-1})|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n, \boldsymbol{\tau})}{\|(\boldsymbol{\tau}, q)\|_{\boldsymbol{\Sigma}}} \\
&\le \sup_{(\boldsymbol{\tau}, q)\in\boldsymbol{\Sigma}} \frac{|([\mu(\theta) - \mu(\theta^{n-1})]|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}, \boldsymbol{\tau})|}{\|(\boldsymbol{\tau}, q)\|_{\boldsymbol{\Sigma}}} \\
&\quad + \sup_{(\boldsymbol{\tau}, q)\in\boldsymbol{\Sigma}} \frac{|(\mu(\theta^{n-1})[|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n], \boldsymbol{\tau})|}{\|(\boldsymbol{\tau}, q)\|_{\boldsymbol{\Sigma}}} \\
&= T_1 + T_2.
\end{aligned}
$$

Note that(2.9),

$$
\begin{aligned}
T_1 &\le \|[\mu(\theta) - \mu(\theta^{n-1})]|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}\|_{\boldsymbol{L}^r} \\
&\le L\|\boldsymbol{\sigma}\|_{\boldsymbol{L}^{r'(1+\delta)}}^{r'-1}\|\theta - \theta^{n-1}\|_{L^{r(1+\delta)/\delta}} \\
&\le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1}\|\nabla(\theta - \theta^{n-1})\|_{L^2}.
\end{aligned} \tag{3.10}
$$

Taking $(\boldsymbol{\tau}, q) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}^n, p - p^n)$, $(\boldsymbol{v}, \boldsymbol{\chi}) = (\boldsymbol{u} - \boldsymbol{u}^n, \boldsymbol{\omega} - \boldsymbol{\omega}^n)$ in (3.6a) and (3.6b), respectively, we know that

$$
(\mu(\theta^{n-1})|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n - \mu(\theta)|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}, \boldsymbol{\sigma} - \boldsymbol{\sigma}^n) = 0.
$$

So, we have

$$
\begin{aligned}
& |([\mu(\theta) - \mu(\theta^{n-1})]|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}, \boldsymbol{\sigma} - \boldsymbol{\sigma}^n)| \\
= {} & |(\mu(\theta^{n-1})[|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n, \boldsymbol{\sigma} - \boldsymbol{\sigma}^n)| \\
\ge {} & \mu_*|(|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n, \boldsymbol{\sigma} - \boldsymbol{\sigma}^n)| \\
\ge {} & C\frac{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}}^2}{\|\boldsymbol{\sigma}\|_{\boldsymbol{L}^{r'}}^{2-r'} + \|\boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}}^{2-r'}} + C(||\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n|, |\boldsymbol{\sigma} - \boldsymbol{\sigma}^n|)
\end{aligned} \tag{3.11}
$$

where we have used a similar technique as in [27]. Thus, we have

$$
\begin{aligned}
& (||\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n|, |\boldsymbol{\sigma} - \boldsymbol{\sigma}^n|) \\
\le {} & C|([\mu(\theta) - \mu(\theta^{n-1})]|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma}, \boldsymbol{\sigma} - \boldsymbol{\sigma}^n)| \\
\le {} & CL\|\boldsymbol{\sigma}\|_{\boldsymbol{L}^{r'(1+\delta)}}^{r'-1}\|\theta - \theta^{n-1}\|_{L^{r(1+\delta)/\delta}}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}} \\
\le {} & C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1}\|\nabla(\theta - \theta^{n-1})\|_{L^2}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}}.
\end{aligned} \tag{3.12}
$$

Using the fact that (cf. [27]), for all $\boldsymbol{\tau} \in [\boldsymbol{L}^{r'}(\Omega)]^2$

(3.13)
$$(|\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n), \boldsymbol{\tau}) \le C(||\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n|, |\boldsymbol{\sigma} - \boldsymbol{\sigma}^n|)^{1/r}\|\boldsymbol{\tau}\|_{\boldsymbol{L}^{r'}},$$

we can get

$$\begin{aligned} T_2 &\le C(||\boldsymbol{\sigma}|^{r'-2}\boldsymbol{\sigma} - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n|, |\boldsymbol{\sigma} - \boldsymbol{\sigma}^n|)^{1/r} \\ &\le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{(r'-1)/r}\|\nabla(\theta - \theta^{n-1})\|_{L^2}^{1/r}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}}^{1/r} \\ &\le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1}\|\nabla(\theta - \theta^{n-1})\|_{L^2}^{2/r}. \end{aligned} \tag{3.14}$$

