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Uniform RIP Bounds for Recovery of Signals with Partial Support Information by Weighted ℓ_p-Minimization

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Abstract. In this paper, we consider signal recovery in both noiseless and noisy cases via weighted ℓ_p $(0<p\leq 1)$ minimization when some partial support information on the signals is available. The uniform sufficient condition based on restricted isometry property (RIP) of order tk for any given constant $t>d$ ($d\geq 1$ is determined by the prior support information) guarantees the recovery of all k-sparse signals with partial support information. The new uniform RIP conditions extend the state-of-the-art results for weighted ℓ_p-minimization in the literature to a complete regime, which fill the gap for any given constant $t>2d$ on the RIP parameter, and include the existing optimal conditions for the ℓ_p-minimization and the weighted ℓ_1-minimization as special cases.

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1 Introduction

In compressed sensing, a central goal is to efficiently recover sparse signals $\boldsymbol{x}\in\mathbb{R}^n$ from a relatively small number of linear measurements, i.e.

$$\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{e}, \tag{1.1}$$

where $\boldsymbol{y}\in\mathbb{R}^m$, $\boldsymbol{A}\in\mathbb{R}^{m\times n}(m\ll n)$ is a sensing matrix and $\boldsymbol{e}\in\mathbb{R}^m$ denotes a vector of measurement errors. It has been a research focus in applied mathematics, statistics, and machine

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learning, with abundant applications ranging from medical imaging to speech recognition and video coding. A series of fast algorithms have been developed to recover the signal $\boldsymbol{x}$ from a relatively small number of linear measurements (1.1). The ℓ_p-minimization with $0<p\leq 1$ is among the most well-known algorithms for the reconstruction of the signal $\boldsymbol{x}$

$$\begin{aligned}&\min_{\boldsymbol{x}\in\mathbb{R}^n}\|\boldsymbol{x}\|_p^p\\ &\text{s.t.}\quad \boldsymbol{A}\boldsymbol{x}-\boldsymbol{y}\in\mathcal{B},\end{aligned}\tag{1.2}$$

where $\mathcal{B}$ is a set determined by the noise structure and $\|\boldsymbol{x}\|_p=(\sum_{i=1}^n|x_i|^p)^{1/p}$. For the noiseless case, $\mathcal{B}=\{\mathbf{0}\}$.

In this paper, we consider the weighted ℓ_p-minimization $(0<p\leq 1)$ [7–9, 11–15, 17, 18, 20] to recover the signal $\boldsymbol{x}$ from (1.1), when some prior information is included in the estimates of the support of $\boldsymbol{x}$ or some estimates of largest coefficients of $\boldsymbol{x}$. For instance, video and audio signals exhibit strong correlation over temporal frames, which can be used to estimate a portion of the support based on previously decoded frames. The main idea inherited in the weighted ℓ_p-minimization is to make the entries of $\boldsymbol{x}$, which are expected to be large, be penalized less in the weighted objective function by introducing a weight vector $\mathbf{w}\in[0,1]^n$. The weighted ℓ_p-minimization is formulated as follows:

$$\begin{aligned}&\min_{\boldsymbol{x}\in\mathbb{R}^n}\|\boldsymbol{x}\|_{p,\mathbf{w}}^p\\ &\text{s.t.}\quad \boldsymbol{A}\boldsymbol{x}-\boldsymbol{y}\in\mathcal{B},\end{aligned}\tag{1.3}$$

where

$$\|\boldsymbol{x}\|_{p,\mathbf{w}}=\left(\sum_{i=1}^n \mathrm{w}_i|x_i|^p\right)^{\frac{1}{p}}.$$

In particular, the weighted ℓ_p-minimization (1.3) reduces to the well-known weighted ℓ_1-minimization used for the signal recovery when $p=1$, i.e.

$$\begin{aligned}&\min_{\boldsymbol{x}\in\mathbb{R}^n}\|\boldsymbol{x}\|_{1,\mathbf{w}}\\ &\text{s.t.}\quad \boldsymbol{A}\boldsymbol{x}-\boldsymbol{y}\in\mathcal{B}.\end{aligned}\tag{1.4}$$

Let $\widetilde{T}\subseteq[n]=\{1,2,\ldots,n\}$ be a known support estimate of $\boldsymbol{x}$. The weight vector $\mathbf{w}$ in this paper is taken by

$$\mathrm{w}_i=\begin{cases}\omega, & i\in\widetilde{T},\\ 1, & i\in\widetilde{T}^c\end{cases}\tag{1.5}$$

for some fixed $\omega\in[0,1]$.

The signal recovery based on partially known support is introduced in [2, 15, 20]. In [2, 14, 16, 19, 20], the known support information is incorporated using weighted ℓ_1-minimization with zero weights on the known support $\widetilde{T}$, i.e. $\omega=0$ in (1.5). Friedlander *et al.* [9] extended the weighted ℓ_1-minimization to nonzero weights, i.e. $\omega\in[0,1]$

in (1.5), and derived its stable and robust recovery guarantees based on restricted isometry property, which is one of the most widely used frameworks in compressed sensing proposed in [5]. RIP based signal recovery has been extensively studied via the weighted ℓ_p-minimization (1.3) in the literature, see [7–10, 14, 15, 17, 20].

Definition 1.1. *For a matrix $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ and an integer $1\le k\le n$, $\boldsymbol{A}$ is said to satisfy the RIP of order k if there exists a constant $\delta_k\in[0,1)$ such that*

$$(1-\delta_k)\|\boldsymbol{x}\|_2^2\le\|\boldsymbol{A}\boldsymbol{x}\|_2^2\le(1+\delta_k)\|\boldsymbol{x}\|_2^2 \tag{1.6}$$

holds for all k-sparse signals $\boldsymbol{x}\in\mathbb{R}^n$. A signal $\boldsymbol{x}\in\mathbb{R}^n$ is called k-sparse if the number of its nonzero entries is k at most. The smallest constant δ_k is called the restricted isometry constant (RIC) of order k for $\boldsymbol{A}$.

Note that when k is not an integer, δ_k is defined as $\delta_{\lceil k\rceil}$ in [4], where $\lceil k\rceil$ denotes an integer satisfying $k<\lceil k\rceil<k+1$.

This paper is devoted to developing a uniform RIP bound on δ_{tk} for the exact recovery of signals with partial support information via the weighted ℓ_p-minimization (1.3) with $0<p\le 1$ for all $t>d$ where $d\ge 1$ is determined by the prior support information. We provide the state-of-the-art results for weighted ℓ_p-minimization in the literature to a complete regime, which fill the gap for $t>2d$ on δ_{tk} based signal recovery conditions, and include the optimal results for the ℓ_1-minimization in [4] and the ℓ_p-minimization with $0<p<1$ in [21, 23] as special cases. Our main tool is to study a crucial sparse decomposition technique adapted to the RIP analysis of the weighted $\ell_p\,(0<p\le 1)$ minimization.

On the other hand, the stable recovery guarantees based on δ_{tk} for all $t>d$ for noisy observations or non-sparse signals with suitable assumptions are provided. Our results for stable recovery of non-sparse signals are new for the weighted $\ell_p\,(0<p\le 1)$ minimization, compared to the recent work in [10]. Here we deduce an upper error bound using some new transformations.

The rest of the paper is organized as follows. In Section 2, we recall some technical lemmas for the (weighted) ℓ_p-minimization with $0<p\le 1$. In Section 3, we first present uniform sufficient conditions for the recovery of sparse signals with prior support information in the noiseless case. Then the error bounds of signal stable recovery are developed in ℓ_2 bounded noise case or non-sparse signals. Finally, the conclusion of the paper is presented in Section 5.

2 Preliminaries

In this section, we first recall some technical lemmas for the analysis of the weighted ℓ_p-minimization (1.3) with $0<p\le 1$.

The following two lemmas have been used in [10]. The first one concerns elementary ℓ_p inequality.

Lemma 2.1 ([10, Lemma V.1]). *Let p and q be two positive numbers. Then*

(I) $\|\boldsymbol{x}\|_p \le \|\boldsymbol{x}\|_2 |\mathrm{supp}(\boldsymbol{x})|^{(2-p)/(2p)}$, *if* $0<p<2$,

(II) $\|\boldsymbol{x}\|_p^p \le (\|\boldsymbol{x}\|_2^2)^{1/q}(\|\boldsymbol{x}\|_{p_1}^{p_1})^{1-1/q}$, *if* $pq>2$ *and* $q>1$, *where* $p_1=(p-2/q)(q/(q-1))$.

The second lemma states some properties on a function $g(z)=pz^{2/p}/2+z-(2-p)\Lambda/2$.

Lemma 2.2 ([10, Lemma V.2]). *For* $0<p\le 1$ *and* $\Lambda>0$, *the function* $g(z)=pz^{2/p}/2+z-(2-p)\Lambda/2$ *is monotonically increasing in* $(0,\infty)$. *In addition, the following statements hold:*

(I) If $0<\Lambda\le 2/(2-p)$, *there exists a unique point* $z_0\in((1-p)\Lambda,(1-p/2)\Lambda)\subseteq(0,1)$ *such that* $g(z_0)=0$.

(II) If $2/(2-p)<\Lambda<(2+p)/(2-p)$, *there exists a unique point* $z_0\in((1-p)\Lambda,1)\subseteq(0,1)$ *such that* $g(z_0)=0$.

(III) If $\Lambda\ge(2+p)/(2-p)$, *there does not exist a point* $z_0\in(0,1)$ *such that* $g(z_0)=0$.

The third lemma is an important lifting inequality established in [3].

Lemma 2.3 ([3]). *Suppose* $n\ge r,\tau\ge 0,a_1\ge a_2\ge\cdots\ge a_n\ge 0$, *and* $\sum_{i=1}^r a_i+\tau\ge\sum_{i=r+1}^n a_i$. *Then for all* $\sigma\ge 1$,

$$\sum_{i=r+1}^n a_i^\sigma \le r\left(\left(\frac{1}{r}\sum_{i=1}^r a_i^\sigma\right)^{\frac{1}{\sigma}}+\frac{\tau}{r}\right)^\sigma.$$

The cone constraint inequality obtained in [11, Inequality (14)] is an essential extension of [9, Inequality (21)], which will play a key role for analyzing the weighted ℓ_p-minimization (1.3). See the following lemma.

Lemma 2.4. *For any two vectors* $\boldsymbol{x},\hat{\boldsymbol{x}}\in\mathbb{R}^n$ *and* $\boldsymbol{h}=\hat{\boldsymbol{x}}-\boldsymbol{x}$, *if* $\|\hat{\boldsymbol{x}}\|_{p,\mathbf{w}}^p\le\|\boldsymbol{x}\|_{p,\mathbf{w}}^p$ *with the weight vector* $\mathbf{w}$ *defined in* (1.5), *then*

$$\begin{aligned}\|\boldsymbol{h}_{\Gamma^c}\|_p^p \le{}& \omega\|\boldsymbol{h}_\Gamma\|_p^p+(1-\omega)\left\|\boldsymbol{h}_{(\widetilde{T}\cup\Gamma)\setminus(\widetilde{T}\cap\Gamma)}\right\|_p^p\\&+2\left(\omega\|\boldsymbol{x}_{\Gamma^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap\Gamma^c}\right\|_p^p\right)\end{aligned}\tag{2.1}$$

for any index set $\Gamma\subseteq[n]$.

A well-known property on RICs with different orders (see for example [3,Lemma 4.1]) is stated as follows.

Lemma 2.5. *Suppose* $A\in\mathbb{R}^{m\times n},k\ge 2$ *is an integer,* $s>1$ *and* sk *is an integer. Then* $\delta_{sk}\le(2s-1)\delta_k$.

A key tool established in [4,22], which represents points in a polytope by convex combinations of k-sparse signals, initiates a process of improving and sharping RIP bounds for signal recovery via the (weighted) ℓ_1-minimization. The sparse representation of a polytope is extended in [23] to adapt $l_p\,(0<p\le 1)$ case, see the following lemma.

Lemma 2.6 ([23, Lemma 2.2]). *For $\boldsymbol{x}\in\mathbb{R}^n$ which satisfies $|\mathrm{supp}(\boldsymbol{x})|=K, \|\boldsymbol{x}\|_p^p\le L\zeta^p$ and $\|\boldsymbol{x}\|_\infty\le\zeta$ with $L\le K$ being a positive integer, ζ being a positive constant and $0<p\le 1$, then $\boldsymbol{x}$ can be represented as the convex combination of L-sparse vectors, i.e.*

$$\boldsymbol{x}=\sum_{i=1}^N\lambda_i\boldsymbol{u}_i,$$

where $\lambda_i>0,\sum_{i=1}^N\lambda_i=1$ and $\|\boldsymbol{u}_i\|_0\le L$. Furthermore,

$$\sum_{i=1}^N\lambda_i\|\boldsymbol{u}_i\|_2^2\le\min\left\{\frac{n}{L}\|\boldsymbol{x}\|_2^2,\zeta^p\|\boldsymbol{x}\|_{2-p}^{2-p}\right\}. \tag{2.2}$$

We have used the key sparse representation tool with $0<p\le 1$ and obtained the following state-of-the-art RIP condition for sparse signal recovery via the weighted ℓ_p-minimization (1.3), which includes the existing optimal result in [6, Theorem 1].

Theorem 2.1 ([10, Theorem III.1]). *For $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$, let $\boldsymbol{x}\in\mathbb{R}^n$ be k-sparse with $T=\mathrm{supp}(\boldsymbol{x})$ and the support estimate $\widetilde{T}\subseteq[n]$. Define $\rho\ge 0$ and $0\le\alpha\le 1$ with $\alpha\rho\le 1$ such that $|\widetilde{T}|=\rho k$ and $|\widetilde{T}\cap T|=\alpha\rho k$. If $\boldsymbol{A}$ satisfies RIP with*

$$\delta_{tk}<\begin{cases}\dfrac{1}{\sqrt{p^2+(2-p)^2\chi^{\frac{2}{2-p}}/(t-d)}-(1-p)}, & d<t\le d+\dfrac{2-p}{2+p}\chi^{\frac{2}{2-p}},\\[2ex] \dfrac{z_0}{(2-p)\chi^{\frac{2}{2-p}}/(t-d)-z_0}, & d+\dfrac{2-p}{2+p}\chi^{\frac{2}{2-p}}<t\le 2d,\end{cases} \tag{2.3}$$

where

$$d=\begin{cases}1, & \omega=1,\\ 1+\max\{0,1-2\alpha\}\rho, & 0\le\omega<1,\end{cases} \tag{2.4}$$

$$\chi=\omega+(1-\omega)(1+\rho-2\alpha\rho)^{\frac{2-p}{2}}, \tag{2.5}$$

and

$$z_0\in\left(\frac{1-p}{t-d}\chi^{\frac{2}{2-p}},\min\left\{1,\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}\right\}\right)$$

is the only positive solution of the equation

$$\frac{p}{2}z^{\frac{2}{p}}+z-\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}=0, \tag{2.6}$$

then the weighted ℓ_p-minimization (1.3) *with the weight vector* $\mathbf{w}$ *defined in* (1.5) *and* $0<p\le 1$ *recovers $\boldsymbol{x}$ exactly.*

3 Main results

In this section, we present RIP bounds for the signal recovery via the weighted ℓ_p-minimization (1.3) with $0<p\leq 1$ in both noiseless and l_2 bounded noise cases.

3.1 Noiseless case

In noiseless case, we obtain a uniform recovery condition based on δ_{tk} with $t>d$ for the exact recovery of the sparse signals $\boldsymbol{x}$ from $\boldsymbol{y}=\boldsymbol{Ax}$ via the weighted ℓ_p-minimization (1.3) with $0<p\leq 1$ and $\mathcal{B}=\{\mathbf{0}\}$.

