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The Boundedness Below of 2×2 Upper Triangular Linear Relation Matrices

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Abstract. In this note, the boundedness below of linear relation matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \in LR(H\oplus K)$ is considered, where $A\in CLR(H)$, $B\in CLR(K)$,$C\in BLR(K,H)$, H,K are separable Hilbert spaces. By suitable space decompositions, a necessary and sufficient condition for diagonal relations A,B is given so that M_C is bounded below for some $C\in BLR(K,H)$. Besides, the characterization of $\sigma_{ap}(M_C)$ and $\sigma_{su}(M_C)$ are obtained, and the relationship between $\sigma_{ap}(M_0)$ and $\sigma_{ap}(M_C)$ is further presented.

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1 Introduction

The linear relation established by Neumann [1] was first introduced into functional analysis, whose motivation is to consider the adjoint of linear differential operators that are not densely defined. It has extensive applications in nonlinear analysis, differential equations, and optimization and control problems. For instance, the port-Hamiltonian formulation can be conveniently established by the linear relation language, in which the kernel of certain row relations (dually, the range of column relations) and the structure of the involved port-Hamiltonian pencils play significant roles [2]. The simplest naturally occurring example of a linear relation is the inverse of a linear mapping $A:X\to Y$, defined by the set of solutions

$$A^{-1}y := \{x\in X : Ax = y\}$$

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of the equation $Ax=y$. Arens [3] studied the resolvent set and the spectrum of linear relations, and obtained the existence theorem of self-adjoint relations.

Suppose H and K are separable Hilbert spaces with infinite dimension. A linear relation $A:H\to K$ is a mapping from the subspace

$$\mathcal{D}(A)=\{x\in H:Ax\neq\emptyset\}\subset H,$$

called the domain of A, into the set of non-empty subsets of K, and for all non-zero scalars c_1,c_2 and $x_1,x_2\in\mathcal{D}(A)$ such that

$$A(c_1x_1+c_2x_2)=c_1Ax_1+c_2Ax_2.$$

We introduce $LR(H,K)$ to represent the class of linear relation from H into K and write $LR(H)=LR(H,H)$. The graph of A is defined by

$$G(A)=\{(x,y)\in H\oplus K:x\in\mathcal{D}(A),y\in Ax\}.$$

Like operator case, A is called a closed linear relation provided that $G(A)$ is closed. The collection of all of closed linear relations is represented by $CLR(H,K)$. The relation A^{-1} is determined by

$$G(A^{-1})=\{(y,x)\in K\oplus H:(x,y)\in G(A)\}.$$

The range of A is designed by $\operatorname{ran}(A)=A(\mathcal{D}(A))$, and the kernel by $\ker(A)=\{x\in H:(x,0)\in G(A)\}$; write $\alpha(A)=\dim\ker(A)$ and $\beta(A)=\operatorname{codim}\operatorname{ran}(A)$. If $\operatorname{ran}(A)=K$ ($\ker(A)=\{0\}$), A is called surjective (injective). Note that for $x\in\mathcal{D}(A)$,

$$y\in Ax\Leftrightarrow Ax=y+A(0).$$

If A is a linear relation from H into K, then we use Q_A to represent the quotient mapping $Q^K_{\overline{A(0)}}\in L(K,K/\overline{A(0)})$, and hence Q_AA is obviously an operator. For $x\in\mathcal{D}(A)$,

$$Q_AAx=Q_Ay,\quad \text{for all } y\in Ax.$$

For $x\in\mathcal{D}(A)$, we define

$$||Ax||=||Q_AAx||.$$

The norm of A is denoted by $||A||=||Q_AA||$. This quantity is semi-norm, because $||A||=0$ does not imply $A=0$. If $||A||<+\infty$, then it is said that A is bounded. The set of all the bounded linear relations defined everywhere is represented by $BLR(H,K)$. The resolvent set of a linear relation $A\in LR(H)$ can be expressed as

$$\rho(A)=\{\lambda\in\mathbb{C}:\ (A-\lambda)^{-1}\text{ is bounded and single valued}\},$$

and the spectrum is defined by $\sigma(A)=\mathbb{C}\backslash\rho(A)$. Recall that $A\in LR(H,K)$ is called bounded below if there exists $\delta>0$ such that $||Ax||\geq\delta||x||$ for each $x\in\mathcal{D}(A)$. The approximate point spectrum $\sigma_{ap}(A)$ and the surjective spectrum $\sigma_{su}(A)$ are defined respectively by

$$\sigma_{ap}(A)=\{\lambda\in\mathbb{C}:A-\lambda\text{ is not bounded below}\},$$

$$\sigma_{su}(A)=\{\lambda\in\mathbb{C}: A-\lambda \text{ is not surjective}\}.$$

