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A Compact Difference Scheme for Time-Space Fractional Nonlinear Diffusion-Wave Equations with Initial Singularity

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Abstract. In this paper, we present a linearized compact difference scheme for one-dimensional time-space fractional nonlinear diffusion-wave equations with initial boundary value conditions. The initial singularity of the solution is considered, which often generates a singular source and increases the difficulty of numerically solving the equation. The Crank-Nicolson technique, combined with the midpoint formula and the second-order convolution quadrature formula, is used for the time discretization. To increase the spatial accuracy, a fourth-order compact difference approximation, which is constructed by two compact difference operators, is adopted for spatial discretization. Then, the unconditional stability and convergence of the proposed scheme are strictly established with superlinear convergence accuracy in time and fourth-order accuracy in space. Finally, numerical experiments are given to support our theoretical results.

AMS subject classifications: 65M06, 65M12

Key words: Fractional nonlinear diffusion-wave equations, finite difference method, fourth-order compact operator, stability, convergence.

1 Introduction

In this paper, the following time-space fractional nonlinear diffusion-wave equation with initial boundary value conditions will be considered

$$ {}_0^CD_t^{\alpha} u(x,t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^\beta}{\partial |x|^\beta} \right) u(x,t) + g(u) + f(x,t), \tag{1.1a} $$

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$$u(x,0)=0, \quad u_t(x,0)=0, \quad 0<x<L, \tag{1.1b}$$

$$u(0,t)=u(L,t)=0, \qquad 0<t\leq T, \tag{1.1c}$$

where $1<\alpha,\beta\leq 2$, $g(u)$ is a nonlinear function of u that fulfills the Lipschitz condition with $g(0)=0$, $f(x,t)$ is a known function, and ${}_0^CD_t^\alpha u(x,t)$ is the temporal Caputo fractional derivative of order α defined as

$${}_0^CD_t^\alpha u(x,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-s)^{1-\alpha}\frac{\partial^2 u(x,s)}{\partial s^2}ds.$$

And $\frac{\partial^\beta u(x,t)}{\partial|x|^\beta}$ is the Riesz fractional derivative of order β defined as

$$\frac{\partial^\beta u(x,t)}{\partial|x|^\beta}=-\frac{1}{2\cos(\frac{\pi\beta}{2})}\left({}_0^{RL}D_x^\beta u(x,t)+{}_x^{RL}D_L^\beta u(x,t)\right),$$

where ${}_0^{RL}D_x^\beta u(x,t)$ and ${}_x^{RL}D_L^\beta u(x,t)$ are the left and right Riemann-Liouville fractional derivatives of order β defined as

$${}_0^{RL}D_x^\beta u(x,t)=\frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_0^x(x-z)^{1-\beta}u(z,t)dz$$

and

$${}_x^{RL}D_L^\beta u(x,t)=\frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_x^L(z-x)^{1-\beta}u(z,t)dz,$$

respectively.

Remark 1.1. In the case of nonhomogeneous initial conditions, such as $u(x,0)=\varphi(x)\neq 0$ and $u_t(x,0)=\psi(x)\neq 0$. To homogenize the initial value conditions, we can use the following transformation

$$\hat{u}(x,t)=u(x,t)-\varphi(x)-t\psi(x).$$

Clearly, the nonhomogeneous boundary conditions can be similarly homogenized.

The time-space fractional diffusion-wave equation (1.1) can be considered as intermediate between parabolic diffusion equations and hyperbolic wave equations. It has been widely applied in the modeling of oxygen delivery through capillaries and anomalous relaxation in magnetic resonance imaging signal magnitude [1–3]. However, using currently available analytical methods, it is impossible to find an exact solution to Eq. (1.1) [4–6]. As a result, if Eq. (1.1) is to be used in practical modeling, effective numerical methods for solving it in the corresponding numerical simulations must be developed (see [7–11] for examples).

We mention some recent numerical methods that have been developed to solve time-space fractional partial differential equations with initial boundary value conditions [12–23]. Bhrawy and Zaky [12] proposed a fast spectral method to solve the multi-term time-space fractional diffusion-wave equation. Ding [14] presented a global Padé approximation method for time-space fractional diffusion equation. Zhao et al. [17] introduced and analyzed a Galerkin finite element scheme for time-space fractional diffusion equation. Vong et al. [18] considered high order finite difference methods for two-dimensional fractional diffusion equations with temporal Caputo and spatial Riemann-Liouville derivatives. Arshad in [19] applied the trapezoidal method and a fourth-order fractional compact difference operator to solve the time-space fractional diffusion equation. Lin et al. [20] proposed separable preconditioners for solving time-space fractional Caputo-Riesz diffusion equations with Toeplitz-like blocks coefficient matrices. Fan [21] studied the two-dimensional multi-term time-space fractional diffusion-wave equation on an irregular convex domain using the unstructured mesh finite element method.

Very recently, Dehghan et al. [24] presented a new method for solving two-dimensional weakly singular time-space fractional integro-differential equation. Abbaszadeh et al. in [25] proposed and analyzed a high-order numerical scheme for solving the two-dimensional time-space distributed order weakly singular integro-partial differential equation using finite difference and Galekrin spectral methods. Huang et al. [26] proposed and analyzed a superlinear convergence method for solving the multi-term and distribution-order fractional wave equation with initial singularity.