Substituting the estimates (3.10) and (3.14) into (3.9), we can obtain

$$\|\boldsymbol{u}-\boldsymbol{u}^n\|_{\boldsymbol{L}^r}+\|\boldsymbol{\omega}-\boldsymbol{\omega}^n\|_{\boldsymbol{L}^r} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1}\{\|\nabla(\theta-\theta^{n-1})\|_{L^2}+\|\nabla(\theta-\theta^{n-1})\|_{L^2}^{2/r}\}. \tag{3.15}$$

Theorem 3.2. *Under the assumptions of Theorem 2.2, the iterative algorithm (3.1) presents a linear convergence rate if*

$$\tilde{M}(\boldsymbol{f}) = \bar{C}\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{2r'/r} < 1 \tag{3.16}$$

is satisfied. And the following estimates hold:

$$\begin{aligned} &\text{(a)} \quad \|\nabla(\theta - \theta^n)\|_{L^2} \le \tilde{M}(\boldsymbol{f})^n\|\nabla(\theta - \theta^0)\|_{L^2} \\ &\text{(b)} \quad \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}} + \|p - p^n\|_{L^{r'}} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}\tilde{M}(\boldsymbol{f})^{n-1}\|\nabla(\theta - \theta^0)\|_{L^2} \\ &\text{(c)} \quad \|\boldsymbol{u} - \boldsymbol{u}^n\|_{\boldsymbol{L}^r} + \|\boldsymbol{\omega} - \boldsymbol{\omega}^n\|_{\boldsymbol{L}^r} \\ &\qquad \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1}\tilde{M}(\boldsymbol{f})^{n-1}\{\|\nabla(\theta - \theta^0)\|_{L^2} + \|\nabla(\theta - \theta^0)\|_{L^2}^{2/r}\}. \end{aligned} \tag{3.17}$$

4. Mixed finite element approximation

To present the mixed finite element approximation, for simplicity, we assume that Ω is polygonal and is divided into a regular family of triangulations T_h (triangulation of $\bar{\Omega}$ into closed triangles K). Let $P_k(K)$ denote the space of polynomials of degree less than or equal to k on K. Set

$$\boldsymbol{R}(K) = [P_1(K)]^2 + \alpha\, \boldsymbol{curl}\, b_k,$$

where α is a constant, and b_k the "bubble function" defined on K by

$$b_K(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x), \quad \text{with } \lambda_1,\ \lambda_2 \text{ and } \lambda_3 \text{ the barycentric coordinate in } K.$$

We define the finite element spaces (see [15])

$$\boldsymbol{\Sigma}_h = \{(\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\Sigma} \;\Big|\; \boldsymbol{\tau}_h|_K \in [\boldsymbol{R}(K)]^2, q_h|_K \in P_1(K), \forall\, K \in T_h\}$$

$$\boldsymbol{M}_h = \{(\boldsymbol{v}_h, \boldsymbol{\chi}_h) \in \boldsymbol{M} \;\Big|\; \boldsymbol{v}_h|_k \in [P_0(K)]^2, \boldsymbol{\chi}_h|_K = \xi_h\boldsymbol{\zeta} \text{ with } \xi_h|_K \in P_1(K), \forall\, K \in T_h\}$$

where

$$\boldsymbol{\zeta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We also introduce another finite element space

$$\Theta_h \;=\; \Big\{\eta_h \in \Theta \;\Big|\; \eta_h|_K \in P_1(K),\ \forall\, K \in T_h\Big\}.$$

For any given $\theta_h^0 \in \Theta_h$, and $n = 1, 2, \cdots$, we can calculate $((\boldsymbol{\sigma}_h^n, p_h^n); (\boldsymbol{u}_h^n, \boldsymbol{\omega}_h^n); \theta_h^n)$ by:

(4.1)
$$\begin{cases} \textit{Find } ((\boldsymbol{\sigma}_h^n, p_h^n); (\boldsymbol{u}_h^n, \boldsymbol{\omega}_h^n); \theta_h^n) \in \boldsymbol{\Sigma}_h \times \boldsymbol{M}_h \times \Theta_h \textit{ such that} \\ \text{(a)} \quad (\mu(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h) + (\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_h - q_h\boldsymbol{I}), \boldsymbol{u}_h^n) + (\boldsymbol{\tau}_h, \boldsymbol{\omega}_h^n) = 0, \quad \forall\, (\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\Sigma}_h \\ \text{(b)} \quad (\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}_h^n - p_h^n\boldsymbol{I}), \boldsymbol{v}_h) + (\boldsymbol{\sigma}_h^n, \boldsymbol{\chi}_h) + (\boldsymbol{f}, \boldsymbol{v}_h) = 0, \quad \forall\, (\boldsymbol{v}_h, \boldsymbol{\chi}_h) \in \boldsymbol{M}_h \\ \text{(c)} \quad (\nabla\theta_h^n, \nabla\eta_h) - (\mu(\theta_h^{n-1})|{\boldsymbol{\sigma}^n}_h|^{r'}, \eta_h) = 0, \quad \forall\, \eta_h \in \Theta_h. \end{cases}$$