The minimum modulus of a linear relation A is defined by

$$\gamma(A)=\inf_{x\notin\ker(A)}\left\{\frac{||Ax||}{d(x,\ker(A))}\right\},$$

where $d(x,\ker(B))$ indicates the distance from x to $\ker(B)$. For the aforementioned basic knowledge on linear relations, we refer readers to, e.g., [4].

Recently, 2×2 block operator matrices have attracted attention of many scholars (See, e.g., [5-10]). In this work, we consider the linear relations instead of operators and extend some of the existing results. Assuming A and B to be linear relations acting respectively on H and K, we use

$$M_C=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

with $C\in LR(K,H)$ to represent the relation matrix from $H\oplus K$ to $H\oplus K$, which can be determined by

$$G(M_C)=\left\{\left(\begin{pmatrix} x \\ y\end{pmatrix},\begin{pmatrix} u \\ v\end{pmatrix}\right): u\in Ax+Cy, v\in By\right\}.$$

A number of authors have studied properties of M_C in the recent past, including the spectrum, essential spectrum, Browder spectrum, Weyl spectrum, etc. (see [11–20]). In this paper, we consider the boundedness below for M_C, where A,B,C are all linear relations.

2 Auxiliary results

In this section, some lemmas are presented. We will use these results to prove our main results in the sequel.

Lemma 2.1 ([4]). *If $A\in CLR(H,K)$, then a necessary and sufficient condition for A to have a non-zero closed range is $\gamma(A)>0$.*

Lemma 2.2 ([4]). *If $A\in CLR(H,K)$, then* $\ker(A)$ *and* $A(0)$ *are closed.*

Lemma 2.3. *Let $A\in CLR(H,K)$. Then A is bounded below if and only if A is injective and has a closed range.*

Proof. Suppose A is bounded below. For $x\in\mathcal{D}(A)$, if $0\in Ax$ then $Ax=0+A(0)=A(0)$ and

$$0=||Ax||\geq\delta||x||$$

for some $\delta>0$. It can be obtained that $x=0$, that is, A is injective. Set $\{y_n\}\subset\operatorname{ran}(A)$, $y_n\to y_0\,(n\to\infty)$. Then there exists a sequence $\{x_n\}\subset\mathcal{D}(A)$ such that $Ax_n=y_n+A(0)$ and

$$\delta||x_n||\leq||Ax_n||=||Q_Ay_n||.$$

This together with the convergence of $\{y_n\}$ and the continuity of Q_A implies that $\{x_n\}$ is a Cauchy sequence, and hence $x_n \to x_0\,(n\to\infty)$ for some $x_0 \in H$. Note that A is closed, so $x_0 \in \mathcal{D}(A)$ and $y_0 \in Ax_0$. This proves $\mathrm{ran}(A)$ to be closed.

Conversely, suppose that A is injective and has closed range. By Lemma 2.1, $\gamma(A)>0$ and for $x\in\mathcal{D}(A)$,

$$||Ax|| \geq \gamma(A)||x||.$$

Thus A is boundedness below, which ends the proof. □

Lemma 2.4. *For $A\in LR(H,K)$ and $B\in BLR(K,G)$, if BA is bounded below then A is bounded below.*

Proof. For $x\in\mathcal{D}(A)$, if $||x||\leq \delta_1^{-1}||BAx||$, then

$$||x|| \leq \delta_1^{-1}||BAx|| \leq \delta_1^{-1}||B||||Ax|| = \delta^{-1}||Ax||$$

with $\delta=\delta_1||B||^{-1}$, giving the conclusion. □

Lemma 2.5. *Let $A\in CLR(H,K)$. Then A can be expressed as the following block linear relation matrix*

$$A=\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} : \ker(A)\oplus(\mathcal{D}(A)\cap\ker(A)^{\perp}) \to A(0)\oplus A(0)^{\perp}.$$

where A_1 and A_2 are surjective relations such that $A_1x_1=A_2x_2=A(0)$ for $x_1\in\ker(A)$ and $x_2\in\mathcal{D}(A)\cap\ker(A)^{\perp}$; A_3 is injective and single valued.