However, there are still few publications on numerical methods for time-space fractional nonlinear partial differential equations with initial singularity. This motivates us to propose an efficient numerical method for solving the time-space fractional nonlinear diffusion-wave equation with initial boundary value conditions that takes regularity under consideration. In this paper, we consider the analytical solution to Problem (1.1) with the following time regularity assumption:

$$\left|\frac{\partial^i u(x,t)}{\partial t^i}\right| \le Ct^{\sigma-i}, \quad i=0,1,2, \tag{1.2}$$

where $1<\sigma<\alpha$ is a regularity parameter. Herein, we construct high-order accurate linearized compact difference schemes for time-space fractional nonlinear diffusion-wave equation with initial boundary value conditions. Specifically, the considered problem is converted into their equivalent partial integro-differential equations. Then, using the Crank-Nicolson technique in combination with the second-order convolution quadrature formula and the midpoint formula in time, as well as the classical central difference formula and the fourth-order compact operators in space, we will construct a linearized compact finite difference scheme. Next, the linearized compact finite difference scheme is proved to be unconditional stable and convergence.

The remainder of this paper is structured as follows. Section 2 provides and discusses various preparatory and relevant lemmas. The linearized compact finite difference scheme is constructed in Section 3. In Section 4, the stability and convergence of the linearized compact finite difference scheme are proved. Numerical experiments are provided to verify the theoretical results in Section 5. Section 6 concludes this paper with a brief conclusion.

2 Preliminaries

In this section, we introduce certain fundamental notations and key lemmas that will be utilized throughout the remainder of this paper. Assume that both M,N are positive integers. Let $\tau=T/N$ and $t_n=n\tau$ $(n=0,1,\cdots,N)$. Let $h=L/M$ and $x_i=ih$ $(i=0,1,\cdots,M)$. Then, the spatial central difference operator and compact difference operators are defined as

$$
\begin{aligned}
&\delta_x^2 u_i^n=\frac{1}{h^2}\left(u_{i-1}^n-2u_i^n+u_{i+1}^n\right),\\
&\mathcal{A}u_i^n=\begin{cases}\dfrac{1}{12}\left(u_{i-1}^n+10u_i^n+u_{i+1}^n\right), & 1\le i\le M-1,\\ u_i^n, & i=0 \ \text{ or } \ M,\end{cases}\\
&\mathcal{H}u_i^n=\begin{cases}\dfrac{\beta}{24}u_{i-1}^n+\left(1-\dfrac{\beta}{12}\right)u_i^n+\dfrac{\beta}{24}u_{i+1}^n, & 1\le i\le M-1,\\ u_i^n, & i=0 \ \text{ or } \ M.\end{cases}
\end{aligned}
$$

It is obvious that

$$\mathcal{A}u_i^n=\left(1+\frac{h^2}{12}\delta_x^2\right)u_i^n,\quad \mathcal{H}u_i^n=\left(1+\frac{\beta h^2}{24}\delta_x^2\right)u_i^n.$$

For convenience, we introduce a new compact difference operator

$$
\begin{aligned}
\mathcal{L}u_i^n=\mathcal{A}\mathcal{H}u_i^n&=\left(1+\frac{h^2}{12}\delta_x^2\right)\left(1+\frac{\beta h^2}{24}\delta_x^2\right)u_i^n\\
&=\left(1+\frac{\beta h^2}{24}\delta_x^2\right)\left(1+\frac{h^2}{12}\delta_x^2\right)u_i^n=\mathcal{H}\mathcal{A}u_i^n.
\end{aligned}
$$

Lemma 2.1. *If $u(t)$ satisfies* (1.2), *then the following results*

$$
\begin{aligned}
u_t(t_{n+1/2})&=\frac{u(t_{n+1})-u(t_n)}{\tau}+\mathcal{O}(t_{n+1}^{\sigma-3}\tau^2)\\
&=\delta_t u^{n+\frac{1}{2}}+\mathcal{O}(t_{n+1}^{\sigma-3}\tau^2)
\end{aligned}\tag{2.1}
$$

and

$$
{}_0J_t^{\alpha-1}u(t_{n+1/2})=\frac{1}{2}\left[{}_0J_t^{\alpha-1}u(t_{n+1})+{}_0J_t^{\alpha-1}u(t_n)\right]+\mathcal{O}(t_{n+1}^{\sigma+\alpha-3}\tau^2)\tag{2.2}
$$

hold, where ${}_0J_t^{\alpha}$ is the Riemann-Liouville integral operator defined by

$${}_0J_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-z)^{\alpha-1}u(z)dz.$$

Proof. For $-1\leq\gamma\leq 1$ and $n=0,1,\cdots,N-1$, we can easily find that by the Taylor expansion

$$\left(t_{n+\frac{1}{2}}\right)^{\sigma-\gamma}=\frac{1}{2}\left[\left(t_{n+1}^{\sigma-\gamma}\right)+\left(t_n^{\sigma-\gamma}\right)\right]+\mathcal{O}\left(t_{n+1}^{\sigma-\gamma-2}\tau^2\right). \tag{2.3}$$