Similarly to Proposition 3.1 and Proposition 3.2 in [15], we can easily get the result:

Lemma 4.1. *There exists a positive constant β^* such that*

$$\inf_{(\boldsymbol{v}_h,\boldsymbol{\chi}_h)\in \boldsymbol{M}_h} \sup_{(\boldsymbol{\tau}_h,q_h)\in\boldsymbol{\Sigma}_h} \frac{(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_h - q_h\boldsymbol{I}),\boldsymbol{v}_h)+(\boldsymbol{\tau}_h,\boldsymbol{\chi}_h)}{\|(\boldsymbol{v}_h,\boldsymbol{\chi}_h)\|_{\boldsymbol{M}}\|(\boldsymbol{\tau}_h,q_h)\|_{\Sigma}} \geq \beta^*. \tag{4.2}$$

Lemma 4.2. *There exists a positive constant C indenpent of h, such that*

$$\|q_h\|_{L^{r'}} \leq C\|\boldsymbol{\tau}_h\|_{\boldsymbol{L}^{r'}}, \quad \forall\,(\boldsymbol{\tau}_h,q_h)\in \boldsymbol{X}_h,$$

where

$$\boldsymbol{X}_h = \{(\boldsymbol{\tau}_h,q_h)\in\boldsymbol{\Sigma}_h; (\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_h + q_h\boldsymbol{I}),\boldsymbol{v}_h)+(\boldsymbol{\tau}_h,\boldsymbol{\chi}_h)=0, \quad \forall\,(\boldsymbol{v}_h,\boldsymbol{\chi}_h)\in\boldsymbol{M}_h\}.$$

Remark 4.1. *Similarly to Theorem* 2.1 *and Theorem* 2.2*, we can easily show that the existence and uniqueness of problem* (4.1)*.*

To analyze problem (4.1), for given θ_h^{n-1}, we define $((\tilde{\boldsymbol{\sigma}}_h^n,\tilde{p}_h^n);(\tilde{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{\omega}}_h^n))\in\boldsymbol{\Sigma}_h\times\boldsymbol{M}_h$ such that

(4.3)

$$\begin{array}{ll}
\text{(a)} & (\mu(\theta_h^{n-1})|\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n,\boldsymbol{\tau}_h)+(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_h-q_h\boldsymbol{I}),\tilde{\boldsymbol{u}}_h^n)+(\boldsymbol{\tau}_h,\tilde{\boldsymbol{\omega}}_h^n)=0, \quad \forall\,(\boldsymbol{\tau}_h,q_h)\in\boldsymbol{\Sigma}_h,\\
\text{(b)} & (\boldsymbol{\nabla}\cdot(\tilde{\boldsymbol{\sigma}}_h^n-\tilde{p}_h^n\boldsymbol{I}),\boldsymbol{v}_h)+(\tilde{\boldsymbol{\sigma}}_h^n,\boldsymbol{\chi}_h)+(\boldsymbol{f},\boldsymbol{v}_h)=0, \quad \forall\,(\boldsymbol{v}_h,\boldsymbol{\chi}_h)\in\boldsymbol{M}_h.
\end{array}$$

Using a similar technique as in [15], we can get the following result:

Lemma 4.3. *For given θ_h^{n-1}, if $(\boldsymbol{\sigma},p)\in[\boldsymbol{W}^{m,r'}(\Omega)]^2\times W^{m,r'}(\Omega)$ and $(\boldsymbol{u},\boldsymbol{\omega})\in\boldsymbol{W}^{1,r}(\Omega)\times[\boldsymbol{W}^{m,r}(\Omega)]^2$, $m=1,2$, then there exists a constant C, independent on h, such that*

$$\begin{array}{lrcl}
\text{(a)} & \|\boldsymbol{\sigma}^n-\tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}}+\|p^n-\tilde{p}^n\|_{L^{r'}} & \leq & Ch^{mr'/2}\\
\text{(b)} & \|\boldsymbol{\omega}^n-\tilde{\boldsymbol{\omega}}_h^n\|_{\boldsymbol{L}^r} & \leq & Ch^{m(r'-1)}\\
\text{(c)} & \|\boldsymbol{u}^n-\tilde{\boldsymbol{u}}_h^n\|_{\boldsymbol{L}^r} & \leq & \left\{\begin{array}{ll} Ch^{r'-1}, & \text{if} \quad m=1\\ Ch^{\min\{1,2(r'-1)\}}, & \text{if} \quad m=2.\end{array}\right.
\end{array} \tag{4.4}$$