Proof. The block representation of A holds obviously by using Lemma 2.2. We now prove the properties of $A_i\,(1=1,2,3)$. Apparently, the result holds for $x_1=x_2=0$. If $x_1\neq 0, x_1\in\ker(A)$, then

$$A_1x_1 = 0 + A_1(0) = A_1(0) = A(0)$$

since $0\in A_1x_1$ and $0\in A_1(0)$. For $x_2\neq 0, x_2\in\ker(A)^{\perp}$, suppose

$$Ax_2 = y + A(0), \tag{2.1}$$

where $y\in\mathrm{ran}(A)$. Denote by P_1 and P_2 the orthogonal projection from K onto $A(0)$ and $A(0)^{\perp}$, respectively. Then

$$A_2x_2 = P_1Ax_2 = A(0), \qquad A_3x_2 = P_2Ax_2 = P_2y. \tag{2.2}$$

Note that the single valued property of A_3 is immediate from (2.2), and then it remains to show that A_3 is injective. Indeed, if $A_3x_2=0$, then $P_2y=0$ (for convenience, still use (2.1) and (2.2)), which is equivalent to $y\in A(0)$. This in combination with (2.1) implies $Ax_2=A(0)$, namely $x_2\in\ker(A)$, whence $x_2=0$. Thus A_3 is shown to be injective. □

In [11] and [19], it is shown that the usual matrix multiplication formula need not hold for relation matrices and some special cases are also mentioned. Here we have a simple observation.

Lemma 2.6. *Let A,A' and B,B' be, respectively, linear relations in H and K, and C be a linear relation form K into H. Then*

(i) $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}=\begin{pmatrix} AA' & CB' \\ 0 & BB' \end{pmatrix}$*, when B' is single valued;*

(ii) $\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}=\begin{pmatrix} A'A & A'C \\ 0 & B'B \end{pmatrix}$.

Lemma 2.7. *Let $A\in CLR(H)$, $B\in CLR(K)$ and $C\in CLR(K,H)$ with $\mathcal{D}(B)\subset\mathcal{D}(C)$ and $\mathcal{D}(A^*)\subset\mathcal{D}(C^*)$. Then the adjoint of $M_C=\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is of the form*

$$M_C^*=\begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix}:\mathcal{D}(A^*)\oplus\mathcal{D}(B^*)\to H\oplus K.$$

Proof. Let $\left(\left(\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right),\left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)\right)\in G\left(\left(\begin{smallmatrix} A^* & 0 \\ C^* & B^* \end{smallmatrix}\right)\right)$. Obviously, $u_2\in\mathcal{D}(A^*)$, $v_2\in\mathcal{D}(B^*)$, $x_2\in A^*u_2$ and $y_2\in C^*u_2+B^*v_2$. For $y_2\in C^*u_2+B^*v_2$, there exist $y_{2,1}\in C^*u_2$ and $y_{2,2}\in B^*v_2$ such that $y_2=y_{2,1}+y_{2,2}$. Then, for all $(x_1,u_{1,1})\in G(A)$, $(y_1,u_{1,2})\in G(C)$ and $(y_1,v_1)\in G(B)$, we have

$$\langle u_{1,1},u_2\rangle=\langle x_1,x_2\rangle,\ \langle u_{1,2},u_2\rangle=\langle y_1,y_{2,1}\rangle,\ \langle v_1,v_2\rangle=\langle y_1,y_{2,2}\rangle.$$

Taking $u_1=u_{1,1}+u_{1,2}$ yields

$$\left\langle\begin{pmatrix} u_1 \\ v_1 \end{pmatrix},\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\right\rangle=\left\langle\begin{pmatrix} x_1 \\ y_1 \end{pmatrix},\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right\rangle.$$