Since $u(t)=\mathcal{O}(t^\sigma)$, we deduce that

$$_0J_t^{\alpha-1}u(t_{n+\frac{1}{2}})=\mathcal{O}\left(t_{n+\frac{1}{2}}^{\sigma+\alpha-1}\right).$$

Therefore, (2.2) is obtained by setting $\gamma=1-\alpha$ in (2.3). Similarly, (2.1) can be obtained by letting $\gamma=1$ in (2.3). □

Lemma 2.2 ([27,28]). *Let $1<\sigma<\alpha<2$ and $\omega_k^{(\alpha-1)}$ are the weights associated with the generating function*

$$\left(\frac{3}{2}-2z+\frac{z^2}{2}\right)^{1-\alpha},$$

under the Assumption (1.2)*, then*

$$\left|{}_0J_{t_{n+1}}^{\alpha-1}u(t)-\tau^{\alpha-1}\sum_{k=0}^{n+1}\omega_{n+1-k}^{(\alpha-1)}u(t_k)\right|\leq Ct_{n+1}^{\sigma+\alpha-3}\tau^2.$$

For linearizing the nonlinear function $g(u)$, the following lemma is necessary.

Lemma 2.3 ([29]). *Suppose $u(t)$ satisfies the Assumption* (1.2)*, then it holds*

$$u(t_{n+1})=2u(t_n)-u(t_{n-1})+\mathcal{O}(t_n^{\sigma-2}\tau^2).$$

The following two lemmas are listed in order to show the truncation errors of two compact difference operators, which generates the fourth-order accuracy approximation in space.

Lemma 2.4 (Lemma 1.2 in [30]). *Suppose $u(x)\in C^6([x_{i-1},x_{i+1}])$ and $\zeta(s)=5(1-s)^3-3(1-s)^3$, we obtain*

$$\mathcal{A}u''(x_i)-\delta_x^2u(x_i)=\frac{h^4}{360}\int_0^1\left[u^{(6)}(x_i-sh)+u^{(6)}(x_i+sh)\right]\zeta(s)ds.$$

Lemma 2.5 (Theorem 2.4 in [31]). *Let $1<\beta<2$ and $u(x)$ is defined in a finite interval $[0,L]$. If $u(x)\in C^7(\mathbb{R})$ and all its derivative up to the order five belong to $L(\mathbb{R})$, then*

$$-\delta_x^\beta u(x)=\mathcal{H}\left(\frac{\partial^\beta u(x)}{\partial|x|^\beta}\right)+\mathcal{O}(h^4),$$

where

$$\delta_x^\beta u(x)=\frac{1}{h^\beta}\sum_{j=-\lceil\frac{L-x}{h}\rceil}^{\lceil\frac{x}{L}\rceil}\frac{(-1)^j\Gamma(\beta+1)}{\Gamma(\beta/2-j+1)\Gamma(\beta/2+j+1)}u(x-jh),$$

where $\frac{\partial^\beta u(x)}{\partial|x|^\beta}$ is the Riesz derivative with order β.

3 Derivation of a linearized compact difference scheme

In this section, a linearized compact finite difference scheme for Problem (1.1) will be derived under the Assumption (1.2). After multiplying ${}_0J_t^{\alpha-1}$ on both sides of Eq. (1.1), we get the following partial integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} = {}_0J_t^{\alpha-1}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^\beta}{\partial |x|^\beta}\right)u(x,t)+g(u)\right]+F(x,t), \tag{3.1}$$

where $F(x,t)={}_0J_t^{\alpha-1}f(x,t)$.

Assume $u(x,\cdot)\in C^7([0,L])$ with $u(0,\cdot)=u(L,\cdot)=0$ and consider Eq. (3.1) at the point $(x_i,t_{n+1/2})$, that is

$$\left.\frac{\partial u(x_i,t)}{\partial t}\right|_{t=t_{n+\frac{1}{2}}} = {}_0J_{t_{n+\frac{1}{2}}}^{\alpha-1}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^\beta}{\partial |x|^\beta}\right)u(x_i,t)+g(u(x_i,t))\right]+F(x_i,t_{n+\frac{1}{2}}).$$

Using the Crank-Nicolson method and Lemma 2.1, we obtain

$$\begin{aligned}
&\frac{u(x_i,t_{n+1})-u(x_i,t_n)}{\tau}\\
=&\frac{1}{2}\left({}_0J_{t_{n+1}}^{\alpha-1}\frac{\partial^2 u(x_i,t)}{\partial x^2}+{}_0J_{t_n}^{\alpha-1}\frac{\partial^2 u(x_i,t)}{\partial x^2}\right)+\frac{1}{2}\left({}_0J_{t_{n+1}}^{\alpha-1}\frac{\partial^\beta u(x_i,t)}{\partial |x|^\beta}+{}_0J_{t_n}^{\alpha-1}\frac{\partial^\beta u(x_i,t)}{\partial |x|^\beta}\right)\\
&+\frac{1}{2}\left({}_0J_{t_{n+1}}^{\alpha-1}g(u(x_i,t))+{}_0J_{t_n}^{\alpha-1}g(u(x_i,t))\right)+F(x_i,t_{n+\frac{1}{2}})+\mathcal{O}(t_{n+1}^{\sigma-3}\tau^2).
\end{aligned}$$