To analyze the convergence of the finite element approximation (4.1), we introduce another projection operator $R_h:\Theta\to\Theta_h$ satisfying:

$$\begin{array}{ll}
\text{(a)} & (\nabla(\theta^n-R_h\theta^n),\nabla\eta_h)=0, \quad \forall\,\eta_h\in\Theta_h\\
\text{(b)} & \|\theta^n-R_h\theta^n\|_{L^2}+h\|\nabla(\theta^n-R_h\theta^n)\|_{L^2}\leq Ch^2\|\theta^n\|_{H^2}.
\end{array} \tag{4.5}$$

Subtracting (3.1) from (4.1) and using (4.3) and (4.5a), we can get the residual equations as follows:

(4.6)

$$\begin{array}{lrcl}
\text{(a)} & (\mu(\theta_h^{n-1})[|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n-|\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n],\boldsymbol{\tau}_h) & & \\
 & +([\mu(\theta_h^{n-1})-\mu(\theta^{n-1})]|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n,\boldsymbol{\tau}_h) & & \\
 & +(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_h-q_h\boldsymbol{I}),\boldsymbol{u}_h^n-\tilde{\boldsymbol{u}}_h^n)+(\boldsymbol{\tau}_h,\boldsymbol{\omega}_h^n-\tilde{\boldsymbol{\omega}}_h^n) & = & 0, \quad \forall\,(\boldsymbol{\tau}_h,q_h)\in\boldsymbol{\Sigma}_h\\
\text{(b)} & (\boldsymbol{\nabla}\cdot[\boldsymbol{\sigma}_h^n-\tilde{\boldsymbol{\sigma}}_h^n-(p_h^n-\tilde{p}_h^n)\boldsymbol{I}],\boldsymbol{v}_h)+(\boldsymbol{\sigma}_h^n-\tilde{\boldsymbol{\sigma}}_h^n,\boldsymbol{\chi}_h)+(\boldsymbol{f},\boldsymbol{v}_h) & = & 0, \quad \forall\,(\boldsymbol{v}_h,\boldsymbol{\chi}_h)\in\boldsymbol{M}_h\\
\text{(c)} & (\nabla(\theta_h^n-R_h\theta^n),\nabla\eta_h)-(\mu(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^{r'}-\mu(\theta^{n-1})|\boldsymbol{\sigma}^n|^{r'},\eta_h) & = & 0, \quad \forall\,\eta_h\in\Theta_h.
\end{array}$$

Taking $(\boldsymbol{\tau}_h,q_h)=(\boldsymbol{\sigma}_h^n-\tilde{\boldsymbol{\sigma}}_h^n,p_h^n-\tilde{p}_h^n)$ in (4.6a) and using (4.6b), we have

$$\begin{array}{rl}
 & (\mu(\theta_h^{n-1})[|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n-|\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n],\boldsymbol{\sigma}_h^n-\tilde{\boldsymbol{\sigma}}_h^n)\\
= & ([\mu(\theta^{n-1})-\mu(\theta_h^{n-1})]|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n,\boldsymbol{\sigma}_h^n-\tilde{\boldsymbol{\sigma}}_h^n).
\end{array} \tag{4.7}$$

By (2.11) and (3.4), we can obtain

$$
\begin{aligned}
&\mu_* C_1 \|\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}}^2 (\|\tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} + \|\boldsymbol{\sigma}_h^n\|_{\boldsymbol{L}^{r'}})^{r'-2} \\
\le\ & (\mu(\theta_h^{n-1})[|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n - |\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n], \boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n) \\
=\ & ([\mu(\theta^{n-1}) - \mu(\theta_h^{n-1})]|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n, \boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n) \\
\le\ & L\|\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} \|\boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'(1+\delta)}}^{r'-1} \|\theta^{n-1} - \theta_h^{n-1}\|_{L^{r(1+\delta)/\delta}} \\
\le\ & C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1} \|\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} \|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}.
\end{aligned} \tag{4.8}
$$

Hence, using Lemma 4.2, we obtain

$$
\|\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} + \|p_h^n - \tilde{p}_h^n\|_{L^{r'}} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}. \tag{4.9}
$$

where we have used the fact that $\|\tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} + \|\boldsymbol{\sigma}_h^n\|_{\boldsymbol{L}^{r'}} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}$, which is similar to the one for $\boldsymbol{\sigma}$ in Theorem 2.1.