Note that the collection of all such $\left(\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right),\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)\right)$ is exactly $G(M_C)$. It follows that $\left(\left(\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right),\left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)\right)\in G(M_C^*)$. This means $\left(\begin{smallmatrix} A^* & 0 \\ C^* & B^* \end{smallmatrix}\right)\subset M_C^*$. On the other hand, assume $\left(\left(\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right),\left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)\right)\in G(M_C^*)$, i.e.,

$$\left\langle\begin{pmatrix} u_1 \\ v_1 \end{pmatrix},\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\right\rangle=\left\langle\begin{pmatrix} x_1 \\ y_1 \end{pmatrix},\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right\rangle.$$

for all $\left(\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right),\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)\right)\in G(M_C)$. Since $\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)\in M_C\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right)$, it follows that $v_1\in By_1$ and there exist $u_{1,1}\in Ax_1$ and $u_{1,2}\in Cy_1$ such that $u_1=u_{1,1}+u_{1,2}$. Thus

$$\langle u_{1,1},u_2\rangle+\langle u_{1,2},u_2\rangle+\langle v_1,v_2\rangle=\langle x_1,x_2\rangle+\langle y_1,y_2\rangle$$

for all $x_1\in\mathcal{D}(A)$, $y_1\in\mathcal{D}(B)$, $u_{1,1}\in Ax_1$, $u_{1,2}\in Cy_1$ and $v_1\in By_1$. In particular, if $y_1=0$, $u_{1,2}=0$ and $v_1=0$, then

$$\langle u_{1,1},u_2\rangle=\langle x_1,x_2\rangle,$$

which shows $u_2 \in \mathcal{D}(A^*) \subset \mathcal{D}(C^*)$ and $x_2 \in A^* u_2$; if $x_1 = 0$ and $u_{1,1} = 0$, then

$$\begin{aligned}\langle v_1, v_2\rangle &= \langle y_1, y_2\rangle - \langle u_{1,2}, u_2\rangle \\ &= \langle y_1, y_2\rangle - \langle y_1, y_{2,1}\rangle = \langle y_1, (y_2 - y_{2,1})\rangle\end{aligned}$$

for $y_{2,1} \in C^* u_2$, which implies $v_2 \in \mathcal{D}(B^*)$ and $y_2 - y_{2,1} \in B^* v_2$. In conclusion, $\binom{u_2}{v_2} \in \mathcal{D}(A^*) \oplus \mathcal{D}(B^*)$ and

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix},$$

and hence $M_C^* \subset \left(\begin{smallmatrix} A^* & 0 \\ C^* & B^* \end{smallmatrix}\right)$. This completes the proof. $\square$

Remark 2.1. Proposition 2.2 in [14] also studied the adjoint of M_C with everywhere defined entries and it is only proved that $M_C^* \subset \left(\begin{smallmatrix} A^* & 0 \\ C^* & B^* \end{smallmatrix}\right)$.

3 Main results

This section will give the main results of this paper and their proofs. We start with the boundedness below of diagonal relation matrices whose proof is obvious.

Theorem 3.1. *Let $A \in LR(H)$ and $B \in LR(K)$. then $M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is bounded below if and only if A and B are both bounded below.*

We now discuss the boundedness below of upper relation matrices by means of the space decomposition technique and the minimum modulus.

Theorem 3.2. *Let $A \in CLR(H)$ and $B \in CLR(K)$, then M_C is bounded below for some $C \in BLR(K,H)$ if and only if A is bounded below and*

(i) $\alpha(B) \leq \beta(A)$, if B has closed range;
(ii) $\beta(A) = \infty$, if $ran(B)$ is not closed.

Proof. (i) Suppose A is bounded below, B has closed range and $\alpha(B) \leq \beta(A)$. Combining with the closedness of $\ker(A), \ker(B), A(0), B(0)$, we can pick J, an arbitrary isometry from $\ker(B)$ to $\mathrm{ran}(A)^\perp$, and let C be a linear relation from $\ker(B)^\perp \oplus \ker(B)$ to $A(0) \oplus (\mathrm{ran}(A) \ominus A(0)) \oplus \mathrm{ran}(A)^\perp$ defined by