Now, let us act both sides of the above equation with the compact operator $\mathcal{L}$. Then by using Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned}
&\mathcal{L}\left(\frac{u(x_i,t_{n+1})-u(x_i,t_n)}{\tau}\right)\\
=&\frac{1}{2}\left({}_0J_{t_{n+1}}^{\alpha-1}\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^\beta\right)u(x_i,t)+{}_0J_{t_n}^{\alpha-1}\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^\beta\right)u(x_i,t)\right)\\
&+\frac{\mathcal{L}}{2}\left({}_0J_{t_{n+1}}^{\alpha-1}g(u(x_i,t))+{}_0J_{t_n}^{\alpha-1}g(u(x_i,t))\right)\\
&+\mathcal{L}F(x_i,t_{n+\frac{1}{2}})+\mathcal{O}(t_{n+1}^{\sigma-3}\tau^2+h^4).
\end{aligned}$$

Let $u(x_i,t_n)=u_i^n$. By Lemma 2.2, it achieves that

$$\begin{aligned}
\mathcal{L}\left(\frac{u_i^{n+1}-u_i^n}{\tau}\right)=&\frac{\tau^{\alpha-1}}{2}\left(\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^\beta\right)u_i^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^\beta\right)u_i^{n-k}\right)\\
&+\frac{\tau^{\alpha-1}\mathcal{L}}{2}\left(\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}g(u_i^{n+1-k})+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}g(u_i^{n-k})\right)\\
&+\mathcal{L}F_i^{n+\frac{1}{2}}+\mathcal{O}(t_{n+1}^{\sigma-3}\tau^2+h^4).
\end{aligned} \tag{3.2}$$

In terms of the unknown u_i^{n+1}, Eq. (3.2) is a nonlinear system. To linearize Eq. (3.2), we use

$$u_i^1 = u_i^0 + \tau (u_t)_i^0 + \mathcal{O}\left(\tau^2\, t^{\sigma-1}\Big|_{t_0}^{t_1}\right)$$

and Lemma 2.3 for $n=0$ and $1 \le n \le N-1$, respectively, i.e.,

$$\begin{aligned}\mathcal{L}\left(u_i^1 - u_i^0\right) = & \frac{\tau^\alpha}{2}\left(\sum_{k=0}^{1}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)u_i^{1-k} + \omega_0^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)u_i^0\right) \\ & + \frac{\tau^\alpha \mathcal{L}}{2}\left(\omega_0^{(\alpha-1)} g(u_i^0 + \tau (u_t)_i^0) + \omega_1^{(\alpha-1)} g(u_i^0)\right) \\ & + \frac{\tau^\alpha \mathcal{L}}{2}\omega_0^{(\alpha-1)} g(u_i^0) + \tau \mathcal{L} F_i^{n+\frac{1}{2}} + R_i^* \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\mathcal{L}\left(u_i^{n+1} - u_i^n\right) = & \frac{\tau^\alpha}{2}\left(\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)u_i^{n+1-k} + \sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)u_i^{n-k}\right) \\ & + \frac{\tau^\alpha \mathcal{L}}{2}\left(\sum_{k=1}^{n+1}\omega_k^{(\alpha-1)} g(u_i^{n+1-k}) + \sum_{k=0}^{n}\omega_k^{(\alpha-1)} g(u_i^{n-k})\right) \\ & + \frac{\tau^\alpha \omega_0^{(\alpha-1)}}{2}\mathcal{L} g(2u_i^n - u_i^{n-1}) + \tau \mathcal{L} F_i^{n+\frac{1}{2}} + R_i^*, \end{aligned} \tag{3.4}$$

where $R_i^* = \mathcal{O}(t_{n+1}^{\sigma-3}\tau^3 + \tau h^4)$.

Noting $(u_t)_i^0 = 0$, omitting the truncation error term R_i^* in (3.3) and (3.4) and replacing the u_i^n by its numerical solution U_i^n, one can get the following linearized compact finite difference schemes for Eq. (3.1),

$$\begin{aligned}\mathcal{L}\left(U_i^1 - U_i^0\right) = & \frac{\tau^\alpha}{2}\left(\sum_{k=0}^{1}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)U_i^{1-k} + \omega_0^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)U_i^0\right) \\ & + \frac{\tau^\alpha \mathcal{L}}{2}\left(\omega_0^{(\alpha-1)} g(U_i^0) + \omega_1^{(\alpha-1)} g(U_i^0)\right) \\ & + \frac{\tau^\alpha \mathcal{L}}{2}\omega_0^{(\alpha-1)} g(U_i^0) + \tau \mathcal{L} F_i^{n+\frac{1}{2}} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\mathcal{L}\left(U_i^{n+1} - U_i^n\right) = & \frac{\tau^\alpha}{2}\left(\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)U_i^{n+1-k} + \sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)U_i^{n-k}\right) \\ & + \frac{\tau^\alpha \mathcal{L}}{2}\left(\sum_{k=1}^{n+1}\omega_k^{(\alpha-1)} g(U_i^{n+1-k}) + \sum_{k=0}^{n}\omega_k^{(\alpha-1)} g(U_i^{n-k})\right) \\ & + \frac{\tau^\alpha \omega_0^{(\alpha-1)}}{2}\mathcal{L} g(2U_i^n - U_i^{n-1}) + \tau \mathcal{L} F_i^{n+\frac{1}{2}}. \end{aligned} \tag{3.6}$$