Theorem 3.1. *Let $\boldsymbol{y}=\boldsymbol{Ax}$ for a k-sparse vector $\boldsymbol{x}\in\mathbb{R}^n$ with $T=\mathrm{supp}(\boldsymbol{x})$, and $\widetilde{T}\subseteq[n]$ be a support estimate of $\boldsymbol{x}$. Define $\rho\geq 0$ and $0\leq\alpha\leq 1$ with $\alpha\rho\leq 1$ such that $|\widetilde{T}|=\rho k$ and $|\widetilde{T}\cap T|=\alpha\rho k$. Given the weight vector $\mathbf{w}\in[0,1]^n$ as defined in* (1.5) *and $0<p\leq 1$, if tk is an integer and*

$$\begin{aligned}1-\delta_{tk}^2-p\chi^{\frac{2}{p}}\left(2(t-d)\right)^{-\frac{2-p}{p}}&\left(\frac{\sqrt{p^2\delta_{2(t-d)k}^2+4(1-p)\delta_{tk}^2}+(2-p)\delta_{2(t-d)k}}{1+\delta_{2(t-d)k}}\right)^{\frac{2-2p}{p}}\\&\times\left(2\delta_{tk}^2-p\delta_{2(t-d)k}^2+\delta_{2(t-d)k}\sqrt{p^2\delta_{2(t-d)k}^2+4(1-p)\delta_{tk}^2}\right)>0 \qquad (3.1)\end{aligned}$$

for some $t>d$, where d and χ are defined in (2.4) *and* (2.5)*, respectively, then the weighted ℓ_p-minimization* (1.3) *with $\mathcal{B}=\{\mathbf{0}\}$ recovers $\boldsymbol{x}$ exactly.*

The proof of Theorem 3.1 can be found in Section 4.2. We first provide some remarks for the case $d<t\leq 2d$.

For $d<t\leq 2d,\delta_{2(t-d)k}\leq\delta_{tk}$ by the monotonicity of RICs. By some simple calculation, it is easy to see that the quantity

$$\begin{aligned}&\left(\frac{(2-p)\delta_{2(t-d)k}+\sqrt{p^2\delta_{2(t-d)k}^2+4(1-p)\delta_{tk}^2)}}{1+\delta_{2(t-d)k}}\right)^{\frac{2-2p}{p}}\\&\times\left(2\delta_{tk}^2-p\delta_{2(t-d)k}^2+\delta_{2(t-d)k}\sqrt{p^2\delta_{2(t-d)k}^2+4(1-p)\delta_{tk}^2}\right)\end{aligned}$$

is monotonically increasing in $\delta_{2(t-d)k}$. Then

$$1-\delta_{tk}^2-p\chi^{\frac{2}{p}}\left(2(t-d)\right)^{-\frac{2-p}{p}}\left(\frac{2(2-p)\delta_{tk}}{1+\delta_{tk}}\right)^{\frac{2-2p}{p}}\left(2(2-p)\delta_{tk}^2\right)>0$$

guarantees the condition (3.1) holds. Therefore, we have the following corollary.

Corollary 3.1. *Let $\boldsymbol{y}=\boldsymbol{Ax}$ for a k-sparse vector $\boldsymbol{x}\in\mathbb{R}^n$ with $T=\mathrm{supp}(\boldsymbol{x})$ and the support estimate $\widetilde{T}\subseteq[n]$. Let α and ρ be the same as in Theorem* 3.1*. If $\boldsymbol{A}$ satisfies RIP with*

$$\delta_{tk}<\tilde{\delta}(p,t,d,\chi) \qquad (3.2)$$

for $d<t\le 2d$, where d and χ are respectively defined in (2.4) *and* (2.5)*, and*

$$\tilde{\delta}(p,t,d,\chi)\in\left[\left(1+2p\chi^{\frac{2}{p}}\left(\frac{2-p}{2-p}\right)^{\frac{2-p}{p}}\right)^{-\frac{1}{2}},1\right)$$

is the unique position solution of the equation

$$\delta=\left(1+p\chi^{\frac{2}{p}}\left(\frac{2-p}{t-d}\right)^{\frac{2-p}{p}}\left(\frac{\delta}{1+\delta}\right)^{\frac{2-2p}{p}}\right)^{-\frac{1}{2}},\tag{3.3}$$

then the weighted ℓ_p-minimization (1.3) *with the weight vector* $\mathbf{w}$ *defined in* (1.5) *and* $0<p\le 1$ *recovers* $\boldsymbol{x}$ *exactly.*

Remark 3.1. As pointed out before, the state-of-the-art result based on δ_{tk} with $d<t\le 2d$ for the exact recovery of the sparse signal $\boldsymbol{x}$ from $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$ has been developed in our previous paper [10]. See Theorem 2.1. The following facts have a direct bearing on the matter and deserve our careful discussion. When $d<t\le d+(2-p)\chi^{2/(2-p)}/(2+p)$, the condition (2.3) is weaker than (3.2). When $d+(2-p)\chi^{2/(2-p)}/(2+p)<t\le 2d$, the condition (2.3) is equivalent to (3.2). In fact, the Eq. (3.3) can be written as

$$\frac{p}{2}\left(\frac{2-p}{t-d}\frac{\delta}{1+\delta}\chi^{\frac{2}{2-p}}\right)^{\frac{2}{p}}+\frac{2-p}{t-d}\frac{\delta}{1+\delta}\chi^{\frac{2}{2-p}}-\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}=0.$$

Then $\tilde{\delta}(p,t,d,\chi)$ in (3.2) satisfies

$$\tilde{\delta}(p,t,d,\chi)=\frac{z_0}{(2-p)\chi^{\frac{2}{2-p}}/(t-d)-z_0}$$

for the unique positive solution

$$z_0\in\left(\frac{1-p}{t-d}\chi^{\frac{2}{2-p}},\min\left\{1,\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}\right\}\right)$$

of (2.6), which infers that (3.2) is exactly the condition (2.3). When $d<t\le d+(2-p)/(2+p)\times\chi^{2/(2-p)}$, we will prove that

$$\tilde{\delta}(p,t,d,\chi)=\frac{z_0}{(2-p)\chi^{\frac{2}{2-p}}/(t-d)-z_0}<\frac{1}{\sqrt{p^2+(2-p)^2\chi^{\frac{2}{2-p}}/(t-d)}-(1-p)}.$$

That is to show that

$$z_0<\frac{2-p}{(t-d)\left(\sqrt{p^2+(2-p)^2\chi^{\frac{2}{2-p}}/(t-d)}+p\right)}\chi^{\frac{2}{2-p}},$$

which is obvious since $z_0<1$ and

$$\frac{\chi^{\frac{2}{2-p}}}{t-d}\frac{2-p}{\sqrt{p^2+(2-p)^2\chi^{\frac{2}{2-p}}/(t-d)}+p}\geq 1$$

for $d<t\leq d+(2-p)\chi^{2/(2-p)}/(2+p)$.

When $\omega=1$, we have $\chi=1$ in (2.5) and $d=1$ in (2.4), then the condition (3.2) reduces to a sufficient condition in [6, Theorem 1] for sparse signal recovery via the ℓ_p-minimization (1.2), which includes the sharp sufficient condition [23, Theorem 1.2].

Corollary 3.2. *Let $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$ for a k-sparse vector $\boldsymbol{x}\in\mathbb{R}^n$ with $T=\operatorname{supp}(\boldsymbol{x})$. If $\boldsymbol{A}$ satisfies RIP with*

$$\delta_{tk}<\tilde{\delta}(p,t,1,1) \tag{3.4}$$

for $1<t\leq 2$, where

$$\tilde{\delta}(p,t,1,1)\in\left[\left(1+2p\left(\frac{2-p}{2(t-1)}\right)^{\frac{2-p}{p}}\right)^{-\frac{1}{2}},1\right)$$

is the unique positive solution of the equation

$$\delta=\left(1+p\left(\frac{2-p}{t-1}\right)^{\frac{2-p}{p}}\left(\frac{\delta}{1+\delta}\right)^{\frac{2-2p}{p}}\right)^{-\frac{1}{2}},$$

then the ℓ_p-minimization (1.2) with $0<p\leq 1$ and $\mathcal{B}=\mathbf{0}$ recovers $\boldsymbol{x}$ exactly.

Remark 3.2. When $\omega=1$, the condition (3.2) reduces to (3.4), and the condition (2.3) reduces to

$$\delta_{tk}<\begin{cases}\dfrac{1}{\sqrt{p^2+(2-p)^2/(t-1)}-(1-p)}, & 1<t\leq 1+\dfrac{2-p}{2+p},\\ \dfrac{z_0}{(2-p)/(t-d)-z_0}, & 1+\dfrac{2-p}{2+p}<t\leq 2\end{cases} \tag{3.5}$$

for the unique positive solution

$$z_0\in\left(\frac{1-p}{t-d}\chi^{\frac{2}{2-p}},\min\left\{1,\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}\right\}\right)$$

of the equation

$$\frac{p}{2}z^{\frac{2}{p}}+z-\frac{2-p}{2(t-1)}=0, \tag{3.6}$$

which is sufficient for sparse signal recovery via the ℓ_p-minimization. By Remark 3.1, the condition (3.5) in [10] is equivalent to (3.4), which is sharp for sparse signal recovery via the ℓ_p-minimization (1.2) when $1+(2-p)/(2+p)<t\leq 2$, see [6, Remark 10]. When $1<t\leq 1+(2-p)/(2+p)$, the condition (3.5) in [10] is weaker than (3.4).

Remark 3.3. For $t=2$, the condition (3.5) or (3.4) is the sharp sufficient condition [23, Theorem 1.2]. That is,

$$\delta_{2k} < \frac{\eta_0}{2-p-\eta_0},$$

where $\eta_0 \in (1-p, 1-p/2)$ is the only positive solution of the equation

$$\frac{p}{2} z^{\frac{p}{2}} + z - 1 + \frac{p}{2} = 0.$$

It is worth to point out that the uniform condition (3.1) in Theorem 3.1 involves both δ_{tk} and $\delta_{2(t-d)k}$ for $0<p\le 1$. It is a little surprise that the uniform condition (3.1) involves only δ_{tk} for $p=1$ and it reduces to the state-of-the-art condition in [8, Theorem 3.1, Remark 3.1] for the exact recovery of $\boldsymbol{x}$. See the following Corollary 3.3, which can be easily inferred from Theorem 3.1.

Corollary 3.3. *For $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$, let $\boldsymbol{x}\in\mathbb{R}^n$ be k-sparse with $T=\operatorname{supp}(\boldsymbol{x})$ and the support estimate $\widetilde{T}\subseteq[n]$. Let α and ρ be the same as in Theorem* 3.1. *If $\boldsymbol{A}$ satisfies RIP with*

$$\delta_{tk} < \sqrt{\frac{t-d}{t-d+\chi^2}} \tag{3.7}$$

for some $t>d$, where d is defined in (2.4) *and χ is defined in* (2.5) *with $p=1$, then the weighted ℓ_1-minimization* (1.4) *with $\mathcal{B}=\{\mathbf{0}\}$ exactly recovers $\boldsymbol{x}$.*

Remark 3.4. Note that the sufficient condition (3.7) is tight under certain cases, see [8, Theorem 3.2].

For the most classical case $p=1$ and $\omega=1$, then $\chi=1$ in (2.5), $d=1$ in (2.4) and the uniform condition (3.1) in Theorem 3.1 reduces to the sharp sufficient condition [4, Theorem 1.1].

Corollary 3.4. *Let $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$ for a k-sparse vector $\boldsymbol{x}\in\mathbb{R}^n$. If*

$$\delta_{tk} < \sqrt{\frac{t-1}{t}} \tag{3.8}$$

for some $t>1$, then $\boldsymbol{x}$ can be exactly recovered by the ℓ_p-minimization (1.2) *with $p=1$ and $\mathcal{B}=\{\mathbf{0}\}$.*

Now we consider the general case $t>d$. When $\omega=1$ or $\alpha=1/2$, it is clear that $\chi=1$ in (2.5) and $d=1$ in (2.4). And the uniform RIP conditions (3.1) reduces to the uniform result for the ℓ_p-minimization [21, Theorem 1].

Corollary 3.5. *Assume that $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$ where $\boldsymbol{x}\in\mathbb{R}^n$ is a k-sparse signal. If tk is an integer and*

$$\begin{aligned}
&1-\delta_{tk}^2 - p\big(2(t-1)\big)^{-\frac{2-p}{p}} \left(\frac{(2-p)\delta_{2(t-1)k}+\sqrt{p^2\delta_{2(t-1)k}^2+4(1-p)\delta_{tk}^2}}{1+\delta_{2(t-1)k}}\right)^{\frac{2-2p}{p}} \\
&\quad\times\Big(2\delta_{tk}^2 - p\delta_{2(t-1)k}^2 + \delta_{2(t-1)k}\sqrt{p^2\delta_{2(t-1)k}^2+4(1-p)\delta_{tk}^2}\Big) > 0
\end{aligned} \tag{3.9}$$

for some $t>1$, then the ℓ_p-minimization (1.2) *with $\mathcal{B}=\{\mathbf{0}\}$ and $0<p\le 1$ exactly recovers $\boldsymbol{x}$.*

When $\alpha > 1/2$ and $\omega \in [0,1)$, it is clear that $\chi < 1$ from (2.5) and $d = 1$ in (2.4). If $\chi < 1$ and $d = 1$, then the sufficient condition (3.9) implies the condition (3.1). Therefore, we have the following proposition.

Proposition 3.1. *If $\alpha > 1/2$ and $\omega \in [0,1)$, then the sufficient condition* (3.1) *of the weighted ℓ_p-minimization* (1.3) *is weaker than the condition* (3.9) *of the ℓ_p-minimization* (1.2) *for exact sparse recovery.*

Here, we provide a frame diagram (Fig. 1) to summarize the remarks and corollaries following Theorem 3.1. And the conditions on δ_{tk} contain several quantities in the remarks and corollaries. Baraniuk *et al.* [1] provide a bound on RICs for a set of random matrices from concentration of measure. For these random measurement matrices, [1, Theorem 5.2] shows that

$$P(\delta_k < \lambda) \geq 1 - 2\left(\frac{12en}{k\lambda}\right)^k \exp\left(-m\left(\frac{\lambda^2}{16} - \frac{\lambda^3}{48}\right)\right)$$

holds for positive integer $k < m$ and $0 < \lambda < 1$. Then, for any known bound $\delta_k < \delta_0 < 1$, $\delta_k < \delta_0$ hold in high probability when

$$m \geq \frac{k \log(n/k)}{\delta_0^2/16 - \delta_0^3/48}.$$

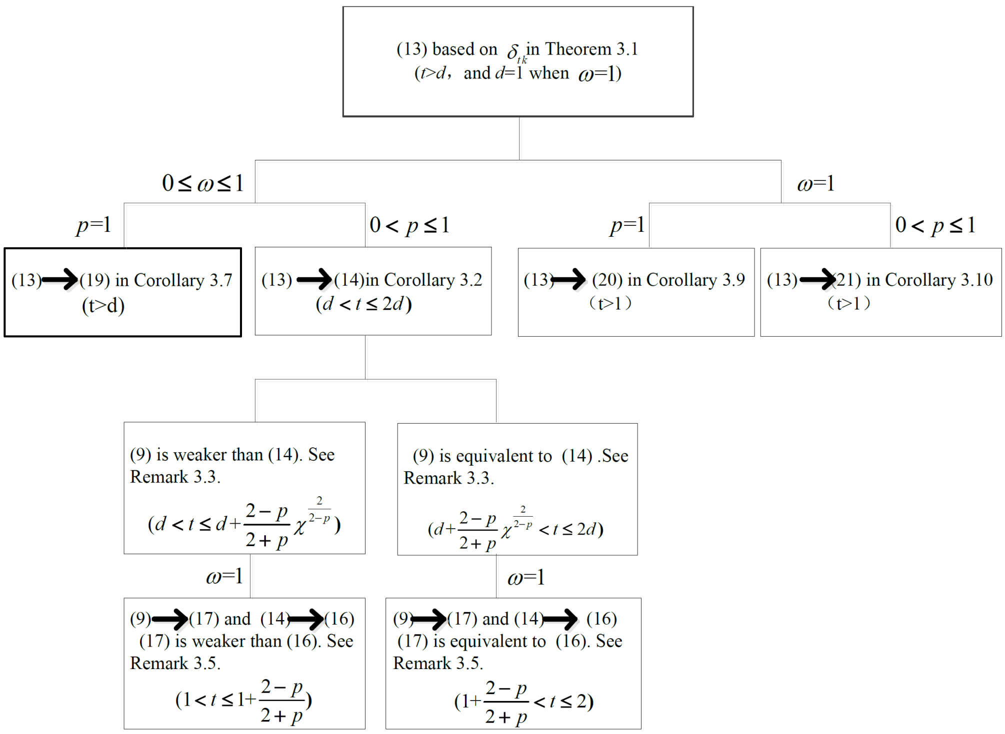

Figure 1: The whole structure of bounds on δ_{tk} follows from Theorem 3.1.