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & J \end{pmatrix}.$$

Applying the decomposition similar to Lemma 2.5 yields

$$M_C = \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ A_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & J \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{D}(A)\cap\ker(A)^{\perp} \\ \ker(A) \\ \mathcal{D}(B)\cap\ker(B)^{\perp} \\ \ker(B) \end{pmatrix} \to \begin{pmatrix} A(0) \\ \mathrm{ran}(A)\ominus A(0) \\ \mathrm{ran}(A)^{\perp} \\ B(0) \\ B(0)^{\perp} \end{pmatrix}, \tag{3.1}$$

where

$$A_1(x_1)=A_2(x_2)=A(0) \qquad \text{for } x_1\in\ker(A) \quad \text{and} \quad x_2\in\mathcal{D}(A)\cap\ker(A)^{\perp}, \tag{4a}$$

$$B_1(y_1)=B_2(y_2)=B(0) \qquad \text{for } y_1\in\ker(B) \quad \text{and} \quad y_2\in\mathcal{D}(B)\cap\ker(B)^{\perp}; \tag{4b}$$

A_3 is an invertible operator, and B_3 as an operator from $\mathcal{D}(B)\cap\ker(B)^{\perp}$ to $\mathrm{ran}(B)\ominus B(0)$ is bijective. From (3.1), it can be easily seen that M_C is closed. Let $(x_1,x_2,y_1,y_2)^T\in\ker(M_C)\subset \ker(A)^{\perp}\oplus\ker(A)\oplus\ker(B)^{\perp}\oplus\ker(B)$, then

$$\begin{cases} 0\in A_1x_1+A_2x_2, \\ 0=A_3x_1, \\ 0=Jy_2, \\ 0\in B_1x_1+B_2x_2, \\ 0=B_3y_1. \end{cases}$$

Thus

$$x_2\in\ker(A), \quad x_1=0, \quad y_1=0, \quad y_2=0,$$

which leads to

$$\ker(M_C)\subset\{0\}\oplus\ker(A)\oplus\{0\}\oplus\{0\}. \tag{3.5}$$

From Lemma 2.3, it follows that A is injective, whence $\ker(A)=\{0\}$. This together with (3.5) implies that M_C is injective. We now show that $\mathrm{ran}(M_C)$ is closed. Indeed, let $(u_{n,1},u_{n,2},u_{n,3},v_{n,1},v_{n,2})^T\in\mathrm{ran}(M_C)$ such that $(u_{n,1},u_{n,2},u_{n,3},v_{n,1},v_{n,2})^T\to(u_1,u_2,u_3,v_1,v_2)^T$ as $n\to\infty$. Then

$$\begin{cases} u_1\in A(0), \\ u_2\in R(A)\ominus A(0), \\ u_3\in R(A)^{\perp}, \\ v_1\in B(0), \\ v_2\in R(B)\ominus B(0). \end{cases}$$

Noticing that these sets are closed, we have:

- $u_1\in A_2x_2=A(0)$ for any $x_2\in\ker(A)$ from (4a);

- There exists $x_1 \in \mathcal{D}(A) \cap \ker(A)^{\perp}$ such that $u_2 = A_3 x_1$, since A_3 is invertible;
- There exists $y_2 \in \ker(B)$ such that $u_3 = Jy_2$, since there exists $y_{n,2} \in \ker(B)$ satisfying $Jy_{n,2} = u_{n,3} \to u_3$, which together with the isometry of J leads to $y_{n,2} \to y_2$ for some $y_2 \in \ker(B)$ and hence $u_3 = Jy_2$;
- $v_1 \in B_2 y_2 = B(0)$ for any $y_2 \in \ker(B)$ from (4b);
- There exists $y_1 \in \mathcal{D}(B) \cap \ker(B)^{\perp}$ such that $v_2 = B_3 y_1$, since the operator $B_3 : \mathcal{D}(B) \cap \ker(B)^{\perp} \to \operatorname{ran}(B) \ominus B(0)$ is bijective.

Obviously,

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \end{pmatrix} \in M_C \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix},$$

which proves $R(M_C)$ to be closed. Therefore M_C is bounded below by Lemma 2.3.