By Lemma 4.1 and (4.6a), we know that

$$
\begin{aligned}
&\beta^* \|(\boldsymbol{u}_h^n - \tilde{\boldsymbol{u}}_h^n, \boldsymbol{\omega}_h^n - \tilde{\boldsymbol{\omega}}_h^n)\|_M \\
\le\ & \sup_{(\boldsymbol{\tau}_h, q_h)\in\boldsymbol{\Sigma}_h} \frac{(\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_h - q_h\boldsymbol{I}), \boldsymbol{u}_h^n - \tilde{\boldsymbol{u}}_h^n) + (\boldsymbol{\tau}_h, \boldsymbol{\omega}_h^n - \tilde{\boldsymbol{\omega}}_h^n)}{\|(\boldsymbol{\tau}_h, q_h)\|_\Sigma} \\
=\ & \sup_{(\boldsymbol{\tau}_h, q_h)\in\boldsymbol{\Sigma}_h} \frac{-([\mu(\theta_h^{n-1}) - \mu(\theta^{n-1})]|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n + \mu(\theta_h^{n-1})[|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n - |\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n], \boldsymbol{\tau}_h)}{\|(\boldsymbol{\tau}_h, q_h)\|_\Sigma} \\
\le\ & \sup_{(\boldsymbol{\tau}_h, q_h)\in\boldsymbol{\Sigma}_h} \frac{|([\mu(\theta^{n-1}) - \mu(\theta_h^{n-1})]|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n, \boldsymbol{\tau}_h)|}{\|(\boldsymbol{\tau}_h, q_h)\|_\Sigma} \\
&\quad + \sup_{(\boldsymbol{\tau}_h, q_h)\in\boldsymbol{\Sigma}_h} \frac{|(\mu(\theta_h^{n-1})[|\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n - |\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n], \boldsymbol{\tau}_h)|}{\|(\boldsymbol{\tau}_h, q_h)\|_\Sigma} \\
=\ & E_1 + E_2.
\end{aligned} \tag{4.10}
$$

Using (2.9), we have

$$
\begin{aligned}
E_1 &\le L\|\boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'(1+\delta)}}^{r'-1} \|\theta^{n-1} - \theta_h^{n-1}\|_{L^{r(1+\delta)/\delta}} \\
&\le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1} \|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}.
\end{aligned} \tag{4.11}
$$

Now, we estimate E_2. Using a similar technique as in [15, 27], we can get

$$
\begin{aligned}
&((\mu(\theta_h^{n-1})[|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n - |\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n], \boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n) \\
\ge\ & \mu_*(|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n - |\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n, \boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n) \\
\ge\ & C\frac{\|\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}}^2}{\|\tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}}^{2-r'} + \|\boldsymbol{\sigma}_h^n\|_{\boldsymbol{L}^{r'}}^{2-r'}} + C(||\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n - |\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n|, |\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n|).
\end{aligned} \tag{4.12}
$$

By Hölder inequality, Sobolev inequality and (4.9), we can get

$$
\begin{aligned}
&([\mu(\theta^{n-1}) - \mu(\theta_h^{n-1})]|\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n, \boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n) \\
\le\ & L\|\boldsymbol{\sigma}^n\|_{L^{r'(1+\delta)}}^{r'-1} \|\theta^{n-1} - \theta_h^{n-1}\|_{L^{r(1+\delta)/\delta}} \|\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} \\
\le\ & C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1} \|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} \|\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} \\
\le\ & C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'} \|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^2.
\end{aligned} \tag{4.13}
$$

By (4.12), and notice that (4.7) and (4.13), we get

$$
(||\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n - |\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n|, |\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n|) \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'} \|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^2. \tag{4.14}
$$

Using the fact that (cf. [27]), for all $\boldsymbol{\tau}_h \in [\boldsymbol{L}^{r'}(\Omega)]^2 \bigcap \{\boldsymbol{\tau}_h|_K \in [\boldsymbol{R}(K)]^2, \forall K \in T_h\}$

$$\begin{aligned} &([|\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n - |\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n], \boldsymbol{\tau}_h) \\ \le\ & C(|\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n - |\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n|, |\tilde{\boldsymbol{\sigma}}_h^n - \boldsymbol{\sigma}_h^n|)^{1/r}\|\boldsymbol{\tau}_h\|_{\boldsymbol{L}^{r'}}, \end{aligned} \tag{4.15}$$

thus, by (4.14)

$$\begin{aligned} E_2 \ &\le\ C(|\tilde{\boldsymbol{\sigma}}_h^n|^{r'-2}\tilde{\boldsymbol{\sigma}}_h^n - |\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n|, |\tilde{\boldsymbol{\sigma}}_h^n - \boldsymbol{\sigma}_h^n|)^{1/r} \\ &\le\ C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^{2/r}. \end{aligned} \tag{4.16}$$

Hence, we have

$$\|\boldsymbol{u}_h^n - \tilde{\boldsymbol{u}}_h^n\|_{\boldsymbol{L}^r} + \|\boldsymbol{\omega}_h^n - \tilde{\boldsymbol{\omega}}_h^n\|_{\boldsymbol{L}^r} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'-1}\{\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + \|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^{2/r}\}. \tag{4.17}$$