For example, the lower bound of m to ensure $\delta_{tk}<\tilde{\delta}(p,t,d,\chi)$ in (3.2) to hold in high probability is

$$m\geq\frac{k\log(n/k)}{\tilde{\delta}^2(p,t,d,\chi)/16-\tilde{\delta}^3(p,t,d,\chi)/48}.$$

Next, we devote to developing a general RIP condition on δ_{tk} for $t\geq 2d$ to achieve the recovery of sparse signals via the weighted ℓ_p minimization, which will fill the gap on δ_{tk} based signal recovery condition for $t>2d$, compared with the work in [10].

Theorem 3.2. *Let $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$ where $\boldsymbol{x}\in\mathbb{R}^n$ is a k-sparse vector with $T=\operatorname{supp}(\boldsymbol{x})$, and $\widetilde{T}\subseteq[n]$ be a support estimate of $\boldsymbol{x}$. Let α and ρ be the same as in Theorem* 3.1. *Given the weight vector $\mathbf{w}\in[0,1]^n$ defined in* (1.5) *and $0<p\leq 1$, if*

$$\delta_{tk}<\delta(p,t,d,\chi) \tag{3.10}$$

for some $t\geq 2d$, where d and χ are respectively defined in (2.4) *and* (2.5) *and $\delta(p,t,d,\chi)$ satisfying*

$$\begin{aligned}&\left[1+p\left(\frac{\chi^{\frac{2}{2-p}}}{2(t-d)}\right)^{\frac{2-p}{p}}\left(\frac{s(2-p)+\sqrt{s^2p^2+4(1-p)}}{4(t-d)}t\right)^{\frac{2-2p}{p}}\right.\\&\qquad\left.\times\left(2-ps^2+s\sqrt{s^2p^2+4(1-p)}\right)\right]^{-\frac{1}{2}}\leq\delta(p,t,d,\chi)<1,\end{aligned}$$

where $s=(3t-4d)/t$, is the unique positive solution of the following equation:

$$\begin{aligned}z=&\left[1+p\left(\frac{\chi^{\frac{2}{2-p}}}{2(t-d)}\right)^{\frac{2-p}{p}}\left(\frac{s(2-p)+\sqrt{s^2p^2+4(1-p)}}{1+sz}z\right)^{\frac{2-2p}{p}}\right.\\&\qquad\left.\times\left(2-ps^2+s\sqrt{s^2p^2+4(1-p)}\right)\right]^{-\frac{1}{2}},\end{aligned} \tag{3.11}$$

then the weighted ℓ_p-minimization (1.3) *with $\mathcal{B}=\{\mathbf{0}\}$ recovers $\boldsymbol{x}$ exactly.*

The proof of Theorem 3.2 can be found in Section 4.3.

Remark 3.5. Let

$$\begin{aligned}Q(d,z)=&\frac{2}{p}(t-d)^{\frac{2-p}{p}}\frac{1-z^2}{z^2}\left(\frac{s(2-p)+\sqrt{s^2p^2+4(1-p)}}{2(1+sz)}z\right)^{-\frac{2-2p}{p}}\\&\times\left(2-ps^2+s\sqrt{s^2p^2+4(1-p)}\right)^{-1}.\end{aligned} \tag{3.12}$$

Then the Eq. (3.11) can be written as $Q(d,z)=\chi^{p/2}$.

Remark 3.6. For $\omega=1$ or $\alpha=1/2$, Theorem 3.2 reduces to [21, Theorem 2]. That is, when $\omega=1$ or $\alpha=1/2$, the condition (3.10) reduces to $\delta_{tk}<\delta(p,t,1,1)$ for some $t\geq 2$, which is a state-of-the-art sufficient condition in [21, Theorem 2] for the sparse signal recovery via the ℓ_p-minimization, where $\delta(p,t,1,1)$ is the unique positive solution of the equation $Q(1,z)=1$.

When $\alpha>1/2$ and $\omega\in[0,1)$, we have $\chi<1$ in (2.5) and $d=1$ in (2.4). Then, the condition (3.10) reduces to $\delta_{tk}<\delta(p,t,1,\chi)$ where $\delta(p,t,1,\chi)$ is the unique positive solution of the equation $Q(1,z)=\chi^{p/2}$. By some simple calculation, the function $Q(1,z)$ is monotonically decreasing on $z\in(0,1]$. Therefore, $\delta(p,t,1,1)<\delta(p,t,1,\chi)$ when $\chi<1$. We establish the following proposition.

Proposition 3.2. *If $\alpha>1/2$ and $\omega\in[0,1)$, then the sufficient condition $\delta_{tk}<\delta(p,t,1,\chi)$ of the weighted ℓ_p-minimization* (1.3) *is weaker than the condition $\delta_{tk}<\delta(p,t,1,1)$ of the ℓ_p-minimization* (1.2) *in* [21, *Theorem* 2] *for exact sparse recovery.*

3.2 Noisy or non-sparse signal case

In the subsection, the origin signal $\boldsymbol{x}$ is not limited to be k-sparse, which is different from the sparse signals considered in [10]. We derive the following results, which complete the RIP based characterization for the recovery of signals via the weighted ℓ_p-minimization (1.3).

First, we consider the stable recovery based on δ_{tk} with $d<t\leq d+(2-p)/(2+p)\chi^{2/(2-p)}$ in the following theorem.

Theorem 3.3. *Let $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{e}$, where $\boldsymbol{x}\in\mathbb{R}^n$ and $\|\boldsymbol{e}\|_2\leq\varepsilon$. Let $T=\mathrm{supp}(\boldsymbol{x}_k)$ where $\boldsymbol{x}_k$ is the best k-term approximation of $\boldsymbol{x}$ which only keeps the largest k entries in magnitude, and $\widetilde{T}\subseteq[n]$ be a support estimate of $\boldsymbol{x}_k$. Define $\rho\geq 0$ and $0\leq\alpha\leq 1$ with $\alpha\rho\leq 1$ such that $|\widetilde{T}|=\rho k$ and $|\widetilde{T}\cap T|=\alpha\rho k$. Given the weight vector $\mathbf{w}\in[0,1]^n$ defined in* (1.5) *and $0<p\leq 1$, suppose $\hat{\boldsymbol{x}}^{\ell_2}$ is a minimizer of* (1.3) *with $\mathcal{B}=\mathcal{B}^{l_2}(\varepsilon)=\{\boldsymbol{z}\in\mathbb{R}^m:\|\boldsymbol{z}\|_2\leq\varepsilon\}$. If $\boldsymbol{A}$ satisfies RIP with*

$$\delta_{tk}<\frac{1}{\sqrt{p^2+(2-p)^2\chi^{\frac{2}{2-p}}}-(1-p)} \tag{3.13}$$

for $d<t\leq d+(2-p)\chi^{2/(2-p)}/(2+p)$ where χ and d are respectively defined in (2.5) *and* (2.4)*, then*

$$\begin{aligned}\|\boldsymbol{x}-\hat{\boldsymbol{x}}^{\ell_2}\|_2\leq&\sqrt{1+2^{\frac{2-2p}{p}}}C_1\left(\sqrt{p^2+q\chi^{\frac{2}{2-p}}}+p\right)^{-1}\left(1-\delta_{tk}\left(\sqrt{p^2+q\chi^{\frac{2}{2-p}}}+p-1\right)\right)^{-1}\varepsilon\\&+\sqrt{C_2^2+2^{\frac{2-2p}{p}}\left(C_2+\left(2(dk)^{-\frac{2-p}{2}}\right)^{\frac{1}{p}}\right)^2}\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)^{\frac{1}{p}},\end{aligned} \tag{3.14}$$

where

$$
\begin{aligned}
C_1 =& 2\left(\sqrt{p^2+q\chi^{\frac{2}{2-p}}}+2p-2\right)\sqrt{1+\delta_{tk}} \\
&+\left(4\left(\sqrt{p^2+q\chi^{\frac{2}{2-p}}}+2p-2\right)^2(1+\delta_{tk})+8\left(\sqrt{p^2+q\chi^{\frac{2}{2-p}}}+p\right)\right. \\
&\left.\times\left(1-\delta_{tk}\left(\sqrt{p^2+q\chi^{\frac{2}{2-p}}}+p-1\right)\right)(1-p)\right)^{\frac{1}{2}},
\end{aligned} \tag{3.15}
$$

$q=q(t,d)=(2-p)^2/(t-d)$, *and*

$$
C_2=\left(\left(\frac{2}{p}\left(\frac{\sqrt{p^2+q\chi^{\frac{2}{2-p}}}+p}{2}-1\right)\right)^{\frac{p}{2}}\left(\frac{(2-p)\chi^{\frac{2}{2-p}}\delta_{tk}}{(t-d)(1+\delta_{tk})}\right)^{-\frac{2-p}{2}}-1\right)^{-\frac{1}{p}}\left(\frac{2}{k^{\frac{2-p}{2}}\chi^{\frac{2}{p}}}\right)^{\frac{1}{p}}. \tag{3.16}
$$

The proof of Theorem 3.3 can be found in Section 4.4.

Next, the stable recovery result based on δ_{tk} with $d+(2-p)\chi^{2/(2-p)}/(2+p)\le t<2d$ is developed in the following theorem.

Theorem 3.4. *Let $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{e}$ where $\boldsymbol{x}\in\mathbb{R}^n$ and $\|\boldsymbol{e}\|_2\le\varepsilon$. Let $T=\operatorname{supp}(\boldsymbol{x}_k)$ where $\boldsymbol{x}_k$ is the best k-term approximation of $\boldsymbol{x}$, and $\widetilde{T}\subseteq[n]$ be a support estimate of $\boldsymbol{x}_k$. Define $\rho\ge 0$ and $0\le\alpha\le 1$ with $\alpha\rho\le 1$ such that $|\widetilde{T}|=\rho k$ and $|\widetilde{T}\cap T|=\alpha\rho k$. Given the weight vector $\mathbf{w}\in[0,1]^n$ defined in* (1.5) *and $0<p\le 1$, suppose $\hat{\boldsymbol{x}}^{\ell_2}$ is a minimizer of* (1.3) *with $\mathcal{B}=\mathcal{B}^{l_2}(\varepsilon)=\{\boldsymbol{z}\in\mathbb{R}^m:\|\boldsymbol{z}\|_2\le\varepsilon\}$. If $\boldsymbol{A}$ satisfies RIP with*

$$
\delta_{tk}<\frac{z_0}{(2-p)\chi^{\frac{2}{2-p}}-z_0} \tag{3.17}
$$

for $d+(2-p)\chi^{2/(2-p)}/(2+p)<t\le 2d$, where

$$
z_0\in\left(\frac{1-p}{t-d}\chi^{\frac{2}{2-p}},\min\left\{1,\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}\right\}\right)
$$

is the only positive solution of the equation

$$
\frac{p}{2}z^{\frac{2}{p}}+z-\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}=0, \tag{3.18}
$$

then

$$
\begin{aligned}
\|\boldsymbol{x}-\hat{\boldsymbol{x}}^{\ell_2}\|_2\le&\frac{\sqrt{1+2^{\frac{2-2p}{p}}}D_1}{1-\left((2-p)\chi^{\frac{2}{2-p}}-z_0(t-d)\right)\delta_{tk}/\left(z_0(t-d)\right)}\varepsilon \\
&+\sqrt{D_2^2+2^{\frac{2-2p}{p}}\left(D_2+\left(2(dk)^{-\frac{2-p}{2}}\right)^{\frac{1}{p}}\right)^2}\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)^{\frac{1}{p}},
\end{aligned} \tag{3.19}
$$

where

$$D_1=2\left(1-\frac{z_0(t-d)}{\chi^{\frac{2}{2-p}}}\right)\sqrt{1+\delta_{tk}} \tag{3.20}$$

$$+2\left(\left(1-\frac{z_0(t-d)}{\chi^{\frac{2}{2-p}}}\right)^2(1+\delta_{tk})+2\left(1-\frac{(2-p)\chi^{\frac{2}{2-p}}-z_0(t-d)}{z_0(t-d)}\delta_{tk}\right)\frac{1-p}{2-p}\frac{z_0(t-d)}{\chi^{\frac{2}{2-p}}}\right)^{\frac{1}{2}},$$

$$D_2=\left(z_0^{\frac{2-p}{2}}\left(\frac{(2-p)\chi^{\frac{2}{2-p}}\delta_{tk}}{(1+\delta_{tk})(t-d)}\right)^{-\frac{2-p}{2}}-1\right)^{-\frac{1}{p}}\left(\frac{2}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{1}{p}}. \tag{3.21}$$

The proof of Theorem 3.4 can be found in Section 4.5.

Finally, we consider the stable recovery of the signal $\boldsymbol{x}$ on the high order RIP δ_{tk} with $t\geq 2d$.

Theorem 3.5. *Let $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{e}$ where $\boldsymbol{x}\in\mathbb{R}^n$ and $\|\boldsymbol{e}\|_2\leq\varepsilon$. Let $T=\mathrm{supp}(\boldsymbol{x}_k)$ where $\boldsymbol{x}_k$ is the best k-term approximation of $\boldsymbol{x}$, and $\widetilde{T}\subseteq[n]$ be a support estimate of $\boldsymbol{x}_k$. Define $\rho\geq 0$ and $0\leq\alpha\leq 1$ with $\alpha\rho\leq 1$ such that $|\widetilde{T}|=\rho k$ and $|\widetilde{T}\cap T|=\alpha\rho k$. Given the weight vector $\mathbf{w}\in[0,1]^n$ defined in* (1.5) *and $0<p\leq 1$, suppose $\hat{\boldsymbol{x}}^{\ell_2}$ is a minimizer of* (1.3) *with $\mathcal{B}=\mathcal{B}^{l_2}(\varepsilon)=\{\boldsymbol{z}\in\mathbb{R}^m:\|\boldsymbol{z}\|_2\leq\varepsilon\}$. If $\boldsymbol{A}$ satisfies RIP with* (3.10)*, then*

$$\|\boldsymbol{x}-\hat{\boldsymbol{x}}^{\ell_2}\|_2\leq\frac{\sqrt{1+2^{\frac{2-2p}{p}}}E_1\delta^2(p,t,d,\chi)}{2\big(\delta(p,t,d,\chi)-\delta_{tk}\big)}\varepsilon+\sqrt{E_2^2+\left(E_2+\left(2(dk)^{-\frac{2-p}{2}}\right)^{\frac{1}{p}}\right)^2}$$

$$\times\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)^{\frac{1}{p}}, \tag{3.22}$$

where

$$E_1=\left(\frac{1}{\delta(p,t,d,\chi)}-\frac{1}{2}\left(\sqrt{(sp)^2+4(1-p)}-sp\right)\right)\sqrt{1+\delta_{tk}}$$

$$+\left(\left(\frac{1}{\delta(p,t,d,\chi)}-\frac{1}{2}\left(\sqrt{(sp)^2+4(1-p)}-sp\right)\right)^2(1+\delta_{tk})\right.$$

$$\left.+2\frac{\delta(p,t,d,\chi)-\delta_{tk}}{\delta^2(p,t,d,\chi)}\left(\sqrt{(sp)^2+4(1-p)}-sp\right)\right)^{\frac{1}{2}}, \tag{3.23}$$

$$E_2=\left(\left(\frac{\delta(p,t,d,\chi)\big(1+s\delta_{tk}\big)}{\delta_{tk}\big(1+s\delta(p,t,d,\chi)\big)}\right)^{1-p}-1\right)^{-\frac{1}{p}}\left(\frac{2}{k^{\frac{2-p}{2}}\chi^{\frac{2}{p}}}\right)^{\frac{1}{p}}, \tag{3.24}$$

where $s=(3t-4d)/t$.