Conversely, suppose M_C is boundedness below for some $C \in BLR(K,H)$. Note that

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

by Lemma 2.6. From Lemma 2.4, it follows that A is bounded below. Write M_C as

$$M_C = UV,$$

where

$$U = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \qquad V = \begin{pmatrix} A & C \\ 0 & I \end{pmatrix},$$

By Theorem 4.1 in [22], we can get

$$\begin{aligned} &\alpha(M_C) + \beta(A) + \beta(B) + \dim(V(0)) + \dim(U(0)) - \dim(M_C(0)) \\ =&\beta(M_C) + \alpha(A) + \alpha(B). \end{aligned} \tag{3.6}$$

Applying $\alpha(M_C) = \alpha(A) = 0$ and

$$\begin{aligned} \dim(M_C(0)) &= \dim((A(0)+C(0)) \oplus B(0)) \\ &= \dim(A(0)+C(0)) + \dim(B(0)) \\ &= \dim(V(0)) + \dim(U(0)) \end{aligned}$$

to (3.6), we get

$$\beta(M_C) + \alpha(B) = \beta(A) + \beta(B).$$

In view of $\beta(M_C)\geq\beta(B)$, then $\beta(A)\geq\alpha(B)$. In conclusion, the proof of assertion (i) is finished.

(ii) Assume that A is bounded below, $\mathrm{ran}(B)$ is not closed and $\beta(A)=\infty$. Define $C=\binom{0}{J}:K\to\mathrm{ran}(A)\oplus\mathrm{ran}(A)^{\perp}$, where J is an isometry from K to $\mathrm{ran}(A)^{\perp}$. The linear relation matrix M_C can be expressed as

$$M_C=\begin{pmatrix} A_1 & 0\\ 0 & J\\ 0 & B\end{pmatrix}$$

defined from $H\oplus K$ to $\mathrm{ran}(A)\oplus\mathrm{ran}(A)^{\perp}\oplus K$. If $(x,y)^T\in\ker(M_C)$, then

$$\begin{cases} 0\in A_1x,\\ 0=Jy,\\ 0\in By.\end{cases}$$

Obviously, $x\in\ker(A)$ and $y=0$, which implies

$$\ker(M_C)\subset\ker(A)\oplus\{0\}=\{0\},$$

whence M_C is injective. Next, we will show that $\mathrm{ran}(M_C)$ is closed. Assuming $(x,y)^T\in\mathcal{D}(A)\oplus\mathcal{D}(B)$, we have

$$\begin{aligned}||M_C\tbinom{x}{y}||^2&=||Ax||^2+||Jy||^2+||By||^2\\&\geq||Ax||^2+||Jy||^2\\&\geq\gamma^2(A)||x||^2+||y||^2\\&\geq\min(\gamma^2(A),1)(||x||^2+||y||^2).\end{aligned}$$

Thus

$$\gamma(M_C)=\inf\{||M_C\tbinom{x}{y}||:\tbinom{x}{y}\in\mathcal{D}(A)\oplus\mathcal{D}(B),||x||^2+||y||^2=1\}>0.$$

From Lemma 2.1, it follows that M_C has closed range. Note that M_C is closed, so M_C is bounded below by Lemma 2.3.

Conversely, suppose M_C is boundedness below for some $C\in BLR(K,H)$. As above, A is bounded below. We now suppose $\beta(A)<\infty$. Since $R(B)$ is not closed,

$$\gamma(B)=\inf\left\{\frac{||By||}{d(y,\ker(B))}:y\in\mathcal{D}(B)\backslash\ker(B)\right\}=0.$$

For any $y\in\mathcal{D}(B)\backslash\ker(B)$,

$$\frac{||By||}{d(y,\ker(B))}=\frac{||By_1||}{\inf\limits_{v\in\ker(B)}\{||y_1+y_2-v||\}}=\frac{||By_1||}{\inf\limits_{v\in\ker(B)}\{||y_1-v||\}}$$

$$=\frac{||By_1||}{\inf\limits_{v\in\ker(B)}\sqrt{||y_1||^2+||v||^2}}=\frac{||By_1||}{||y_1||},$$

where $y=y_1+y_2$ with $y_1\in\mathcal{D}(B)\cap\ker(B)^\perp$ and $y_2\in\ker(B)$. Then

$$\inf\{||By||:y\in\mathcal{D}(B)\cap\ker(B)^\perp,||y||=1\}=\gamma(B)=0,$$

which ensure that there exists a sequence $\{y_n\}\subset\mathcal{D}(B)\cap\ker(B)^\perp$ of orthogonal unit vectors such that $By_n\to 0$ as $n\to\infty$. This makes us claim that for some $\varepsilon_0>0$,

$$d(\mathrm{ran}(A),Cy_{n_k})\geq\varepsilon_0 \qquad \text{for all } k\in\mathbb{N}, \tag{3.7}$$