4 Analysis of the linearized compact difference schemes (3.5) and (3.6)

To begin, we define the grid function space Θ_h as follows,

$$\Theta_h = \{u_i^n | 0 \le n \le N,\ 0 \le i \le M \ \text{ and } \ u_0^n = u_M^n = 0\}.$$

For two vectors $u^n, v^n \in \Theta_h$, we denote

$$\langle u^n, v^n \rangle = h \sum_{i=1}^{M-1} u_i^n v_i^n, \quad ||u^n||^2 = \langle u^n, u^n \rangle.$$

Lemma 4.1 (Lemma 3.4 in [32]). *For $1<\beta<2$ and the operator δ_x^β defined in Lemma 2.5, there exists a linear difference operator, denoted by $\delta_x^{\beta/2}$, such that*

$$\langle \delta_x^\beta u^n, v^n \rangle = \langle \delta_x^{\beta/2} u^n, \delta_x^{\beta/2} v^n \rangle,$$

where $u^n, v^n \in \Theta_h$.

Lemma 4.2 (Lemma 4.2.2 in [33]). *For $u^n, v^n \in \Theta_h$, it holds*

$$\langle \delta_x^2 u^n, v^n \rangle = -\langle \delta_x u^n, \delta_x v^n \rangle.$$

Lemma 4.3. *The operators $\mathcal{A}$ and $\mathcal{H}$ are symmetric, positive and commutative. Thus, the operator $\mathcal{L} = \mathcal{A}\mathcal{H}$ is symmetric and positive. Furthermore, there exist invertible matrices* **A**, **B**, *and* **Q** *such that*

$$\langle \mathcal{A} u^n, v^n \rangle = \langle \mathbf{B} u^n, \mathbf{B} v^n \rangle, \quad \langle \mathcal{H} u^n, v^n \rangle = \langle \mathbf{A} u^n, \mathbf{A} v^n \rangle \quad \text{and} \quad \langle \mathcal{L} u^n, v^n \rangle = \langle \mathbf{Q} u^n, \mathbf{Q} v^n \rangle,$$

where $u^n, v^n \in \Theta_h$.

Proof. This result is straightforward to obtain using the definitions of the operators $\mathcal{A}$, $\mathcal{H}$, and $\mathcal{L}$. □

Now we can define a new norm

$$\|u^n\|_{\mathcal{L}}^2 = \langle \mathcal{L} u^n, u^n \rangle = \langle \mathbf{Q} u^n, \mathbf{Q} u^n \rangle$$

and establish the equivalence of two norms $\|\cdot\|_{\mathcal{L}}$ and $\|\cdot\|$.

Lemma 4.4. *For any grid function $u^n \in \Theta_h$, it holds*

$$\frac{1}{3}\|u^n\|^2 \le \|u^n\|_{\mathcal{L}}^2 \le \|u^n\|^2.$$

Proof. Note that

$$\begin{aligned}\mathcal{L}=\mathcal{A}\mathcal{H}=&\left(1+\frac{\beta h^2}{24}\delta_x^2\right)\left(1+\frac{h^2}{12}\delta_x^2\right)\\ =&1+\frac{h^2}{12}\delta_x^2+\frac{\beta h^2}{24}\delta_x^2+\frac{\beta h^4}{288}\delta_x^2\delta_x^2,\end{aligned}$$

then

$$\begin{aligned}\langle\mathcal{L}u^n,u^n\rangle=&\left\langle\left(1+\frac{h^2}{12}\delta_x^2+\frac{\beta h^2}{24}\delta_x^2+\frac{\beta h^4}{288}\delta_x^2\delta_x^2\right)u^n,u^n\right\rangle\\ =&\|u^n\|^2+\frac{h^2}{12}\langle\delta_x^2u^n,u^n\rangle+\frac{\beta h^2}{24}\langle\delta_x^2u^n,u^n\rangle+\frac{\beta h^4}{288}\langle\delta_x^2\delta_x^2u^n,u^n\rangle\\ =&\|u^n\|^2-\frac{h^2}{12}\|\delta_xu^n\|^2-\frac{\beta h^2}{24}\|\delta_xu^n\|^2+\frac{\beta h^4}{288}\|\delta_x\delta_xu^n\|^2.\end{aligned}$$

Using the inverse estimate

$$\|\delta_xu\|^2\le\frac{4}{h^2}\|u^n\|^2,$$

we have

$$\langle\mathcal{L}u^n,u^n\rangle\ge\|u^n\|^2-\frac{1}{3}\|u^n\|^2-\frac{\beta}{6}\|u^n\|^2.$$