The proof of Theorem 3.5 can be found in Section 4.6.

4 Proofs of main results

To simplify the proof of the main results, we develop in advance some elementary estimates based on technical lemmas in Section 2 for the analysis of the weighted ℓ_p-minimization with $0<p\leq 1$ and the weight vector $\mathbf{w}\in[0,1]^n$ defined in (1.5).

4.1 Some elementary estimates

For any vector $\boldsymbol{x}\in\mathbb{R}^n$, define $\boldsymbol{x}_{\max(k)}$ as $\boldsymbol{x}$ with all but the largest k entries in absolute value set to zero, and $\boldsymbol{x}_{-\max(k)}=\boldsymbol{x}-\boldsymbol{x}_{\max(k)}$. For any index set $S\subset\{1,2,\ldots,n\}$, $\boldsymbol{x}_S$ is defined to be the vector which equals to $\boldsymbol{x}$ on S, and zero elsewhere.

Combining Lemma 2.4 with Lemma 2.6, we first introduce the following estimates which will play a crucial role in establishing recovery conditions.

Lemma 4.1. *For the vectors $\hat{\boldsymbol{x}}$ and $\boldsymbol{x}$, suppose that $\|\hat{\boldsymbol{x}}\|_{p,\mathbf{w}}^p\leq\|\boldsymbol{x}\|_{p,\mathbf{w}}^p$. Let $\boldsymbol{x}_k$ be the best k-term approximation of $\boldsymbol{x}$ with $T=\operatorname{supp}(\boldsymbol{x}_k)$, and $\widetilde{T}\subseteq[n]$ be a known support estimate of $\boldsymbol{x}$. Define $\rho\geq 0$ and $0\leq\alpha\leq 1$ with $\alpha\rho\leq 1$ such that $|\widetilde{T}|=\rho k$ and $|\widetilde{T}\cap T|=\alpha\rho k$. Let $\boldsymbol{h}=\hat{\boldsymbol{x}}-\boldsymbol{x}$ and*

$$\nu^p=\omega\|\boldsymbol{h}_T\|_p^p+(1-\omega)\left\|\boldsymbol{h}_{(\widetilde{T}\cup T)\setminus(\widetilde{T}\cap T)}\right\|_p^p+2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\Big). \tag{4.1}$$

For $t>d$ and a positive integer tk, define two index sets

$$\Upsilon_1=\left\{i\in\operatorname{supp}\big(\boldsymbol{h}_{-\max(dk)}\big):|h_i|>\frac{\nu}{\big((t-d)k\big)^{\frac{1}{p}}}\right\}, \tag{4.2}$$

$$\Upsilon_2=\left\{i\in\operatorname{supp}\big(\boldsymbol{h}_{-\max(dk)}\big):|h_i|\leq\frac{\nu}{\big((t-d)k\big)^{\frac{1}{p}}}\right\}, \tag{4.3}$$

where d is defined in (2.4)*. Then*

(i) The vector $\boldsymbol{h}_{\Upsilon_2}$ can be represented as a convex combination of $((t-d)k-|\Upsilon_1|)$-sparse vectors $\boldsymbol{u}^{(i)}$ with $\operatorname{supp}(\boldsymbol{u}^{(i)})\subseteq\Upsilon_2$, i.e.

$$\boldsymbol{h}_{\Upsilon_2}=\sum_{i=1}^N\lambda_i\boldsymbol{u}^{(i)}, \tag{4.4}$$

where N is a positive integer, $\lambda_i>0$, $\sum_{i=1}^N\lambda_i=1$, and

$$\sum_{i=1}^N\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2\leq\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}}\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{2-p}}\big(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\big)^{\frac{2-2p}{2-p}}, \tag{4.5}$$

where $T_{dk}^{\boldsymbol{h}}=\operatorname{supp}(\boldsymbol{h}_{\max(dk)})$ and χ is defined in (2.5)*.*

(ii) For the vectors $\boldsymbol{h}_{-\max(dk)}$ and $\boldsymbol{h}_{\Upsilon_2}$, the following estimates hold:

$$\|\boldsymbol{h}_{-\max(dk)}\|_2^2 \le \left(\|\boldsymbol{h}_{\max(dk)}\|_2^p + \frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\big)}{(dk)^{\frac{2-p}{2}}} \right)^{\frac{2}{p}}, \tag{4.6}$$

$$\|\boldsymbol{h}_{\Upsilon_2}\|_2^2 \le \frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}} \left(\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}}\|_2^p + \frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi} \right)^{\frac{2}{p}}. \tag{4.7}$$

Proof. (i) By $\|\hat{\boldsymbol{x}}\|_{p,\mathbf{w}}^p \le \|\boldsymbol{x}\|_{p,\mathbf{w}}^p, \boldsymbol{h} = \hat{\boldsymbol{x}} - \boldsymbol{x}$ and (2.1) in Lemma 2.4 with $\Gamma = T$, one has

$$\begin{aligned} \|\boldsymbol{h}_{T^c}\|_p^p \le& \,\omega\|\boldsymbol{h}_T\|_p^p + (1-\omega)\big\|\boldsymbol{h}_{(\widetilde{T}\cup T)\setminus(\widetilde{T}\cap T)}\big\|_p^p \\ &+ 2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\Big) = \nu^p \end{aligned} \tag{4.8}$$

for $0 \le \omega \le 1$, where the equality is due to the definition of ν^p in (4.1).

For the k-sparse signal $\boldsymbol{x}_k$ and $T = \mathrm{supp}(\boldsymbol{x}_k)$, we have $|T| \le k$ and $d \ge 1$ and $dk \in \mathbb{N}^+$ from the definition of d in (2.4). Then it follows that

$$\|\boldsymbol{h}_{-\max(dk)}\|_p^p \le \|\boldsymbol{h}_{T^c}\|_p^p \le \nu^p, \tag{4.9}$$

by the inequality (4.8).

First, we will show a convex combination of sparse vectors for $\boldsymbol{h}_{\Upsilon_2} \in \mathbb{R}^n$. By $T^{\boldsymbol{h}}_{dk} = \mathrm{supp}(\boldsymbol{h}_{\max(dk)})$, the definitions of Υ_1 in (4.2) and Υ_2 in (4.3), it is obvious that

$$(T^{\boldsymbol{h}}_{dk})^c = \Upsilon_1 \cup \Upsilon_2, \quad \Upsilon_1 \cap \Upsilon_2 = \emptyset. \tag{4.10}$$

For the vector $\boldsymbol{h}_{\Upsilon_1}$

$$\|\boldsymbol{h}_{\Upsilon_1}\|_p^p = \sum_{i\in\Upsilon_1} |h_i|^p \ge |\Upsilon_1| \frac{\nu^p}{(t-d)k}. \tag{4.11}$$

By (4.9) and (4.10), one has

$$\|\boldsymbol{h}_{\Upsilon_1}\|_p^p \le \|\boldsymbol{h}_{\Upsilon_1}\|_p^p + \|\boldsymbol{h}_{\Upsilon_2}\|_p^p = \|\boldsymbol{h}_{-\max(dk)}\|_p^p \le \nu^p.$$

Combining (4.11) with the above inequality, we deduce that for $\nu > 0$,

$$|\Upsilon_1| \le (t-d)k.$$

For the vector $\boldsymbol{h}_{\Upsilon_2}$, it is easy to see that

$$\begin{aligned} &\|\boldsymbol{h}_{\Upsilon_2}\|_\infty \overset{(a)}{\le} \frac{\nu}{\big((t-d)k\big)^{\frac{1}{p}}}, \\ &\|\boldsymbol{h}_{\Upsilon_2}\|_p^p \overset{(b)}{=} \|\boldsymbol{h}_{-\max(dk)}\|_p^p - \|\boldsymbol{h}_{\Upsilon_1}\|_p^p \overset{(c)}{\le} \big((t-d)k - |\Upsilon_1|\big)\frac{\nu^p}{(t-d)k}, \end{aligned} \tag{4.12}$$

where $(a),(b)$ respectively follows from (4.3), (4.10), (c) is due to (4.9) and (4.11). Applying Lemma 2.6 with $L=(t-d)k-|\mathrm{Y}_1|$ and $\zeta=\nu/((t-d)k)^{1/p}$, we obtain the sparse expression (4.4)

$$\boldsymbol{h}_{\mathrm{Y}_2}=\sum_{i=1}^{N}\lambda_i\boldsymbol{u}^{(i)},$$

where $\lambda_i>0,\sum_{i=1}^N\lambda_i=1$, every $\boldsymbol{u}^{(i)}$ is $((t-d)k-|\mathrm{Y}_1|)$-sparse and $\mathrm{supp}(\boldsymbol{u}^{(i)})\subseteq\mathrm{Y}_2$. Furthermore, by (2.2),

$$\begin{aligned}\sum_{i=1}^{N}\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2&\le\min\left\{\frac{n}{L}\|\boldsymbol{h}_{\mathrm{Y}_2}\|_2^2,\frac{\nu^p}{k(t-d)}\|\boldsymbol{h}_{\mathrm{Y}_2}\|_{2-p}^{2-p}\right\}\le\frac{\nu^p}{k(t-d)}\|\boldsymbol{h}_{\mathrm{Y}_2}\|_{2-p}^{2-p}\\&\le\frac{\nu^p}{k(t-d)}\left(\|\boldsymbol{h}_{\mathrm{Y}_2}\|_2^2\right)^{\frac{2-2p}{2-p}}\left(\|\boldsymbol{h}_{\mathrm{Y}_2}\|_p^p\right)^{\frac{p}{2-p}}\\&\le\frac{\nu^p}{k(t-d)}\left(\|\boldsymbol{h}_{\mathrm{Y}_2}\|_2^2\right)^{\frac{2-2p}{2-p}}\left(\left((t-d)k-|\mathrm{Y}_1|\right)\frac{\nu^p}{k(t-d)}\right)^{\frac{p}{2-p}}\\&\le\frac{\nu^{\frac{2p}{2-p}}}{k(t-d)}\left(\|\boldsymbol{h}_{\mathrm{Y}_2}\|_2^2\right)^{\frac{2-2p}{2-p}},\end{aligned}\tag{4.13}$$

where the third inequality is from Lemma 2.1(II), and the fourth inequality follows from (4.12).

For ν^p in (4.1), we deduce that

$$\begin{aligned}\nu^p&=\omega\|\boldsymbol{h}_T\|_p^p+(1-\omega)\left\|\boldsymbol{h}_{(T\cup\widetilde{T})\setminus(T\cap\widetilde{T})}\right\|_p^p+2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\Big)\\&\le\omega|T|^{\frac{2-p}{2}}\|\boldsymbol{h}_T\|_2^p+(1-\omega)|(T\cup\widetilde{T})\setminus(T\cap\widetilde{T})|^{\frac{2-p}{2}}\left\|\boldsymbol{h}_{(T\cup\widetilde{T})\setminus(T\cap\widetilde{T})}\right\|_2^p\\&\quad+2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\Big)\\&\le k^{\frac{2-p}{2}}\Big(\omega+(1-\omega)(1+\rho-2\alpha\rho)^{\frac{2-p}{2}}\Big)\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}}\right\|_2^p+2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\Big)\\&=k^{\frac{2-p}{2}}\chi\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}}\right\|_2^p+2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\Big),\end{aligned}\tag{4.14}$$

where the first inequality is from $0<p\le1$ and Lemma 2.1(I) and the second inequality follows from $|T|\le k$ and $|(T\cup\widetilde{T})\setminus(T\cap\widetilde{T})|\le(1+\rho-2\alpha\rho)k\le dk$ and $T^{\boldsymbol{h}}_{dk}=\mathrm{supp}(\boldsymbol{h}_{\max(dk)})$, and the last equality is due to the definition of χ in (2.5).

Then, substituting (4.14) into (4.13), we obtain

$$\sum_i\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2\le\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}}\right\|_2^p+\frac{2(\omega\left\|\boldsymbol{x}_{T^c}\right\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{2-p}}\left(\|\boldsymbol{h}_{\mathrm{Y}_2}\|_2^2\right)^{\frac{2-2p}{2-p}},$$

which is (4.5).

(ii) For the vector $\boldsymbol{h}_{-\max(dk)}$, from (4.9) and ν^p in (4.1) it follows that

$$\begin{aligned}\|\boldsymbol{h}_{-\max(dk)}\|_p^p &\le \omega\|\boldsymbol{h}_T\|_p^p + (1-\omega)\big\|\boldsymbol{h}_{(T\cup\widetilde{T})\setminus(T\cap\widetilde{T})}\big\|_p^p + 2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\Big) \\ &\le \|\boldsymbol{h}_{\max(dk)}\|_p^p + 2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\Big),\end{aligned}$$

where we use the facts that $|T|\le k$ and $|(T\cup\widetilde{T})\setminus(T\cap\widetilde{T})|\le(1+\rho-2\alpha\rho)k\le dk$ in the second inequality. By the above inequality and Lemma 2.3, we obtain that

$$\begin{aligned}\|\boldsymbol{h}_{-\max(dk)}\|_2^2 &\le dk\left(\frac{\|\boldsymbol{h}_{\max(dk)}\|_2^p}{(dk)^{\frac{p}{2}}} + \frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\big)}{dk}\right)^{\frac{2}{p}} \\ &= \left(\|\boldsymbol{h}_{\max(dk)}\|_2^p + \frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\big)}{(dk)^{\frac{2-p}{2}}}\right)^{\frac{2}{p}},\end{aligned}$$

which is (4.6). For the vector $\boldsymbol{h}_{\Upsilon_2}$, there is

$$\begin{aligned}\|\boldsymbol{h}_{\Upsilon_2}\|_2^2 &\le \|\boldsymbol{h}_{\Upsilon_2}\|_\infty^{2-p}\|\boldsymbol{h}_{\Upsilon_2}\|_p^p \le \|\boldsymbol{h}_{\Upsilon_2}\|_\infty^{2-p}\|\boldsymbol{h}_{-\max(dk)}\|_p^p \le \left(\frac{\nu^p}{(t-d)k}\right)^{\frac{2-p}{p}}\nu^p \\ &\le \frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}}\left(\|\boldsymbol{h}_{T^h_{dk}}\|_2^p + \frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}},\end{aligned}$$

where the second and third inequalities are respectively due to $\|\boldsymbol{h}_{\Upsilon_2}\|_p^p\le\|\boldsymbol{h}_{-\max(dk)}\|_p^p\le\nu^p$ and $\|\boldsymbol{h}_{\Upsilon_2}\|_\infty\le\nu/((t-d)k)^{1/p}$ in (4.12), and the last inequality is due to (4.14). $\square$

The following two lemmas contains useful facts on RIP, whose prototype has been used in [21] for the analysis of the ℓ_p-minimization (1.2). The first one is based on Lemma 2.5 and its proof is omitted since it is very simple.