where $\{y_{n_k}\}$ is some subsequence of $\{y_n\}$. Assume not and, without loss of generality, let $d(\mathrm{ran}(A),Cy_n)\to 0$. Therefore, there exists a sequence $\{x_n\}\subset\mathcal{D}(A)$ such that $d(Ax_n,Cy_n)\to 0$. Since M_C is injective, $(x_n,y_n)^T\notin\ker(M_C)$ and

$$d(\left(\begin{smallmatrix}x_n\\ y_n\end{smallmatrix}\right),\ker(M_C))=\sqrt{||x_n||^2+||y_n||^2}\geq||y_n||=1,$$

which indicates

$$\frac{||M_C\left(\begin{smallmatrix}x_n\\ y_n\end{smallmatrix}\right)||}{d(\left(\begin{smallmatrix}x_n\\ y_n\end{smallmatrix}\right),\ker(M_C))}\leq||M_C\left(\begin{smallmatrix}x_n\\ y_n\end{smallmatrix}\right)||=\sqrt{||Ax_n-Cy_n||^2+||By_n||^2}\to 0.$$

This contradicts

$$\gamma(M_C)=\inf\left\{\frac{||M_C\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right)||}{d(\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right),\ker(M_C))}:\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right)\in\mathcal{D}(M_C)\setminus\ker(M_C)\right\}>0$$

since M_C has closed range by Lemma 2.3. Thus (3.7) holds true as expected. In what follows, suppose $\{e_1,\ldots,e_{\beta(A)}\}$ is a basis of $\mathrm{ran}(A)^\perp$, and write P_k for the orthogonal projection from H to $span\{e_i\}$ $(i=1,\ldots,\beta(A))$. For $k\in\mathbb{N}$, let $u_k\in Cy_{n_k}$ and $u_k=u_{k,1}+u_{k,2}$ with $u_{k,1}\in\mathrm{ran}(A)^\perp$ and $u_{k,2}\in\mathrm{ran}(A)$. Since $||u_{k,1}||\geq\varepsilon_0$ $(n\in\mathbb{N})$ by (3.7), it can be seen that

$$\sum_{k=1}^{\infty}\left\|\frac{1}{k}u_{k,1}\right\|=\infty,$$
$$\left\|\sum_{k=1}^{\infty}P_{i_0}\left(\frac{1}{k}u_{k,1}e^{i\theta_k}\right)\right\|=\infty$$

for a certain $i_0\in\{1,\ldots,\beta(A)\}$ and $0\leq\theta_k<2\pi$ $(k\in\mathbb{N})$. Let

$$y=\sum_{k=1}^{\infty}\frac{1}{n}y_{n_k}e^{i\theta_k},$$

and then $||y||^2=\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty$. However

$$||Cy||\geq||P_{i_0}(Cy)||=\left\|\sum_{k=1}^{\infty}P_{i_0}C\left(\frac{1}{k}y_{n_k}e^{i\theta_k}\right)\right\|=\infty,$$

which is a contradiction. So $\beta(A)=\infty$. This ends the proof. □

Remark 3.1. Note that the operator matrix version of Theorem 3.2 can be found in [6], in which A, B, C and M_C are all operators of the corresponding Hilbert spaces.

Example 3.1. Let $H=K=l^2$, and the linear relations A,B are defined as

$$\begin{aligned}
&Ax=(0,0,0,x_1,0,0,0,x_2,0,0,0,\dots)+A(0), \qquad x=(x_1,x_2,\dots)\in l^2,\\
&A(0)=\{(y_1,0,0,0,y_2,0,0,0,y_3,0,0,0,\dots):(y_1,y_2,\dots)\in l^2\},\\
&Bx=(0,x_1,x_2,\dots), \qquad x=(x_1,x_2,\dots)\in l^2.
\end{aligned}$$

It is obvious that A is bounded below with $\beta(A)=\infty$, and B is a bounded operator with closed range and $\alpha(B)=0$. From Theorem 3.2, there exists $C\in BLR(K,H)$ such that M_C is bounded below. On the other hand, defining the relation C by

$$\begin{aligned}
&Cx=(0,0,x_1,0,0,0,x_2,0,0,0,\dots)+C(0), \qquad x=(x_1,x_2,\dots)\in l^2,\\
&C(0)=(0,z_1,0,0,0,z_2,0,0,0,z_3,0,0,0,\dots), \qquad (z_1,z_2,\dots)\in l^2,
\end{aligned}$$

we then claim that C is such a candidate relation. In fact, if $\binom{x}{y}\in\ker(M_C)$, then

$$\begin{cases} 0\in Ax+Cy;\\ 0\in By,\end{cases}$$

which implies $x\in\ker(A)$ and $y=0$ since B is injective. Thus

$$\ker(M_C)\subset\ker(A)\oplus\{0\}=\{0\},$$

which in combination with $\operatorname{ran}(M_C)=H\oplus\operatorname{ran}(B)$ leads to boundedness below of M_C.