Now we are going to estimate $\|\nabla(\theta_h^n - R_h\theta^n)\|_{L^2}$. Choosing $\eta_n = \theta_h^n - R_h\theta^n$ in (4.6c), we have

$$\begin{aligned} \|\nabla(\theta_h^n - R_h\theta^n)\|_{L^2}^2 \ &=\ (\nabla(\theta_h^n - R_h\theta^n), \nabla(\theta_h^n - R_h\theta^n)) \\ &=\ (\mu(\theta_h^{n-1})|\boldsymbol{\sigma}_h^n|^{r'} - \mu(\theta^{n-1})|\boldsymbol{\sigma}^n|^{r'}, \theta_h^n - R_h\theta^n) \\ &=\ (\mu(\theta_h^{n-1})[|\boldsymbol{\sigma}_h^n|^{r'} - |\boldsymbol{\sigma}^n|^{r'}], \theta_h^n - R_h\theta^n) \\ &\quad\ +([\mu(\theta_h^{n-1}) - \mu(\theta^{n-1})]|\boldsymbol{\sigma}^n|^{r'}, \theta_h^n - R_h\theta^n) \\ &=\ R_1 + R_2. \end{aligned} \tag{4.18}$$

Notice that

$$\begin{aligned} |\boldsymbol{\sigma}_h^n|^{r'} - |\boldsymbol{\sigma}^n|^{r'} \ &=\ \frac{1}{2}(\boldsymbol{\sigma}^n + \boldsymbol{\sigma}_h^n)(|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n) \\ &\quad +\frac{1}{2}(\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}^n)(|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n + |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n), \end{aligned}$$

then, by Hölder inequality, (2.12), Sobolev inequality and (4.9), we have

$$\begin{aligned} R_1 \ &=\ \frac{1}{2}(\mu(\theta_h^{n-1})(\boldsymbol{\sigma}^n + \boldsymbol{\sigma}_h^n)(|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n - |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n), \theta_h^n - R_h\theta^n) \\ &\quad +\frac{1}{2}(\mu(\theta_h^{n-1})(\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}^n)(|\boldsymbol{\sigma}_h^n|^{r'-2}\boldsymbol{\sigma}_h^n + |\boldsymbol{\sigma}^n|^{r'-2}\boldsymbol{\sigma}^n), \theta_h^n - R_h\theta^n) \\ &\le\ C\|\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}}(\|\boldsymbol{\sigma}_h^n\|_{\boldsymbol{L}^{r'(1+\delta)}}^{r'-1} + \|\boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'(1+\delta)}}^{r'-1})\|\theta_h^n - R_h\theta^n\|_{L^{r(1+\delta)/\delta}} \\ &\le\ C\|\boldsymbol{f}\|_{\boldsymbol{L}^{r'}}^{r'-1}\|\nabla(\theta_h^n - R_h\theta^n)\|_{L^2}[\|\boldsymbol{\sigma}_h^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} + \|\tilde{\boldsymbol{\sigma}}_h^n - \boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'}}]. \end{aligned} \tag{4.19}$$

By Hölder inequality, Sobolev inequality and (4.9),

$$\begin{aligned} R_2 \ &\le\ L\|\theta^{n-1} - \theta_h^{n-1}\|_{L^{2(1+\delta)/\delta}}\|\boldsymbol{\sigma}^n\|_{\boldsymbol{L}^{r'(1+\delta)}}^{r'}\|\theta_h^n - R_h\theta^n\|_{L^{2(1+\delta)/\delta}} \\ &\le\ C\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}\|\nabla(\theta_h^n - R_h\theta^n)\|_{L^2}. \end{aligned} \tag{4.20}$$

Combining (4.18)-(4.20) and using (4.9), we obtain

$$\|\nabla(\theta_h^n - R_h\theta^n)\|_{L^2} \le \hat{C}\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + Ch^{mr'/2}, \tag{4.21}$$

thus

$$\begin{aligned} \|\nabla(\theta^n - \theta_h^n)\|_{L^2} \ &\le\ \|\nabla(\theta^n - R_h\theta^n)\|_{L^2} + \|\nabla(\theta_h^n - R_h\theta^n)\|_{L^2} \\ &\le\ \hat{C}\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{r'}\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + C\{h^{mr'/2} + h\}. \end{aligned} \tag{4.22}$$

Hence, we have the following theorem:

Theorem 4.1. *Let* $((\boldsymbol{\sigma}^n, p^n); (\boldsymbol{u}^n, \boldsymbol{\omega}^n); \theta^n)$ *and* $((\boldsymbol{\sigma}_h^n, p_h^n); (\boldsymbol{u}_h^n, \boldsymbol{\omega}_h^n); \theta_h^n)$ *be the solutions to problems* (3.1) *and* (4.1) *respectively, and that* $(\boldsymbol{\sigma}^n, p^n) \in [\boldsymbol{W}^{m,r'}(\Omega)]^2 \times W^{m,r'}(\Omega)$, $(\boldsymbol{u}^n, \boldsymbol{\omega}^n) \in \boldsymbol{W}^{1,r}(\Omega) \times [\boldsymbol{W}^{m,r}(\Omega)]^2$, $m = 1, 2$, *and* $\theta^n \in H^2(\Omega)$. *if*

$$\hat{C}\|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{2r'/r} = \hat{M}(\boldsymbol{f}) < 1 \tag{4.23}$$

holds, then the following error estimates hold
(4.24)

$$\begin{aligned}
&\text{(a)} & \|\nabla(\theta^n - \theta_h^n)\|_{L^2} &\leq \hat{M}(\boldsymbol{f})^n \|\nabla(\theta^0 - \theta_h^0)\|_{L^2} + \frac{C}{1-\hat{M}(\boldsymbol{f})}\{h^{mr'/2} + h\} \\
&\text{(b)} & \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{\boldsymbol{L}^{r'}} + \|p^n - p_h^n\|_{L^{r'}} &\leq C\hat{M}(\boldsymbol{f})^{n-1}\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} \\
&&&\quad + \frac{C}{1-\hat{M}(\boldsymbol{f})}\{h^{mr'/2} + h\} \\
&\text{(c)} & \|\boldsymbol{\omega}^n - \boldsymbol{\omega}_h^n\|_{\boldsymbol{L}^r} &\leq C\hat{M}(\boldsymbol{f})^{n-1}\{\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2}^{2/r}\} \\
&&&\quad + \frac{C}{1-\hat{M}(\boldsymbol{f})}\{h^{m(r'-1)} + h^{2/r}\} \\
&\text{(d)} & \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{\boldsymbol{L}^r} &\leq C\hat{M}(\boldsymbol{f})^{n-1}\{\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2}^{2/r}\} \\
&&&\quad + \frac{C}{1-\hat{M}(\boldsymbol{f})} \cdot \begin{cases} h^{r'-1}, & \text{if} \quad m = 1 \\ h^{2/r}, & \text{if} \quad m = 2. \end{cases}
\end{aligned}$$

where C *is a constant independent of* n *and* h.

Proof. By (4.22), we know that

$$\begin{aligned}
&\|\nabla(\theta^n - \theta_h^n)\|_{L^2} \\
\leq\ & \hat{M}(\boldsymbol{f})\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + C\{h^{mr'/2} + h\} \\
\leq\ & \hat{M}(\boldsymbol{f})^n\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} + C\sum_{i=1}^{n}\hat{M}(\boldsymbol{f})^{n-i}\{h^{mr'/2} + h\} \\
\leq\ & \hat{M}(\boldsymbol{f})^n\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} + \frac{C}{1-\hat{M}(\boldsymbol{f})}\{h^{mr'/2} + h\}.
\end{aligned} \tag{4.25}$$

Notice that (4.4a), (4.9) and (4.24a)

$$\begin{aligned}
&\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{\boldsymbol{L}^{r'}} + \|p^n - p_h^n\|_{L^{r'}} \\
\leq\ & \|\boldsymbol{\sigma}^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{\boldsymbol{L}^{r'}} + \|p^n - \tilde{p}_h^n\|_{L^{r'}} + \|\tilde{\boldsymbol{\sigma}}_h^n - \boldsymbol{\sigma}_h^n\|_{\boldsymbol{L}^{r'}} + \|\tilde{p}_h^n - p_h^n\|_{L^{r'}} \\
\leq\ & C\{\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + h^{mr'/2}\} \\
\leq\ & C\hat{M}(\boldsymbol{f})^{n-1}\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} + \frac{C}{1-\hat{M}(\boldsymbol{f})}\{h^{mr'/2} + h\}.
\end{aligned} \tag{4.26}$$

Finally, by (4.4), (4.17) and (4.25)

$$\begin{aligned}
&\|\boldsymbol{\omega}^n - \boldsymbol{\omega}_h^n\|_{\boldsymbol{L}^r} \\
\leq\ & \|\boldsymbol{\omega}^n - \tilde{\boldsymbol{\omega}}_h^n\|_{\boldsymbol{L}^r} + \|\boldsymbol{\omega}_h^n - \tilde{\boldsymbol{\omega}}_h^n\|_{\boldsymbol{L}^r} \\
\leq\ & C\{\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + \|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^{2/r}\} + Ch^{m(r'-1)} \\
\leq\ & C\hat{M}(\boldsymbol{f})^{n-1}\{\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2}^{2/r}\} \\
&+ \tfrac{C}{1-\hat{M}(\boldsymbol{f})}\{h^{m(r'-1)} + h^{2/r}\}
\end{aligned} \tag{4.27}$$