Lemma 4.2. *Suppose the sense matrix $\boldsymbol{A}\in\mathbb{R}^{m\times n}, t\ge 2d, k$ and tk are positive integers. Then*

$$\delta_{tk}\le\delta_{2(t-d)k}\le\frac{3t-4d}{t}\delta_{tk}. \tag{4.15}$$

Lemma 4.3. *Suppose $\delta_{tk}<B(t)$ can guarantee the exact recovery of k-sparse signals via some minimization method when tk is a positive integer. If the RIC bound $B(t)$ is monotonically nondecreasing for $t>0$, then $\delta_{tk}<B(t)$ can also guarantee the exact recovery of k-sparse signals via the same minimization method when tk is not an integer.*

Proof. For completeness, we give the proof although it seems routine as in [4, 21]. When tk is not an integer, denote $t'=\lceil tk\rceil/k$. Then $t'k$ is an integer and $t\le t'$. Based on the definition of RIP for non-integer tk, one has $\delta_{tk}=\delta_{\lceil tk\rceil}=\delta_{t'k}$. We deduce that the condition $\delta_{tk}<B(t)$ implies $\delta_{t'k}<B(t')$ since $B(t)$ is monotonically nondecreasing with $t>0$ and then $\delta_{t'k}=\delta_{tk}<B(t)\le B(t')$. Therefore, the desired result holds since $t'k$ is an integer. $\square$

4.2 Proof of Theorem 3.1

Proof. Suppose that $\hat{\boldsymbol{x}}$ is a solution of the weighted ℓ_p-minimization (1.3) with $\mathfrak{B}=\{\boldsymbol{0}\}$, then $\|\hat{\boldsymbol{x}}\|_{p,\mathbf{w}}^p \le \|\boldsymbol{x}\|_{p,\mathbf{w}}^p$ and $\boldsymbol{A}\hat{\boldsymbol{x}}=\boldsymbol{y}$. For the k-sparse signal $\boldsymbol{x}$ and $T=\mathrm{supp}(\boldsymbol{x})$, we have $|T|\le k$ and $\boldsymbol{x}_{T^c}=\boldsymbol{0}$.

Let $\boldsymbol{h}=\hat{\boldsymbol{x}}-\boldsymbol{x}$. To prove that the weighted ℓ_p-minimization (1.3) exactly recovers the k-sparse signal $\boldsymbol{x}$ reduces to proving $\boldsymbol{h}=\boldsymbol{0}$. Suppose that $\boldsymbol{h}\neq\boldsymbol{0}$ and tk is an integer, we next show there is a contradiction under the condition (3.1). It deduces that the weighted ℓ_p-minimization (1.3) exactly recovers the k-sparse signal $\boldsymbol{x}$.

If $\boldsymbol{h}_{T^c}=\boldsymbol{0}$, then $\boldsymbol{h}$ is a k-sparse vector. Since the sensing matrix $\boldsymbol{A}$ satisfies the RIP of order tk with $t>d\ge 1$ and (3.1) implies $\delta_{tk}<1$, we deduce that $\boldsymbol{h}=\boldsymbol{0}$. It is in contradiction with the assumption $\boldsymbol{h}\neq\boldsymbol{0}$. Therefore, in the following proof, we assume that $\boldsymbol{h}_{T^c}\neq\boldsymbol{0}$. In this case, the proof is divided into three main steps as follows.

Step 1: Using Lemma 4.1, we present a convex combination of some sparse vectors for $\boldsymbol{h}_{\Upsilon_2}$, where Υ_2 is defined in (4.3). By $\|\hat{\boldsymbol{x}}\|_{p,\mathbf{w}}^p\le\|\boldsymbol{x}\|_{p,\mathbf{w}}^p$ and Lemma 4.1, one has that

$$\boldsymbol{h}_{\Upsilon_2}=\sum_{i=1}^N \lambda_i \boldsymbol{u}^{(i)}, \tag{4.16}$$

where $\lambda_i>0,\sum_{i=1}^N\lambda_i=1,\boldsymbol{u}^{(i)}$ is $((t-d)k-|\Upsilon_1|)$-sparse and $\mathrm{supp}(\boldsymbol{u}^{(i)})\subseteq\Upsilon_2$. And

$$\begin{aligned}\sum_{i=1}^N\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2 &\le \frac{\chi^{\frac{2}{2-p}}}{t-d}\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}}\|_2^p\right)^{\frac{2}{2-p}}\left(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\right)^{\frac{2-2p}{2-p}}\\ &\le \frac{\chi^{\frac{2}{2-p}}}{t-d}\left(\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\left\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\right\|_2^2}\right)^{\frac{2-2p}{2-p}}\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^2,\end{aligned}\tag{4.17}$$

which follows from (4.5) and $\boldsymbol{x}_{T^c}=\boldsymbol{0}$. In addition, the inequality (4.7) with

$$\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p=0$$

holds. Then

$$\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\le\frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}}\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}}\|_2^2\le\frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}}\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^2. \tag{4.18}$$

Step 2: Show an inequality on $\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^2$ to estimate its upper bound based on the following important identity firstly presented in [4]:

$$\begin{aligned}&\sum_{i=1}^N\lambda_i\left\|\boldsymbol{A}\left(\sum_{j=1}^N\lambda_j\boldsymbol{v}^{(j)}-c\boldsymbol{v}^{(i)}\right)\right\|_2^2+\frac{1-2c}{2}\sum_{i=1}^N\sum_{j=1}^N\lambda_i\lambda_j\left\|\boldsymbol{A}(\boldsymbol{v}^{(i)}-\boldsymbol{v}^{(j)})\right\|_2^2\\ &\quad-(1-c)^2\sum_{i=1}^N\lambda_i\left\|\boldsymbol{A}\boldsymbol{v}^{(i)}\right\|_2^2=0,\end{aligned}\tag{4.19}$$

where $c \le 1/2$ and $\boldsymbol{v}^{(i)} = \boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup \Upsilon_1} + \mu \boldsymbol{u}^{(i)}$ for any $\mu \in \mathbb{R}$ is tk-sparse. In fact, $\boldsymbol{v}^{(i)} - \boldsymbol{v}^{(j)} = \mu(\boldsymbol{u}^{(i)} - \boldsymbol{u}^{(j)})$ and

$$\begin{aligned}\sum_{j=1}^{N}\lambda_j \boldsymbol{v}^{(j)} - c\boldsymbol{v}^{(i)} &= (1-c)\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1} + \mu\sum_{j=1}^{N}\lambda_j\boldsymbol{u}^{(j)} - c\mu\boldsymbol{u}^{(i)} \\ &= (1-c)\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1} + \mu\boldsymbol{h}_{\Upsilon_2} - c\mu\boldsymbol{u}^{(i)} \\ &= (1-c-\mu)\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1} + \mu\boldsymbol{h} - c\mu\boldsymbol{u}^{(i)}. \end{aligned} \tag{4.20}$$

Substituting (4.20) into the first term of the identity (4.19) and using the fact that $\boldsymbol{A}\boldsymbol{h} = \boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{y} = \boldsymbol{0}$, one has

$$\begin{aligned}\sum_{i=1}^{N}\lambda_i\left\|\boldsymbol{A}\left(\sum_{j=1}^{N}\lambda_j\boldsymbol{v}^{(j)} - c\boldsymbol{v}^{i}\right)\right\|_2^2 &= \sum_{i=1}^{N}\lambda_i\big\|\boldsymbol{A}\big((1-c-\mu)\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1} - c\mu\boldsymbol{u}^{(i)}\big)\big\|_2^2 \\ &\le (1+\delta_{tk})\sum_{i=1}^{N}\lambda_i\big\|(1-c-\mu)\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1} - c\mu\boldsymbol{u}^{(i)}\big\|_2^2 \\ &= (1+\delta_{tk})\left((1-c-\mu)^2\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2 + c^2\mu^2\sum_{i=1}^{N}\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2\right), \end{aligned} \tag{4.21}$$

where the inequality follows from $(1-c-\mu)\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1} - c\mu\boldsymbol{u}^{(i)}$ is tk-sparse and $\boldsymbol{A}$ satisfies the RIP of order tk, and the last equality is due to $\mathrm{supp}(\boldsymbol{u}^{(i)}) \subseteq \Upsilon_2$ and (4.10) implying

$$\langle \boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}, \boldsymbol{u}^{(i)}\rangle = 0. \tag{4.22}$$

For the second term of the identity (4.19), from the fact

$$\boldsymbol{v}^{(i)} - \boldsymbol{v}^{(j)} = \mu(\boldsymbol{u}^{(i)} - \boldsymbol{u}^{(j)}),$$

it follows that

$$\begin{aligned}&\frac{1-2c}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda_j\big\|\boldsymbol{A}(\boldsymbol{v}^{(i)} - \boldsymbol{v}^{(j)})\big\|_2^2 \\ &= \frac{1-2c}{2}\mu^2\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda_j\big\|\boldsymbol{A}(\boldsymbol{u}^{(i)} - \boldsymbol{u}^{(j)})\big\|_2^2 \\ &\le (1+\delta_{2(t-d)k})\mu^2\frac{1-2c}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda_j\|\boldsymbol{u}^{(i)} - \boldsymbol{u}^{(j)}\|_2^2 \\ &= (1+\delta_{2(t-d)k})\mu^2(1-2c)\left(\sum_{i=1}^{N}\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2 - \left\|\sum_{i=1}^{N}\lambda_i\boldsymbol{u}^{(i)}\right\|_2^2\right) \\ &= (1+\delta_{2(t-d)k})\mu^2(1-2c)\left(\sum_{i=1}^{N}\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2 - \|\boldsymbol{h}_{\Upsilon_2}\|_2^2\right), \end{aligned} \tag{4.23}$$

where the inequality is from that $\boldsymbol{u}_i$ is $(k(t-d)-|\Upsilon_1|)$-sparse and $t>d$, the last equality is due to $\boldsymbol{h}_{\Upsilon_2}=\sum_{i=1}^N \lambda_i \boldsymbol{u}^{(i)}$.

Furthermore, it follows from $\boldsymbol{v}^{(i)}$ is tk-sparse for the third term of the identity (4.19) that

$$\begin{aligned}(1-c)^2\sum_{i=1}^N \lambda_i \|\boldsymbol{A}\boldsymbol{v}^{(i)}\|_2^2 &\geq (1-\delta_{tk})(1-c)^2\sum_{i=1}^N \lambda_i \|\boldsymbol{v}^{(i)}\|_2^2 \\ &= (1-\delta_{tk})(1-c)^2\left(\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2+\mu^2\sum_{i=1}^N\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2\right), \end{aligned} \tag{4.24}$$

where the equality is from the definition of $\boldsymbol{v}^{(i)}$, i.e.

$$\boldsymbol{v}^{(i)}=\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}+\mu\boldsymbol{u}^{(i)}$$

and (4.22). Substituting the inequalities (4.21), (4.23) and (4.24) into the identity (4.19) with any $\mu\in\mathbb{R}$, one has

$$\begin{aligned} 0 \leq & \big((1+\delta_{tk})(1-c-\mu)^2-(1-\delta_{tk})(1-c)^2\big)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2 \\ & +\big((1+\delta_{tk})c^2\mu^2+(1+\delta_{2(t-d)k})(1-2c)\mu^2-(1-\delta_{tk})(1-c)^2\mu^2\big)\sum_{i=1}^N\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2 \\ & -(1+\delta_{2(t-d)k})(1-2c)\mu^2\|\boldsymbol{h}_{\Upsilon_2}\|_2^2 \\ = & \Bigg((1+\delta_{tk})\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2+\big((2c^2-2c+1)\delta_{tk}+(1-2c)\delta_{2(t-d)k}\big)\sum_{i=1}^N\lambda_i\|\boldsymbol{u}_i\|_2^2 \\ & \quad -(1+\delta_{2(t-d)k})(1-2c)\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\Bigg)\mu^2 \\ & -2(1-c)(1+\delta_{tk})\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2\mu+2\delta_{tk}(1-c)^2\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2 \\ \leq & \left((1+\delta_{tk})+\big((2c^2-2c+1)\delta_{tk}+(1-2c)\delta_{2(t-d)k}\big)\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2}\right)^{\frac{2-2p}{2-p}}\right. \\ & \quad \left. -(1+\delta_{2(t-d)k})(1-2c)\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2}\right)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2\mu^2 \\ & -2(1-c)(1+\delta_{tk})\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2\mu+2\delta_{tk}(1-c)^2\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2, \end{aligned} \tag{4.25}$$

where the last inequality is from (4.17).

Step 3: We show that there is a contradiction for (4.25) under the condition (3.1). Set a second-order function $f(\mu)$ for any $\mu\in\mathbb{R}$,

$$\begin{aligned}f(\mu)=&\left((1+\delta_{tk})+\big((2c^2-2c+1)\delta_{tk}+(1-2c)\delta_{2(t-d)k}\big)\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2}\right)^{\frac{2-2p}{2-p}}\right.\\&\left.-(1+\delta_{2(t-d)k})(1-2c)\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2}\right)\mu^2\\&-2(1-c)(1+\delta_{tk})\mu+2\delta_{tk}(1-c)^2,\end{aligned}\tag{4.26}$$

where

$$c=\frac{1}{2}-\frac{\sqrt{p^2\delta_{2(t-d)k}^2+4(1-p)\delta_{tk}^2}-p\delta_{2(t-d)k}}{4\delta_{tk}},\tag{4.27}$$

and $\|\boldsymbol{h}_{\Upsilon_2}\|_2^2/\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2$ is a parameter. By (4.18), then

$$0\leq\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2}\leq(t-d)^{-\frac{2-p}{p}}\chi^{\frac{2}{p}}.$$

And by the assumption that $\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2\neq0$, the above inequality (4.25) is equivalent to

$$f(\mu)\geq0,\quad \mu\in\mathbb{R}.\tag{4.28}$$

The discriminant of the function (4.26) is

$$\begin{aligned}\Delta=4(1-c)^2(1+\delta_{tk})^2-8&\left((1+\delta_{tk})+\big((2c^2-2c+1)\delta_{tk}+(1-2c)\delta_{2(t-d)k}\big)\right.\\&\times\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2}\right)^{\frac{2-2p}{2-p}}\\&\left.-(1+\delta_{2(t-d)k})(1-2c)\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2}\right)\delta_{tk}(1-c)^2\end{aligned}\tag{4.29}$$

with the parameters $\|\boldsymbol{h}_{\Upsilon_2}\|_2^2/\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2\in[0,(t-d)^{-(2-p)/p}\chi^{2/p}]$ and c in (4.27). By some simple analysis, Δ gets minimum value denoting $\Delta_{\min}$ at

$$\frac{\|\boldsymbol{h}_{\Upsilon_2}\|_2^2}{\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\|_2^2}=\left(\frac{\sqrt{p^2\delta_{2(t-d)k}^2+4(1-p)\delta_{tk}^2}+(2-p)\delta_{2(t-d)k}}{2(1+\delta_{2(t-d)k})(t-d)}\chi^{\frac{2}{2-p}}\right)^{\frac{2-p}{p}}\leq\frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}}.$$