The above example illustrates the correctness of Theorem 3.2. Based on Lemma 2.7, the dual result of Theorem 3.2 can be presented as follows.

Theorem 3.3. *Suppose $A\in CLR(H)$, $B\in CLR(K)$, then for some $C\in BLR(K,H)$, M_C is surjective if and only if B is surjective and*

(i) $\beta(A)\leq\alpha(B)$, if the range of A is closed;
(ii) $\beta(B)=\infty$, if the range of A is not closed.

From Theorems 3.2 and 3.3, we can directly get the perturbation results for the approximate point spectrum and surjective spectrum.

Corollary 3.1. *Suppose $A\in CLR(H)$, $B\in CLR(K)$, then*

$$\bigcap_{C\in CR(K,H)}\sigma_{ap}(M_C)=\sigma_{ap}(A)\cup\{\lambda\in\mathbb{C}:R(B-\lambda) \text{ is not closed and } \beta(A-\lambda)<\infty\}$$
$$\cup\{\lambda\in\mathbb{C}:R(B-\lambda) \text{ is closed and } \beta(A-\lambda)<\alpha(B-\lambda)\}.$$

Corollary 3.2. *Suppose $A\in CLR(H)$, $B\in CLR(K)$, then*

$$\bigcap_{C\in CR(K,H)}\sigma_{su}(M_C)=\sigma_{su}(B)\cup\{\lambda\in\mathbb{C}:R(A-\lambda) \text{ is not closed and } \alpha(B-\lambda)<\infty\}$$
$$\cup\{\lambda\in\mathbb{C}:R(A-\lambda) \text{ is closed and } \alpha(B-\lambda)<\beta(A-\lambda)\}.$$

Applying Theorem 3.2, we can discuss the relationship between $\sigma_{ap}(M_0)$ and $\sigma_{ap}(M_C)$.

Theorem 3.4. *Suppose $A\in CLR(H)$, $B\in CLR(K)$, then*

$$\sigma_{ap}(A)\cup\sigma_{ap}(B)=\sigma_{ap}(M_C)\cup(\sigma_{su}(A)\cap\sigma_{ap}(B))$$

holds for every $C\in BLR(K,H)$.

Proof. Obviously,

$$\sigma_{ap}(A)\cup\sigma_{ap}(B)\supset\sigma_{ap}(M_C)\cup(\sigma_{su}(A)\cap\sigma_{ap}(B)).$$

It is sufficient to prove

$$\sigma_{ap}(A)\cup\sigma_{ap}(B)\subset\sigma_{ap}(M_C)\cup(\sigma_{su}(A)\cap\sigma_{ap}(B)). \tag{3.8}$$

Assume $\lambda\in(\sigma_{ap}(A)\cup\sigma_{ap}(B))\setminus\sigma_{ap}(M_C)$. Then $M_C-\lambda$ is boundedness below and, by Theorem 3.2, at least any of the following two conclusions hold:

- $\beta(A-\lambda)=\infty$, if the range of $B-\lambda$ is not closed;
- $\beta(A-\lambda)\geq\alpha(B-\lambda)$, if the range of $B-\lambda$ is closed.

In view of $\lambda\in\sigma_{ap}(B)$, we know that $\alpha(B-\lambda)>0$ if the range of $B-\lambda$ is closed, and hence $A-\lambda$ is not surjective no matter which of the above conditions holds. Thus $\lambda\in\sigma_{su}(A)\cap\sigma_{ap}(B)$. This shows that

$$(\sigma_{ap}(A)\cup\sigma_{ap}(B))\setminus\sigma_{ap}(M_C)\subset\sigma_{su}(A)\cap\sigma_{ap}(B),$$

whence (3.8) is valid. The proof ends here. □

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