and

$$
\begin{aligned}
&\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{\boldsymbol{L}^r} \\
&\le \|\boldsymbol{u}^n - \tilde{\boldsymbol{u}}^n\|_{\boldsymbol{L}^r} + \|\tilde{\boldsymbol{u}}_h^n - \boldsymbol{u}_h^n\|_{\boldsymbol{L}^r} \\
&\le C\{\|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2} + \|\nabla(\theta^{n-1} - \theta_h^{n-1})\|_{L^2}^{2/r}\} + \|\boldsymbol{u}^n - \tilde{\boldsymbol{u}}^n\|_{\boldsymbol{L}^r} \\
&\le C\hat{M}(\boldsymbol{f})^{n-1}\{\|\nabla(\theta^0 - \theta_h^0)\|_{L^2} + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2}^{2/r}\} \\
&\quad + \tfrac{C}{1-\hat{M}(\boldsymbol{f})} \cdot \begin{cases} h^{r'-1}, & \text{if} \quad m = 1 \\ h^{2/r}, & \text{if} \quad m = 2. \end{cases}
\end{aligned}
\tag{4.28}
$$

□

Now, let $C^* = \max\{\bar{C}, \hat{C}\}$ where $\bar{C}$ and $\hat{C}$ are defined by (3.16) and (4.23) respectively. Thus,

$$
C^* \|\boldsymbol{f}\|_{\boldsymbol{L}^2}^{2r'/r} = M^*(\boldsymbol{f}) < 1 \tag{4.29}
$$

implies (3.16) and (4.23). Hence, we get the following main result

Theorem 4.2. *Under the assumptions of Theorems* 3.2 *and* 4.1*, if condition* (4.29) *holds. Then, problem* (2.5) *has a unique solution* $((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega}); \theta)$*, the finite element solution sequence* $\{((\boldsymbol{\sigma}_h^n, p_h^n); (\boldsymbol{u}_h^n, \boldsymbol{\omega}_h^n); \theta_h^n)\}$ *of* (4.1) *converges to* $((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega}); \theta)$ *and the following estimates hold*

(4.30)

$$
\begin{aligned}
\text{(a)}\quad &\|\nabla(\theta - \theta_h^n)\|_{L^2} \\
&\le M^*(\boldsymbol{f})^n\{\|\nabla(\theta - \theta^0)\|_{L^2} + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2}\} + \frac{C}{1 - M^*(\boldsymbol{f})}\{h^{mr'/2} + h\} \\
\text{(b)}\quad &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^n\|_{\boldsymbol{L}^{r'}} + \|p - p_h^n\|_{L^{r'}} \\
&\le CM^*(\boldsymbol{f})^{n-1}\{\|\nabla(\theta - \theta^0)\|_{L^2} + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2}\} + \frac{C}{1 - M^*(\boldsymbol{f})}\{h^{mr'/2} + h\} \\
\text{(c)}\quad &\|\boldsymbol{\omega} - \boldsymbol{\omega}_h^n\|_{\boldsymbol{L}^r} \\
&\le CM^*(\boldsymbol{f})^{n-1}\{\|\nabla(\theta - \theta^0)\|_{L^2} + \|\nabla(\theta - \theta^0)\|_{L^2}^{2/r} + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2} \\
&\quad + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2}^{2/r}\} + \frac{C}{1 - M^*(\boldsymbol{f})}\{h^{m(r'-1)} + h^{2/r}\} \\
\text{(d)}\quad &\|\boldsymbol{u} - \boldsymbol{u}_h^n\|_{\boldsymbol{L}^r} \\
&\le CM^*(\boldsymbol{f})^{n-1}\{\|\nabla(\theta - \theta^0)\|_{L^2} + \|\nabla(\theta - \theta^0)\|_{L^2}^{2/r} + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2} \\
&\quad + \|\nabla(\theta^0 - \theta_h^0)\|_{L^2}^{2/r}\} + \frac{C}{1 - M^*(\boldsymbol{f})} \cdot \begin{cases} h^{r'-1}, & \textit{if} \quad m = 1 \\ h^{2/r}, & \textit{if} \quad m = 2. \end{cases}
\end{aligned}
$$

where C *is a constant independent of* n *and* h*,* $m = 1, 2$*, and* $M^*(\boldsymbol{f}) < 1$ *is defined by* (4.29)*.*

Remark 4.2. *Since* (4.1a) *is still nonlinear equation, then it needs to be solved in practice by an iterative method such as, for example, augmented Lagrangian method* [20] *or conjugate gradient method* [5]*.*

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