Furthermore,

$$\Delta_{\min}=4(1-c)^2\left(1-\delta_{tk}^2-p\chi^{\frac{2}{p}}\left(2(t-d)\right)^{-\frac{2-p}{p}}\right.$$
$$\times\left(\frac{\sqrt{p^2\delta_{2(t-d)k}^2+4(1-p)\delta_{tk}^2}+(2-p)\delta_{2(t-d)k}}{1+\delta_{2(t-d)k}}\right)^{\frac{2-2p}{p}}$$
$$\left.\times\left(2\delta_{tk}^2-p\delta_{2(t-d)k}^2+\delta_{2(t-d)k}\sqrt{p^2\delta_{2(t-d)k}^2}\right)\right)>0,$$

where the last inequality is from the condition (3.1). Therefore,

$$\Delta>0$$

for any $\|\boldsymbol{h}_{\Upsilon_2}\|_2^2/\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^2\in[0,(t-d)^{-(2-p)/p}\chi^{2/p}]$. Then there must be $\mu_0\in\mathbb{R}$ such that $f(\mu_0)<0$ which contradicts with (4.28). The proof is complete. □

4.3 The proof of Theorem 3.2

Proof. By [21, Remark 5], it is clear that the function

$$g(z)=\left(\sqrt{p^2z^2+4(1-p)\delta_{tk}^2}+\frac{(2-p)z}{1+z}\right)^{\frac{2-2p}{p}}\left(2\delta_{tk}^2-pz^2+z\sqrt{p^2z^2+4(1-p)\delta_{tk}^2}\right)$$

is monotonically nondecreasing with $z\geq 0$. For $t\geq 2d$, Lemma 4.2 says that $\delta_{2(t-d)k}\leq s\delta_{tk}$, then

$$g(\delta_{2(t-d)k})\leq g(s\delta_{tk}),$$

where $s=(3t-4d)/t$, and

$$1-\delta_{tk}^2-p\left(\frac{\chi^{\frac{2}{2-p}}}{2(t-d)}\right)^{\frac{2-p}{p}}g(\delta_{2(t-d)k})\geq 1-\delta_{tk}^2-p\left(\frac{\chi^{\frac{2}{2-p}}}{2(t-d)}\right)^{\frac{2-p}{p}}g(s\delta_{tk}).$$

When tk is an integer, we only need to prove that

$$1-\delta_{tk}^2-p\left(\frac{\chi^{2/(2-p)}}{2(t-d)}\right)^{\frac{2-p}{p}}g(s\delta_{tk})>0$$

by Theorem 3.1. We next prove that it is true under the condition (3.10). It reduces to proving that the continuous function

$$h(z)=1-z^2\left(1+p\left(\frac{\chi^{\frac{2}{2-p}}}{2(t-d)}\right)^{\frac{2-p}{p}}\left(\frac{\sqrt{s^2p^2+4(1-p)}+s(2-p)}{1+sz}z\right)^{\frac{2-2p}{p}}\right.$$

$$\times\left(2-ps^2+s\sqrt{s^2p^2+4(1-p)}\right)\Bigg) \tag{4.30}$$

for $z\in[0,1]$ satisfies $h(z)>0$ when $z<\delta(p,t,d,\chi)$. From (3.11), it follows that $h(\delta(p,d,t,\chi))=0$. Next, we prove that $\delta(p,d,t,\chi)$ is the only solution of $h(z)=0$. Since z and $z/(1+sz)$ are both monotonically increasing with $z,h(z)$ is monotonically decreasing with z. Furthermore, one has that $h(0)=1$ and

$$\begin{aligned}h(1)=&-p\left(\frac{\chi^{\frac{2}{2-p}}}{2(t-d)}\right)^{\frac{2-p}{p}}\left(\frac{\sqrt{s^2p^2+4(1-p)}+s(2-p)}{1+s}\right)^{\frac{2-2p}{p}}\\&\times\left(2-ps^2+s\sqrt{s^2p^2+4(1-p)}\right)<0.\end{aligned}$$

Then, there is a unique positive solution of $h(z)=0$ with $z\in(0,1)$. Then, by the fact $h(\delta(p,d,t,\chi))=0,\delta(p,d,t,\chi)$ is the only solution of $h(z)=0$. And $h(z)>h(\delta(p,d,t,\chi))=0$ when $z<\delta(p,d,t,\chi)$. In addition, based on $\delta(p,d,t,\chi)<1$, it is clear that

$$\frac{\delta(p,d,t,\chi)}{1+s\delta(p,d,t,\chi)}<\frac{t}{4(t-d)}.$$

Then, by (3.11), we have that

$$\begin{aligned}\delta(p,d,t,\chi)\geq&\Bigg[1+p\left(\frac{\chi^{\frac{2}{2-p}}}{2(t-d)}\right)^{\frac{2-p}{p}}\left(\frac{\sqrt{s^2p^2+4(1-p)}+s(2-p)}{4(t-d)}t\right)^{\frac{2-2p}{p}}\\&\times\left(2-ps^2+s\sqrt{s^2p^2+4(1-p)}\right)\Bigg]^{-\frac{1}{2}}.\end{aligned}$$

When tk is not an integer, we have the following proof. Since the partial derivative with respect to t in both sides of (3.11)

$$\begin{aligned}\frac{\partial z}{\partial t}=&\frac{1-z^2}{2z(t-d)\left(2-ps^2+s\sqrt{s^2p^2+4(1-p)}\right)}\left(1+\frac{1-z^2}{z^2}\left(\frac{1+psz}{p(1+sz)}\right)\right)^{-1}\\&\times\Bigg(\frac{2-ps^2+s\sqrt{s^2p^2+4(1-p)}}{p}(2-p)-\frac{8d(1-p)(t-d)}{t^2p(1+sz)}\\&\quad\times\left(\frac{2-ps^2+s\sqrt{s^2p^2+4(1-p)}}{s(2-p)+\sqrt{s^2p^2+4(1-p)}}\frac{(2-p)\sqrt{s^2p^2+4(1-p)}+sp^2-4(1-p)z}{\sqrt{s^2p^2+4(1-p)}}\right)\end{aligned}$$

$$
\begin{aligned}
&\quad -\frac{\left(\sqrt{s^2p^2+4(1-p)}-2ps\right)^2}{\sqrt{s^2p^2+4(1-p)}}\cdot\frac{4d(t-d)}{t^2}\Bigg)\\
&\geq \frac{1-z^2}{2z(t-d)\left(2-ps^2+s\sqrt{s^2p^2+4(1-p)}\right)}\left(1+\frac{1-z^2}{z^2}\left(\frac{1+psz}{p(1+sz)}\right)\right)^{-1}\\
&\quad\times\frac{1+(1-p)^2}{\sqrt{s^2p^2+4(1-p)}+sp}\cdot\frac{2}{p}>0 \qquad (4.31)
\end{aligned}
$$

for $t\geq 2d$. Therefore, $\partial\delta(p,d,t,\chi)/\partial t$ is monotonically nondecreasing with $t\geq 2d$. And using Lemma 4.3, it is clear that $\delta_{tk}<\delta(p,t,d,\chi)$ guarantees the exact recovery of k-sparse signals via the weighted ℓ_p-minimization (1.3) with $\mathcal{B}=\{\mathbf{0}\}$. We complete the proof. $\square$

4.4 Proof of Theorem 3.3

Proof. We first assume that tk is an integer. Let $\boldsymbol{h}=\hat{\boldsymbol{x}}^{\ell_2}-\boldsymbol{x}$. Since $\hat{\boldsymbol{x}}^{\ell_2}$ is a solution of the weighted ℓ_p-minimization (1.3) with $\mathcal{B}=\mathcal{B}^{l_2}(\varepsilon)=\{\boldsymbol{z}\in\mathbb{R}^m:\|\boldsymbol{z}\|_2\leq\varepsilon\}$, then $\|\boldsymbol{A}\hat{\boldsymbol{x}}^{\ell_2}-\boldsymbol{y}\|_2\leq\varepsilon$, and $\|\hat{\boldsymbol{x}}^{\ell_2}\|_{p,\mathbf{w}}^p\leq\|\boldsymbol{x}\|_{p,\mathbf{w}}^p$. Furthermore, from $\|\boldsymbol{A}\hat{\boldsymbol{x}}^{\ell_2}-\boldsymbol{y}\|_2\leq\varepsilon$ and $\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{y}\|_2\leq\varepsilon$, it follows that

$$
\|\boldsymbol{A}\boldsymbol{h}\|_2\leq\|\boldsymbol{A}\hat{\boldsymbol{x}}^{\ell_2}-\boldsymbol{y}\|_2+\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{y}\|_2\leq 2\varepsilon. \qquad (4.32)
$$

Next, we will complete the proof by the following two steps.

Step 1: $\boldsymbol{h}_{\Upsilon_2}$ can be presented as a convex combination of some sparse signals by Lemma 4.1 with $\hat{\boldsymbol{x}}=\hat{\boldsymbol{x}}^{\ell_2}$, where Υ_2 is defined in (4.3). Applying $\|\hat{\boldsymbol{x}}^{\ell_2}\|_{p,\mathbf{w}}^p\leq\|\boldsymbol{x}\|_{p,\mathbf{w}}^p$ and Lemma 4.1 with $\hat{\boldsymbol{x}}=\hat{\boldsymbol{x}}^{\ell_2}$, one has $\boldsymbol{h}_{\Upsilon_2}=\sum_{i=1}^N\lambda_i\boldsymbol{u}^{(i)}$, where $\boldsymbol{u}^{(i)}$ is $((t-d)k-|\Upsilon_1|)$-sparse vector and for all i, $\operatorname{supp}(\boldsymbol{u}^{(i)})\subseteq\Upsilon_2,\lambda_i>0$ such that $\sum_{i=1}^N\lambda_i=1$. And

$$
\sum_{i=1}^N\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2\leq\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{2-p}}\left(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\right)^{\frac{2-2p}{2-p}}. \qquad (4.33)
$$

In addition, (4.6) and (4.7) hold.

Step 2: We will develop an upper bound on $\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2$ based on the identity (4.19), where Υ_1 is defined in (4.2). For the first term of the identity (4.19), we deduce

$$
\begin{aligned}
&\sum_{i=1}^N\lambda_i\left\|\boldsymbol{A}\left(\sum_{j=1}^N\lambda_j\boldsymbol{v}^{(j)}-c\boldsymbol{v}^{(i)}\right)\right\|_2^2\\
&\overset{(a)}{=}\sum_{i=1}^N\lambda_i\left\|\boldsymbol{A}\left((1-c-\mu)\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}+\mu\boldsymbol{h}-c\mu\boldsymbol{u}^{(i)}\right)\right\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{i=1}^{N}\lambda_i\big\|A\big((1-c-\mu)h_{T^h_{dk}\cup\Upsilon_1}-c\mu u^{(i)}\big)\big\|_2^2+\mu^2\|Ah\|_2^2\\
&\quad+2\mu\left\langle A\left((1-c-\mu)h_{T^h_{dk}\cup\Upsilon_1}-c\mu\sum_{i=1}^{N}\lambda_i u^{(i)}\right),Ah\right\rangle\\
&\overset{(b)}{=}\sum_{i=1}^{N}\lambda_i\big\|A\big((1-c-\mu)h_{T^h_{dk}\cup\Upsilon_1}-c\mu u^{(i)}\big)\big\|_2^2\\
&\quad+(1-2c)\mu^2\|Ah\|_2^2+2(1-c)\mu(1-\mu)\big\langle Ah_{T^h_{dk}\cup\Upsilon_1},Ah\big\rangle\\
&\overset{(c)}{\le}(1+\delta_{tk})\sum_{i=1}^{N}\lambda_i\big\|(1-c-\mu)h_{T^h_{dk}\cup\Upsilon_1}-c\mu u^{(i)}\big\|_2^2\\
&\quad+(1-2c)\mu^2\|Ah\|_2^2+2(1-c)\mu(1-\mu)\big\|Ah_{T^h_{dk}\cup\Upsilon_1}\big\|_2\|Ah\|_2\\
&\overset{(d)}{\le}(1+\delta_{tk})\left((1-c-\mu)^2\big\|h_{T^h_{dk}\cup\Upsilon_1}\big\|_2^2+c^2\mu^2\sum_{i=1}^{N}\lambda_i\|u^{(i)}\|_2^2\right)\\
&\quad+4(1-2c)\mu^2\varepsilon^2+4(1-c)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|h_{T^h_{dk}\cup\Upsilon_1}\big\|_2,
\end{aligned}\tag{4.34}
$$

where (a) is due to (4.20), (b) follows from $h_{\Upsilon_2}=\sum_{i=1}^N\lambda_i u^{(i)}$ and $h=h_{T^h_{dk}\cup\Upsilon_1}+h_{\Upsilon_2}$, (c) comes from the fact that $(1-c-\mu)h_{T^h_{dk}\cup\Upsilon_1}-c\mu u^{(i)}$ is tk-sparse and Hölder inequality, and (d) is because $h_{T^h_{dk}\cup\Upsilon_1}$ is tk-sparse and $\|Ah\|_2\le 2\varepsilon$ in (4.32).

For the second term of the identity (4.19), by the monotonicity of RIP and $d<t\le d+(2-p)\chi^{2/(2-p)}/(2+p)$, the inequality (4.23) reduces to

$$
\frac{1-2c}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda_j\big\|A\big(v^{(i)}-v^{(j)}\big)\big\|_2^2\le(1+\delta_{tk})\mu^2(1-2c)\left(\sum_{i=1}^{N}\lambda_i\|u^{(i)}\|_2^2-\|h_{\Upsilon_2}\|_2^2\right).
$$

Substituting (4.24), (4.34) and the above inequality into the identity (4.19) with $c=p/2$, we derive that

$$
\begin{aligned}
0&\le(1+\delta_{tk})\left(\left(1-\frac{p}{2}-\mu\right)^2\|h_{T^h_{dk}\cup\Upsilon_1}\|_2^2+\left(\frac{p^2\mu^2}{4}+\mu^2(1-p)\right)\sum_{i=1}^{N}\lambda_i\|u^{(i)}\|_2^2-\mu^2(1-p)\|h_{\Upsilon_2}\|_2^2\right)\\
&\quad-(1-\delta_{tk})\left(1-\frac{p}{2}\right)^2\left(\big\|h_{T^h_{dk}\cup\Upsilon_1}\big\|_2^2+\mu^2\sum_{i=1}^{N}\lambda_i\|u^{(i)}\|_2^2\right)\\
&\quad+2(2-p)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|h_{T^h_{dk}\cup\Upsilon_1}\big\|_2+4(1-p)\mu^2\varepsilon^2\\
&\overset{(a)}{\le}\left((1+\delta_{tk})\left(1-\frac{p}{2}-\mu\right)^2-(1-\delta_{tk})\left(1-\frac{p}{2}\right)^2\right)\big\|h_{T^h_{dk}\cup\Upsilon_1}\big\|_2^2\\
&\quad+2(2-p)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|h_{T^h_{dk}\cup\Upsilon_1}\big\|_2+4(1-p)\mu^2\varepsilon^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(2\delta_{tk}\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(1-\frac{p}{2}\right)^2\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{2-p}}\right.\\
&\qquad\left.\times\left(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\right)^{\frac{2-2p}{2-p}}-(1+\delta_{tk})(1-p)\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\right)\mu^2, \qquad (4.35)
\end{aligned}
$$

where (a) is due to (4.33).