Due to $1<\beta\le 2$, it deduces that

$$\langle\mathcal{L}u^n,u^n\rangle\ge\|u^n\|^2-\frac{1}{3}\|u^n\|^2-\frac{\beta}{6}\|u^n\|^2=\frac{1}{3}\|u^n\|^2.$$

Clearly, it holds that

$$\|\mathcal{L}u^n\|^2\le\|u^n\|^2.$$

This proof is completed. □

Lemma 4.5 (Lemma 2.5 in [34]). *For any positive integer K and any real vector* $(V_1,V_2,\cdots,V_K)$, *then the following inequality holds*

$$\sum_{n=0}^{K-1}\left(\sum_{j=0}^{n}\omega_j^{(\alpha-1)}V_{n+1-j}\right)V_{n+1}\ge 0,$$

where $\{\omega_j^{(\alpha-1)}\}_{j=0}^{\infty}$ *are the weights defined in Lemma* 2.3.

4.1 Convergence

Theorem 4.1. *Let $u(x,t)$ under Assumption* (1.2) *is the exact solution of Eq.* (3.1) *with $u(0,\cdot)=u(L,\cdot)=0$ and $\{U_i^n|0\leq i\leq M,1\leq n\leq N\}$ is the numerical solution of the linearized compact difference Schemes* (3.5) *and* (3.6)*, then it holds*

$$\|e^n\|\leq C\left(\tau^\sigma+h^4\right).$$

Proof. From Eqs. (3.5) and (3.6), we obtain

$$\begin{aligned}\mathcal{L}\left(e_i^{n+1}-e_i^n\right)=&\frac{\tau^\alpha}{2}\left(\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^\beta\right)e_i^{n+1-k}+\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^\beta\right)e_i^{n-k}\right)\\&+\frac{\tau^\alpha\mathcal{L}}{2}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\left(g(u_i^{n-k})-g(U_i^{n-k})\right)\\&+\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\mathcal{L}\left(g(2u_i^n-u_i^{n-1})-g(2U_i^n-U_i^{n-1})\right)+R^*,\end{aligned}\tag{4.1}$$

where $e_i^n=u_i^n-U_i^n$. Multiplying Eq. (4.1) by $h(e_i^{n+1}+e_i^n)$ and summing over $1\leq i\leq M-1$, we obtain

$$\begin{aligned}&\|e^{n+1}\|_{\mathcal{L}}^2-\|e^n\|_{\mathcal{L}}^2\\=&\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left\langle\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^\beta\right)\left(e^{n+1-k}+e^{n-k}\right),e^{n+1}+e^n\right\rangle\\&+\frac{\tau^\alpha\omega_{n+1}^{(\alpha-1)}}{2}\left\langle\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^\beta\right)e^0,e^{n+1}+e^n\right\rangle\\&+\frac{\tau^\alpha}{2}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\left\langle\mathcal{L}\left(g(u^{n-k})-g(U^{n-k})\right),e^{n+1}+e^n\right\rangle\\&+\frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\left\langle\mathcal{L}\left(g(2u^n-u^{n-1})-g(2U^n-U^{n-1})\right),e^{n+1}+e^n\right\rangle+\langle R^*,e^{n+1}+e^n\rangle.\end{aligned}$$

Since $e_i^0=0$ for $0\leq i\leq M$. Summing over n from 1 to $J-1$ and applying Lemmas 4.1 and 4.2, we obtain the following equality

$$\begin{aligned}&\|e^J\|_{\mathcal{L}}^2-\|e^1\|_{\mathcal{L}}^2\\=&-\frac{\tau^\alpha}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\mathbf{A}\delta_x\left(e^{n+1-k}+e^{n-k}\right),\mathbf{A}\delta_x\left(e^{n+1}+e^n\right)\rangle\\&-\frac{\tau^\alpha}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\langle\mathbf{B}\delta_x^{\beta/2}\left(e^{n+1-k}+e^{n-k}\right),\mathbf{B}\delta_x^{\beta/2}\left(e^{n+1}+e^n\right)\rangle\\&+\frac{\tau^\alpha}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\langle\mathcal{L}\left(g(u^{n-k})-g(U^{n-k})\right),e^{n+1}+e^n\rangle\end{aligned}$$

$$
\begin{aligned}
&+\frac{\tau^\alpha \omega_0^{(\alpha-1)}}{2}\sum_{n=1}^{J-1}\langle \mathcal{L}\left(g(2u^n-u^{n-1})-g(2U^n-U^{n-1})\right),e^{n+1}+e^n\rangle\\
&+\sum_{n=1}^{J-1}\langle R^*,e^{n+1}+e^n\rangle. \qquad (4.2)
\end{aligned}
$$

Now, using Eqs. (3.3) and (3.5), and following the same deductions as above, we deduce that

$$
\begin{aligned}
\|e^1\|_{\mathcal{L}}^2=&-\frac{\tau^\alpha \omega_0^{(\alpha-1)}}{2}\left(\|\mathbf{A}\delta_x e^1\|^2+\|\mathbf{B}\delta_x^{\beta/2}e^1\|^2\right)\\
&+\tau^\alpha\omega_0^{(\alpha-1)}\langle \mathcal{L}\left(g(u^0)-g(U^0)\right),e^1\rangle\\
&+\frac{\tau^\alpha \omega_1^{(\alpha-1)}}{2}\langle \mathcal{L}\left(g(u^0)-g(U^0)\right),e^1\rangle+\langle R^*,e^1\rangle. \qquad (4.3)
\end{aligned}
$$