We now consider the last term of the above inequality. Define the function

$$
\begin{aligned}
g_1(\nu)=&2\delta_{tk}\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(1-\frac{p}{2}\right)^2\\
&\times\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{2-p}}\nu^{\frac{2-2p}{2-p}}\\
&-(1+\delta_{tk})(1-p)\nu \qquad (4.36)
\end{aligned}
$$

for

$$
\nu\in\left[0,\frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}}\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\right].
$$

By some simple calculation, we verify that $g_1(\nu)\le g_1(\nu_0)$ with

$$
\nu_0=\left(\frac{(2-p)\delta_{tk}\chi^{\frac{2}{2-p}}}{(1+\delta_{tk})(t-d)}\right)^{\frac{2-p}{p}}\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}. \qquad (4.37)
$$

In addition, by $d<t\le d+((2-p)(2+p))\chi^{\frac{2}{2-p}}$, and

$$
\delta_{tk}<\frac{1}{u-(1-p)}
$$

in (3.13), where

$$
u=\sqrt{p^2+\frac{(2-p)^2\chi^{\frac{2}{2-p}}}{t-d}},
$$

we derive that

$$
\nu_0<\frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}}\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}},
$$

and

$$
\begin{aligned}
g_1(\nu_0)=&\frac{p}{2}(1+\delta_{tk})\left(\frac{(2-p)\chi^{\frac{2}{2-p}}\delta_{tk}}{(1+\delta_{tk})(t-d)}\right)^{\frac{2-p}{p}} \\
&\times\left(\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup \Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}.
\end{aligned}
\tag{4.38}
$$

For (4.38), it follows from

$$
\delta_{tk}<\frac{1}{u-(1-p)}
$$

in (3.13) that

$$
\begin{aligned}
g_{1\max}(\nu_0)<&\frac{p}{2}(1+\delta_{tk})\left(\frac{(2-p)\chi^{\frac{2}{2-p}}}{(t-d)(u+p)}\right)^{\frac{2-p}{p}} \\
&\times\left(\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup \Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}} \\
\le&(1+\delta_{tk})\left(\frac{u+p}{2}-1\right) \\
&\times\left(\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup \Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}},
\end{aligned}
$$

where we use Lemma 2.2(III) with $\Lambda=\chi^{2/(2-p)}/(t-d)$ and

$$
z=\frac{(2-p)\chi^{\frac{2}{2-p}}/(t-d)}{u+p}<1
$$

and

$$
d<t\le d+\frac{2-p}{2+p}\chi^{\frac{2}{2-p}}
$$

in the second inequality. Furthermore, by the above inequality and (4.38), there is the fact that

$$
\frac{p}{2}\left(\frac{(2-p)\chi^{\frac{2}{2-p}}\delta_{tk}}{(1+\delta_{tk})(t-d)}\right)^{\frac{2-p}{p}}<\frac{u+p}{2}-1. \tag{4.39}
$$

Let

$$
\mu=\frac{2-p}{u+p},
$$

we derive that

$$\begin{aligned}
&(1+\delta_{tk})\left(1-\frac{p}{2}-\mu\right)^2-(1-\delta_{tk})\left(1-\frac{p}{2}\right)^2+(1+\delta_{tk})\left(\frac{u+p}{2}-1\right)\mu^2\\
=&\left(\frac{2-p}{u+p}\right)^2-\frac{(2-p)^2}{u+p}+\delta_{tk}\left(\left(1-\frac{p}{2}-\frac{2-p}{u+p}\right)^2+\left(1-\frac{p}{2}\right)^2\right)\\
&+(1+\delta_{tk})\left(\frac{u+p}{2}-1\right)\left(\frac{2-p}{u+p}\right)^2\\
=&\frac{u+p}{2}\big(-1+\delta_{tk}(u+p-1)\big)\mu^2<0,
\end{aligned}\tag{4.40}$$

where the inequality follows from

$$\delta_{tk}<\frac{1}{u-(1-p)}.$$

By (4.35) and the function $g_1(\nu)$ in (4.36), one has that

$$\begin{aligned}
&\left((1+\delta_{tk})\left(1-\frac{p}{2}-\mu\right)^2-(1-\delta_{tk})\left(1-\frac{p}{2}\right)^2+(1+\delta_{tk})\left(\frac{u+p}{2}-1\right)\mu^2\right)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2\\
&\quad+2(2-p)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2+4(1-p)\mu^2\varepsilon^2\\
&\quad-(1+\delta_{tk})\left(\frac{u+p}{2}-1\right)\mu^2\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2+g_1\big(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\big)\mu^2\ge0.
\end{aligned}$$

From (4.40), $g_1(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2)\le g_{1\max}(\nu_0)$, (4.38) and

$$\mu=\frac{2-p}{u+p},$$

the above inequality reduces to

$$\begin{aligned}
&\frac{u+p}{2}\big(-1+\delta_{tk}(u+p-1)\big)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2+2(u+2p-2)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2\\
&\quad+4(1-p)\varepsilon^2-(1+\delta_{tk})\left(\frac{u+p}{2}-1\right)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2+\frac{p}{2}(1+\delta_{tk})\left(\frac{(2-p)\chi^{\frac{2}{2-p}}\delta_{tk}}{(t-d)(1+\delta_{tk})}\right)^{\frac{2-p}{p}}\\
&\quad\times\left(\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\ge0.
\end{aligned}$$

Then, by (4.39),

$$\begin{aligned}
\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2\le&\frac{C_1}{(u+p)\big(1-\delta_{tk}(u+p-1)\big)}\varepsilon\\
&+C_2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\Big)^{\frac{1}{p}},
\end{aligned}\tag{4.41}$$

where the constants C_1 and C_2 are defined in (3.15) and (3.16), respectively. Furthermore,

$$\begin{aligned}\|\boldsymbol{h}\|_2^2 &= \|\boldsymbol{h}_{\max(dk)}\|_2^2 + \|\boldsymbol{h}_{-\max(dk)}\|_2^2 \\ &\leq \left\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \mathrm{Y}_1}\right\|_2^2 + \left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \mathrm{Y}_1}\|_2^p + \frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)}{(dk)^{\frac{2-p}{2}}}\right)^{\frac{2}{p}} \\ &\leq \left\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \mathrm{Y}_1}\right\|_2^2 + 2^{\frac{2-2p}{p}}\left(\left\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup \mathrm{Y}_1}\right\|_2 + \left(\frac{2}{(dk)^{\frac{2-p}{2}}}\right)^{\frac{1}{p}}\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)^{\frac{1}{p}}\right)^2,\end{aligned} \tag{4.42}$$

where we use (4.6) and $T_{dk}^{\boldsymbol{h}} = \mathrm{supp}(\boldsymbol{h}_{\max(dk)})$ in the first inequality, and Jensen inequality in the other inequalities. By (4.41) and (4.42), we have that

$$\begin{aligned}\|\boldsymbol{h}\|_2^2 \leq & \left(\frac{C_1}{(u+p)(1-\delta_{tk}(u+p-1))}\varepsilon + C_2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)^{\frac{1}{p}}\right)^2 \\ & + 2^{\frac{2-2p}{p}}\left(\frac{C_1}{(u+p)\left(1-\delta_{tk}(u+p-1)\right)}\varepsilon + C_2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)^{\frac{1}{p}}\right. \\ & \left. + \left(\frac{2}{(dk)^{\frac{2-p}{2}}}\right)^{\frac{1}{p}}\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)^{\frac{1}{p}}\right)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\boldsymbol{h}\|_2 \leq & \frac{\sqrt{1+2^{\frac{2-2p}{p}}}C_1}{(u+p)\left(1-\delta_{tk}(u+p-1)\right)}\varepsilon + \sqrt{C_2^2 + 2^{\frac{2-2p}{p}}\left(C_2 + \left(2(dk)^{-\frac{2-p}{2}}\right)^{\frac{1}{p}}\right)^2} \\ & \times\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)^{\frac{1}{p}}.\end{aligned}$$

When tk is not an integer, we define $t' = \lceil tk \rceil / k$. Then $t' > t, t'k$ is an integer and

$$\delta_{t'k} = \delta_{tk} < \frac{1}{u-(1-p)}.$$

We obtain the desired result by working on $\delta_{t'k}$. □

4.5 Proof of Theorem 3.4

Proof. By Lemma 2.2,

$$\frac{p}{2}z^{\frac{2}{p}} + z - \frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}} = 0$$

with

$$\frac{2-p}{2+p}\chi^{\frac{p}{2-p}}+d\le t\le 2d$$

has a unique solution in $((1-p)\chi^{2/(2-p)}/(t-d),1)$. If z_0 is the only positive solution of the equation

$$\frac{p}{2}z^{\frac{2}{p}}+z-\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}=0,$$

it is easy to see

$$z_0\in\left(\frac{1-p}{t-d}\chi^{\frac{2}{2-p}},\min\left\{1,\frac{2-p}{2(t-d)}\chi^{\frac{2}{2-p}}\right\}\right).$$

First we assume tk is an integer. When

$$d+\frac{2-p}{2+p}\chi^{\frac{2}{2-p}}<t\le 2d,$$

the inequalities (4.32)-(4.38) still hold as in the proof of Theorem 3.3 in Section 4.4. By the condition $\delta_{tk}<(t-d)z_0/((2-p)\chi^{2/(2-p)}-(t-d)z_0)$ in (3.17), we derive that

$$\frac{(2-p)\delta_{tk}\chi^{\frac{2}{2-p}}}{(1+\delta_{tk})(t-d)}<z_0. \tag{4.43}$$

Then (4.37) and (4.38) respectively change to

$$\begin{aligned}\nu_0&<z_0^{\frac{2-p}{p}}\left(\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\right\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\\&<\left(\frac{2-p}{2}\right)^{\frac{2-p}{p}}\frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}}\left(\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\right\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}},\end{aligned} \tag{4.44}$$

and

$$g_{1\max}(\nu_0)<\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\left(\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\right\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}, \tag{4.45}$$

where we used the facts that $z_0<(2-p)\chi^{2/(2-p)}/(2(t-d))$ and $0<p\le 1$ in the second inequality of (4.44).

In addition, from (4.35) and the function $g_1(\nu)$ defined in (4.36), it is clear that

$$\begin{aligned}&\left((1+\delta_{tk})\left(1-\frac{p}{2}-\mu\right)^2-(1-\delta_{tk})\left(1-\frac{p}{2}\right)^2+\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\mu^2\right)\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\right\|_2^2\\&\quad+2(2-p)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\right\|_2\\&\quad+4(1-p)\mu^2\varepsilon^2-\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\mu^2\left\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\right\|_2^2+g_1\left(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\right)\mu^2\ge 0.\end{aligned} \tag{4.46}$$

Combining the fact that $g_1(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2) \leq g_{1\max}(v_0)$ with (4.7) and (4.38), we derive that

$$\begin{aligned}
&\left[(1+\delta_{tk})\left(1-\frac{p}{2}-\mu\right)^2-(1-\delta_{tk})\left(1-\frac{p}{2}\right)^2+\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\mu^2\right]\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^2 \\
&\quad +2(2-p)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2+4(1-p)\mu^2\varepsilon^2 \\
&\quad -\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\mu^2\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^2+\frac{p}{2}(1+\delta_{tk})\left(\frac{(2-p)\chi^{\frac{2}{2-p}}\delta_{tk}}{(1+\delta_{tk})(t-d)}\right)^{\frac{2-p}{p}} \\
&\quad \times\left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^p+\frac{2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\mu^2\geq 0.
\end{aligned}$$

Let $\mu=z_0(t-d)/\chi^{2/(2-p)}$. Then

$$\begin{aligned}
&(1+\delta_{tk})\left(1-\frac{p}{2}-\mu\right)^2-(1-\delta_{tk})\left(1-\frac{p}{2}\right)^2+\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\mu^2 \\
=&\left(\mu^2-(2-p)\mu\right)+\delta_{tk}\left(\left(1-\frac{p}{2}-\mu\right)^2+\left(1-\frac{p}{2}\right)^2\right)+\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\mu^2 \\
=&\left(\mu^2-(2-p)\mu\right)+\delta_{tk}\left(\left(1-\frac{p}{2}-\mu\right)^2+\left(1-\frac{p}{2}\right)^2\right)+(1+\delta_{tk})\left(\frac{2-p}{2}\mu-\mu^2\right) \\
=&\frac{2-p}{2}\left(-\mu+\delta_{tk}(2-p-\mu)\right) \\
<&\frac{2-p}{2}\left(-\mu+\frac{(t-d)z_0}{(2-p)\chi^{\frac{2}{2-p}}-(t-d)z_0}(2-p-\mu)\right)=0,
\end{aligned}$$

where we used the fact that

$$\frac{p}{2}z_0^{\frac{2}{p}}+z_0-\frac{(2-p)\chi^{\frac{2}{2-p}}}{2(t-d)}=0$$

in the second equality, and the inequality follows from

$$\delta_{tk}<\frac{(t-d)z_0}{(2-p)\chi^{\frac{2}{2-p}}-(t-d)z_0}$$

in (3.17) and

$$z_0<\frac{(2-p)\chi^{\frac{2}{2-p}}}{2(t-d)}.$$

From the above two inequalities, together with $\mu=z_0(t-d)/\chi^{2/(2-p)}$ and (4.46), it follows that

$$\frac{2-p}{2}\left(\mu-\delta_{tk}(2-p-\mu)\right)\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2^2-2(2-p)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\|_2$$

$$
\begin{aligned}
&-4(1-p)\mu^2\varepsilon^2+\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\mu^2\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2-\frac{p}{2}(1+\delta_{tk})\left(\frac{(2-p)\chi^{\frac{2}{2-p}}\delta_{tk}}{(1+\delta_{tk})(t-d)}\right)^{\frac{2-p}{p}}\\
&\times v^2\left(\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\le 0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\frac{2-p}{2}\left(1-\frac{(2-p)\chi^{\frac{2}{2-p}}-z_0(t-d)}{z_0(t-d)}\delta_{tk}\right)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2-2(2-p)(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2\\
&-4(1-p)\mu\varepsilon^2+\frac{p}{2}(1+\delta_{tk})z_0^{\frac{2-p}{p}}\mu\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2-\frac{p}{2}(1+\delta_{tk})\left(\frac{(2-p)\chi^{\frac{2}{2-p}}\delta_{tk}}{(1+\delta_{tk})(t-d)}\right)^{\frac{2-p}{p}}\mu\\
&\times\left(\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\le 0.
\end{aligned}
$$

Then, by (4.43), one has that

$$
\begin{aligned}
\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2\le{}&\frac{D_1}{1-\big((2-p)\chi^{\frac{2}{2-p}}-z_0(t-d)\big)\delta_{tk}/\big(z_0(t-d)\big)}\varepsilon\\
&+D_2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\Big)^{\frac{1}{p}},
\end{aligned}\tag{4.47}
$$

where the constants D_1 and D_2 are defined in (3.20) and (3.21), respectively. Furthermore, combining (4.42) with (4.47) we deduce

$$
\begin{aligned}
\|\boldsymbol{h}\|_2^2\le{}&\left(\frac{D_1}{1-\Big((2-p)\chi^{\frac{2}{2-p}}-z_0(t-d)\Big)\delta_{tk}/\big(z_0(t-d)\big)}\varepsilon\right.\\
&\left.+D_2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\Big)^{\frac{1}{p}}\right)^2\\
&+2^{\frac{2-2p}{p}}\left(\frac{D_1}{1-\big((2-p)\chi^{\frac{2}{2-p}}-z_0(t-d)\big)\delta_{tk}/\big(z_0(t-d)\big)}\varepsilon\right.\\
&\quad+D_2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\big)^{\frac{1}{p}}\\
&\quad\left.+\left(\frac{2}{(dk)^{\frac{2-p}{2}}}\right)^{\frac{1}{p}}\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\Big)^{\frac{1}{p}}\right)^2.
\end{aligned}
$$

As a consequence,

$$\|\boldsymbol{h}\|_2 \leq \frac{\sqrt{1+2^{\frac{2-2p}{p}}}D_1}{1-\big((2-p)\chi^{\frac{2}{2-p}}-z_0(t-d)\big)\delta_{tk}/\big(z_0(t-d)\big)}\varepsilon$$
$$+\sqrt{D_2^2+2^{\frac{2-2p}{p}}\left(D_2+\left(2(dk)^{-\frac{2-p}{2}}\right)^{\frac{1}{p}}\right)^2}\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)^{\frac{1}{p}}.$$

When tk is not an integer, we define $t'=\lceil tk\rceil/k$ as usual. Then $t'>t, t'k$ is an integer and $z_0'<z_0$, where z_0 and z_0' respectively are the unique solution of Eq. (3.18) and

$$\frac{p}{2}z^{\frac{2}{p}}+z-\frac{2-p}{2(t'-d)}\chi^{\frac{2}{2-p}}=0.$$

Therefore,

$$\delta_{t'k}=\delta_{tk}<\frac{(t-d)z_0}{(2-p)\chi^{\frac{2}{2-p}}-(t-d)z_0}<\frac{(t'-d)z_0'}{(2-p)\chi^{\frac{2}{2-p}}-(t'-d)z_0'}.$$

We obtain the desired result by working on $\delta_{t'k}$. We complete the proof. $\square$

4.6 Proof of Theorem 3.5

Proof. Similarly, we first assume tk is an integer. When $t\geq 2d$, the inequalities (4.32)-(4.34) also hold in the proof of Theorem 3.3 in Section 4.4. And let the parameters c and μ in the identity (4.19) be

$$c=\frac{1}{2}-\frac{1}{4}\left(\sqrt{s^2p^2+4(1-p)}-sp\right), \tag{4.48}$$

$$\mu=\frac{2-sp+\sqrt{s^2p^2+4(1-p)}}{2\big(1+\delta(p,t,d,\chi)\big)}\delta(p,t,d,\chi), \tag{4.49}$$

where $s=(3t-4d)/t$, and $\delta(p,t,d,\chi)$ is in (3.10). By $t\geq 2d$ and Lemma 4.2, the inequality (4.23) reduces to

$$\frac{1-2c}{2}\sum_{i=1}^N\sum_{j=1}^N\lambda_i\lambda_j\big\|\boldsymbol{A}(\boldsymbol{v}^{(i)}-\boldsymbol{v}^{(j)})\big\|_2^2$$
$$\leq(1+s\delta_{tk})\mu^2(1-2c)\left(\sum_{i=1}^N\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2-\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\right). \tag{4.50}$$

Substituting the inequalities (4.24), (4.34) and (4.50) into the identity (4.19), we deduce

$$0\leq\big((1+\delta_{tk})(1-c-\mu)^2-(1-\delta_{tk})(1-c)^2\big)\big\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}}\cup\Upsilon_1}\big\|_2^2$$

$$
\begin{aligned}
&+\big((1+\delta_{tk})c^2\mu^2+(1+s\delta_{tk})\mu^2(1-2c)-(1-\delta_{tk})(1-c)^2\mu^2\big)\sum_{i=1}^{N}\lambda_i\|\boldsymbol{u}^{(i)}\|_2^2\\
&-(1+s\delta_{tk})(1-2c)\mu^2\|\boldsymbol{h}_{\Upsilon_2}\|_2^2+4(1-2c)\mu^2\varepsilon^2+4(1-c)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2\\
=&\Big((1+\delta_{tk})(1-c-\mu)^2-(1-\delta_{tk})(1-c)^2\Big)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2\\
&+4(1-c)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2+4(1-2c)\mu^2\varepsilon^2\\
&+\left(2c^2+(1-2c)\frac{4t-4d}{t}\right)\delta_{tk}\mu^2\sum_{i=1}^{N}\lambda_i\|\boldsymbol{u}_i\|_2^2-(1+s\delta_{tk})(1-2c)\mu^2\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\\
\le&\big((1+\delta_{tk})(1-c-\mu)^2-(1-\delta_{tk})(1-c)^2\big)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2\\
&+4(1-c)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2+4(1-2c)\mu^2\varepsilon^2\\
&+\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(2c^2+(1-2c)\frac{4t-4d}{t}\right)\delta_{tk}\mu^2\\
&\times\left(\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{2-p}}\big(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\big)^{\frac{2-2p}{2-p}}\\
&-(1+s\delta_{tk})(1-2c)\mu^2\|\boldsymbol{h}_{\Upsilon_2}\|_2^2,
\end{aligned}
\tag{4.51}
$$

where the inequality is due to (4.33) and $t\ge 2d$. Similarly, we first consider the last term of (4.51). Define a function

$$
\begin{aligned}
g_2(\nu)=&\frac{\chi^{\frac{2}{2-p}}}{t-d}\left(2c^2+(1-2c)\frac{4t-4d}{t}\right)\delta_{tk}\mu^2\\
&\times\left(\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{2-p}}\nu^{\frac{2-2p}{2-p}}-(1+s\delta_{tk})(1-2c)\mu^2\nu
\end{aligned}
$$

for

$$
\nu\in\left[0,\frac{\chi^{\frac{2}{p}}}{(t-d)^{\frac{2-p}{p}}}\left(\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\big\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\big\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\right].
$$

Then the inequality (4.51) can be written as

$$
\begin{aligned}
&\big((1+\delta_{tk})(1-c-\mu)^2-(1-\delta_{tk})(1-c)^2\big)\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2^2\\
&\quad+4(1-c)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^{\boldsymbol{h}}_{dk}\cup\Upsilon_1}\big\|_2\\
&\quad+4(1-2c)\mu^2\varepsilon^2+g_2\big(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2\big)\ge 0.
\end{aligned}
\tag{4.52}
$$

By some elementary calculation, the function $g_2(z)$ attains its supremum at

$$
\begin{aligned}
\nu_0 =& \left(\frac{(2-2p)\big(2c^2+(1-2c)(4t-4d)/t\big)\delta_{tk}}{(2-p)(t-d)(1+s\delta_{tk})(1-2c)} \right)^{\frac{2-p}{p}} \chi^{\frac{2}{p}} \\
&\times \left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}} \cup \Upsilon_1}\|_2^p + \frac{2\big(\omega \|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c \cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi} \right)^{\frac{2}{p}} \\
=& \left(\frac{\sqrt{p^2s^2+4(1-p)}+(2-p)s}{2(1+s\delta_{tk})}\delta_{tk} \right)^{\frac{2-p}{p}} (t-d)^{-\frac{2-p}{p}} \chi^{\frac{2}{p}} \\
&\times \left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}} \cup \Upsilon_1}\|_2^p + \frac{2\big(\omega \|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c \cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi} \right)^{\frac{2}{p}},
\end{aligned}
$$

where we used the definition of c in (4.48). That is

$$
\begin{aligned}
& g_1(\nu) \le g_1(\nu_0) \\
=& \left(\frac{1}{t-d}\left(2c^2+(1-2c)\frac{4t-4d}{t}\right)\delta_{tk} \left(\frac{\sqrt{p^2s^2+4(1-p)}+(2-p)s}{2(1+s\delta_{tk})}\delta_{tk} \right)^{\frac{2-2p}{p}} \right. \\
&\qquad \left. -(1+s\delta_{tk})(1-2c)\left(\frac{\sqrt{p^2s^2+4(1-p)}+(2-p)s}{2(1+s\delta_{tk})}\delta_{tk} \right)^{\frac{2-p}{p}} \right) \\
& \times \mu^2 \chi^{\frac{2}{p}} \left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}} \cup \Upsilon_1}\|_2^p + \frac{2\big(\omega \|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c \cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi} \right)^{\frac{2}{p}} \\
& \times \frac{(t-d)^{-\frac{2-p}{p}}\chi^{\frac{2}{p}} p\big(s\sqrt{p^2s^2+4(1-p)}+2-s^2p\big)\delta_{tk}}{4} \\
& \times \left(\frac{\sqrt{p^2s^2+4(1-p)}+(2-p)s}{2(1+s\delta_{tk})}\delta_{tk} \right)^{\frac{2-2p}{p}} \\
& \times \chi^{\frac{2}{p}} \left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}} \cup \Upsilon_1}\|_2^p + \frac{2\big(\omega \|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c \cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi} \right)^{\frac{2}{p}} \mu^2 \\
=& \frac{\delta_{tk}}{2}\left(\frac{1-h(\delta_{tk})}{\delta_{tk}^2} - 1 \right) \left(\|\boldsymbol{h}_{T_{dk}^{\boldsymbol{h}} \cup \Upsilon_1}\|_2^p + \frac{2\big(\omega \|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c \cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi} \right)^{\frac{2}{p}} \mu^2,
\end{aligned}
$$

where the last two equalities follow from the definition of c in (4.48), and the function $h(z)$ in (4.30), respectively. Then, applying (4.52) and $g_2(\|\boldsymbol{h}_{\Upsilon_2}\|_2^2) \le g_2(\nu_0)$, we derive that

$$\begin{aligned}
&\big((1+\delta_{tk})(1-c-\mu)^2-(1-\delta_{tk})(1-c)^2\big)\big\|\boldsymbol{h}_{T^h_{dk}\cup\Upsilon_1}\big\|_2^2\\
&\quad+4(1-c)\mu(1-\mu)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^h_{dk}\cup\Upsilon_1}\big\|_2+4(1-2c)\mu^2\varepsilon^2+\frac{\delta_{tk}}{2}\left(\frac{1-h(\delta_{tk})}{\delta_{tk}^2}-1\right)\\
&\quad\times\left(\big\|\boldsymbol{h}_{T^h_{dk}\cup\Upsilon_1}\big\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\mu^2\geq 0.
\end{aligned}\tag{4.53}$$

On the other hand, based on the parameters c in (4.48) and μ in (4.49), one has

$$\begin{aligned}
&(1+\delta_{tk})(1-c-\mu)^2-(1-\delta_{tk})(1-c)^2=\left(-\frac{1}{\delta(p,t,d,\chi)}+\left(\frac{1+\delta^2(p,t,d,\chi)}{2\delta^2(p,t,d,\chi)}\right)\delta_{tk}\right)\mu^2,\\
&4(1-c)\mu(1-\mu)=2\left(\frac{1}{\delta(p,t,d,\chi)}-\frac{1}{2}\left(\sqrt{(sp)^2+4(1-p)}-sp\right)\right)\mu^2,\\
&4(1-2c)\mu^2=2\left(\sqrt{(sp)^2+4(1-p)}-sp\right)\mu^2.
\end{aligned}\tag{4.54}$$

Furthermore, since $h(z)$ is monotonically decreasing with z, $h(\delta(p,t,d,\chi))=0$, and $\delta_{tk}<\delta(p,t,d,\chi)$, then

$$\begin{aligned}
&\left(-\frac{1}{\delta(p,t,d,\chi)}+\left(\frac{1+\delta^2(p,t,d,\chi)}{2\delta^2(p,t,d,\chi)}\right)\delta_{tk}\right)-\frac{\delta_{tk}}{2}\left(\frac{1-h(\delta_{tk})}{\delta_{tk}^2}-1\right)\\
\leq&\left(-\frac{1}{\delta(p,t,d,\chi)}+\left(\frac{1+\delta^2(p,t,d,\chi)}{2\delta^2(p,t,d,\chi)}\right)\delta_{tk}\right)+\frac{1-\delta^2(p,t,d,\chi)}{2\delta^2(p,t,d,\chi)}\delta_{tk}\\
=&-\frac{\delta(p,t,d,\chi)-\delta_{tk}}{\delta^2(p,t,d,\chi)}<0.
\end{aligned}$$

Then, using the equalities in (4.54) and the above inequality, (4.53) reduces to

$$\begin{aligned}
&-\frac{\delta(p,t,d,\chi)-\delta_{tk}}{\delta^2(p,t,d,\chi)}\big\|\boldsymbol{h}_{T^h_{dk}\cup\Upsilon_1}\big\|_2^2\\
&\quad+2\left(\frac{1}{\delta(p,t,d,\chi)}-\frac{1}{2}\left(\sqrt{(sp)^2+4(1-p)}-sp\right)\right)\sqrt{1+\delta_{tk}}\varepsilon\big\|\boldsymbol{h}_{T^h_{dk}\cup\Upsilon_1}\big\|_2\\
&\quad+2\left(\sqrt{(sp)^2+4(1-p)}-sp\right)\varepsilon^2-\frac{1-\delta^2(p,t,d,\chi)}{2\delta^2(p,t,d,\chi)}\delta_{tk}\big\|\boldsymbol{h}_{T^h_{dk}\cup\Upsilon_1}\big\|_2^2\\
&\quad+\frac{\delta_{tk}}{2}\left(\frac{1-h(\delta_{tk})}{\delta_{tk}^2}-1\right)\left(\big\|\boldsymbol{h}_{T^h_{dk}\cup\Upsilon_1}\big\|_2^p+\frac{2\big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\big)}{k^{\frac{2-p}{2}}\chi}\right)^{\frac{2}{p}}\mu^2\geq 0.
\end{aligned}$$

As a result, one has

$$\big\|\boldsymbol{h}_{T^h_{dk}\cup\Upsilon_1}\big\|_2\leq\frac{E_1\delta^2(p,t,d,\chi)}{\delta(p,t,d,\chi)-\delta_{tk}}\varepsilon+E_2\Big(\omega\|\boldsymbol{x}_{T^c}\|_p^p+(1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\Big)^{\frac{1}{p}},$$

where the constants E_1 and E_2 are defined in (3.23) and (3.24), respectively. Similarly, using (4.42) and the above inequality, we obtain

$$\begin{aligned}\|\boldsymbol{h}\|_2^2 \le & \left(\frac{E_1\delta^2(p,t,d,\chi)}{\delta(p,t,d,\chi)-\delta_{tk}}\varepsilon + E_2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)^{\frac{1}{p}}\right)^2 \\ & + 2^{\frac{2-2p}{p}}\left(\frac{E_1\delta^2(p,t,d,\chi)}{\delta(p,t,d,\chi)-\delta_{tk}}\varepsilon + E_2\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\|_p^p\right)\right. \\ & \left. + \left(2(dk)^{-\frac{2-p}{2}}\right)^{\frac{1}{p}}\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)^{\frac{1}{p}}\right)^2.\end{aligned}$$

And thus,

$$\begin{aligned}\|\boldsymbol{h}\|_2 \le & \frac{\sqrt{1+2^{\frac{2-2p}{p}}}E_1\delta^2(p,t,d,\chi)}{\delta(p,t,d,\chi)-\delta_{tk}}\varepsilon + \sqrt{E_2^2 + \left(E_2 + \left(2(dk)^{-\frac{2-p}{2}}\right)^{\frac{1}{p}}\right)^2} \\ & \times\left(\omega\|\boldsymbol{x}_{T^c}\|_p^p + (1-\omega)\left\|\boldsymbol{x}_{\widetilde{T}^c\cap T^c}\right\|_p^p\right)^{\frac{1}{p}}.\end{aligned}$$

When tk is not an integer, again we define $t' = \lceil tk \rceil / k$. And $\delta(p,d,t,\chi) \le \delta(p,d,t',\chi)$ since $\partial z/\partial t > 0$ in (4.31) for $t \ge 2d$. Therefore,

$$\delta_{t'k} = \delta_{tk} < \delta(p,d,t,\chi) \le \delta(p,d,t',\chi).$$

We obtain the desired result by working on $\delta_{t'k}$. We complete the proof. □

5 Conclusion

In this paper, we provide a uniform RIP bound for the exact recovery of sparse signals via the weighted ℓ_p-minimization with $0 < p \le 1$ in the noiseless case. In the ℓ_2 bounded noise case, we present the error bound for the stable signal recovery via the weighted ℓ_p-minimization with $0 < p \le 1$, when signals are not limited to sparse signals. The proposed sufficient conditions extend the state-of-the-art results for weighted ℓ_p-minimization in the literature to a complete regime, which fills the gap on δ_{tk} based signal recovery condition for $t > 2d$ and include the existing optimal conditions for the ℓ_p-minimization and the weighted ℓ_1-minimization as special cases.

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