Sum Eqs. (4.2) and (4.3) and use Lemma 4.5, it deduces that

$$
\begin{aligned}
\|e^J\|_{\mathcal{L}}^2\le&\frac{\tau^\alpha}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\left\langle \mathcal{L}\left(g(u^{n-k})-g(U^{n-k})\right),e^{n+1}+e^n\right\rangle\\
&+\frac{\tau^\alpha \omega_0^{(\alpha-1)}}{2}\sum_{n=1}^{J-1}\left\langle \mathcal{L}\left(g(2u^n-u^{n-1})-g(2U^n-U^{n-1})\right),e^{n+1}+e^n\right\rangle\\
&+\tau^\alpha\omega_0^{(\alpha-1)}\langle \mathcal{L}\left(g(u^0)-g(U^0)\right),e^1\rangle\\
&+\frac{\tau^\alpha \omega_1^{(\alpha-1)}}{2}\langle \mathcal{L}\left(g(u^0)-g(U^0)\right),e^1\rangle+\sum_{n=1}^{J-1}\langle R^*,e^{n+1}+e^n\rangle. \qquad (4.4)
\end{aligned}
$$

Exchanging the summation order for the first term on the right-hand side of Inequality (4.4), using the Lipschitz condition of g and Lemma 4.4, we obtain

$$
\begin{aligned}
\|e^J\|_{\mathcal{L}}^2\le& C\tau^\alpha\sum_{k=0}^{J-1}\sum_{n=k}^{J-1}\left(w_{n+1-k}^{(\alpha-1)}+w_{n-k}^{(\alpha-1)}\right)\|e^k\|\|e^{n+1}+e^n\|\\
&+C\tau^\alpha\sum_{n=1}^{J-1}\|e^n\|\|e^{n+1}+e^n\|+C\sum_{n=1}^{J-1}R^*\|e^{n+1}+e^n\|.
\end{aligned}
$$

Since $\tau^{\alpha-1}\sum_{n=k}^{J-1}(\omega_{n+1-k}^{(\alpha-1)}+\omega_{n-k}^{(\alpha-1)})$ is bounded and assume that $\|e^P\|_{\mathcal{L}}=\max_{0\le J\le N}\|e^J\|_{\mathcal{L}}$, then it holds

$$
\begin{aligned}
\|e^P\|_{\mathcal{L}}\le& C\sum_{n=0}^{P-1}\left(t_{n+1}^{\sigma-3}\tau^3+\tau h^4\right)\\
\le& C\left(\sum_{n=0}^{P-1}(n+1)^{\sigma-3}\tau^\sigma+h^4\right).
\end{aligned}
$$

Since $\sum_{n=0}^{P-1}(n+1)^{\sigma-3}$ is bounded. Using Lemma 4.4, we arrive at the estimate

$$\|e^P\| \leq C\left(\tau^\sigma + h^4\right).$$

The proof is completed. □

Remark 4.1. The linearized compact difference Schemes (3.5) and (3.6), according to Theorem 4.1, have temporal accuracy $\mathcal{O}(\tau^\sigma)$. However, Eq. (3.2) has a global truncation error in the temporal direction of $\mathcal{O}(t_{n+1}^{\sigma-3}\tau^2)$. This means that the global convergence order in temporal direction can be 2 if t_{n+1} is far from t_0. As a result, we may conclude that linearized compact difference Schemes (3.5) and (3.6) have temporal accuracy $\mathcal{O}(\tau^\sigma)$ near some first time steps, and become $\mathcal{O}(\tau^2)$ when t_{n+1} is far from t_0. This assertion will be strictly verified by numerical experiments in Section 4.

4.2 Stability

Theorem 4.2. *Suppose $\{U_i^n\}$ and $\{\hat{U}_i^n\}$ are the numerical solutions of linearized compact finite difference Schemes* (3.5) *and* (3.6) *with different initial conditions, then it can be obtained the following unconditional stability result,*

$$\|\xi^P\| \leq C\left(\|\xi^0\| + \tau\|\mathcal{H}\delta_x^2\xi^0\| + \tau\|\mathcal{A}\delta_x^\beta\xi^0\| + \max_{0\leq n\leq P}\|\mathcal{L}\left(F^{n+\frac{1}{2}} - \hat{F}^{n+\frac{1}{2}}\right)\|\right),$$

where $\xi_i^n = U_i^n - \hat{U}_i^n$.

Proof. Note that $\hat{U}_i^n$ is also the numerical solution of the linearized difference difference scheme, thus we have

$$\begin{aligned}
\mathcal{L}\left(\hat{U}_i^{n+1} - \hat{U}_i^n\right) =& \frac{\tau^\alpha}{2}\left(\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)\hat{U}_i^{n+1-k} + \sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)\hat{U}_i^{n-k}\right) \\
&+ \frac{\tau^\alpha\mathcal{L}}{2}\left(\sum_{k=1}^{n+1}\omega_k^{(\alpha-1)}g(\hat{U}_i^{n+1-k}) + \sum_{k=0}^{n}\omega_k^{(\alpha-1)}g(\hat{U}_i^{n-k})\right) \\
&+ \frac{\tau^\alpha\omega_0^{(\alpha-1)}}{2}\mathcal{L}g(2\hat{U}_i^n - \hat{U}_i^{n-1}) + \tau\mathcal{L}\hat{F}_i^{n+\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

Subtracting Eq. (4.5) from Eq. (3.6), we obtain

$$\begin{aligned}
\mathcal{L}\left(\xi_i^{n+1} - \xi_i^n\right) =& \frac{\tau^\alpha}{2}\left(\sum_{k=0}^{n+1}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)\xi_i^{n+1-k} + \sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left(\mathcal{H}\delta_x^2 - \mathcal{A}\delta_x^\beta\right)\xi_i^{n-k}\right) \\
&+ \frac{\tau^\alpha\mathcal{L}}{2}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)} + \omega_k^{(\alpha-1)}\right)\left(g(U_i^{n-k}) - g(\hat{U}_i^{n-k})\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\mathcal{L}\left(g(2U_i^n-U_i^{n-1})-g(2\hat{U}_i^n-\hat{U}_i^{n-1})\right)\\
&+\tau\mathcal{L}\left(F_i^{n+\frac{1}{2}}-\hat{F}_i^{n+\frac{1}{2}}\right).
\end{aligned} \tag{4.6}
$$

Multiplying Eq. (4.6) by $h(\xi_i^{n+1}+\xi_i^n)$, and summing over $1\le i\le M-1$, we have

$$
\begin{aligned}
\|\xi^{n+1}\|_{\mathcal{L}}^2-\|\xi^n\|_{\mathcal{L}}^2=&\frac{\tau^{\alpha}}{2}\sum_{k=0}^{n}\omega_k^{(\alpha-1)}\left\langle\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^{\beta}\right)\left(\xi^{n+1-k}+\xi^{n-k}\right),\xi^{n+1}+\xi^n\right\rangle\\
&+\frac{\tau^{\alpha}}{2}\omega_{n+1}^{(\alpha-1)}\left\langle\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^{\beta}\right)\xi^0,\xi^{n+1}+\xi^n\right\rangle\\
&+\frac{\tau^{\alpha}}{2}\sum_{k=0}^{n}\left(\omega_{k+1}^{(\alpha-1)}+\omega_k^{(\alpha-1)}\right)\left\langle\mathcal{L}\left(g(U^{n-k})-g(\hat{U}^{n-k})\right),\xi^{n+1}+\xi^n\right\rangle\\
&+\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\left\langle\mathcal{L}\left(g(2U^n-U^{n-1})-g(2\hat{U}^n-\hat{U}^{n-1})\right),\xi^{n+1}+\xi^n\right\rangle\\
&+\tau\left\langle\mathcal{L}\left(F^{n+\frac{1}{2}}-\hat{F}^{n+\frac{1}{2}}\right),\xi^{n+1}+\xi^n\right\rangle.
\end{aligned}
$$

Use the same deductions to get Eq. (4.4) and apply Lemma 4.4, it achieves

$$
\begin{aligned}
\|\xi^J\|_{\mathcal{L}}^2-\|\xi^0\|_{\mathcal{L}}^2\le& C\tau\left\|\left(\mathcal{H}\delta_x^2-\mathcal{A}\delta_x^{\beta}\right)\xi^0\right\|\left(\|\xi^{n+1}\|+\|\xi^n\|\right)\\
&+C\tau\sum_{k=0}^{J-1}\|g(U^k)-g(\hat{U}^k)\|\left(\|\xi^{n+1}\|+\|\xi^n\|\right)\\
&+\frac{\tau^{\alpha}\omega_0^{(\alpha-1)}}{2}\sum_{n=1}^{J-1}\|g(2U^n-U^{n-1})-g(2\hat{U}^n-\hat{U}^{n-1})\|\left(\|\xi^{n+1}\|+\|\xi^n\|\right)\\
&+\tau\sum_{n=1}^{J-1}\left\|\mathcal{L}\left(F^{n+\frac{1}{2}}-\hat{F}^{n+\frac{1}{2}}\right)\right\|\left(\|\xi^{n+1}\|+\|\xi^n\|\right).
\end{aligned} \tag{4.7}
$$

Using the Lipschitz condition of g, assuming $\|\xi^P\|_{\mathcal{L}}=\max_{0\le J\le N}\|\xi^J\|_{\mathcal{L}}$ and applying Lemma 4.4, we obtain

$$
\begin{aligned}
\|\xi^P\|\le C\Big(&\|\xi^0\|+\tau\|\mathcal{H}\delta_x^2\xi^0\|+\tau\|\mathcal{A}\delta_x^{\beta}\xi^0\|\\
&+\max_{0\le n\le P}\left\|\mathcal{L}\left(F^{n+\frac{1}{2}}-\hat{F}^{n+\frac{1}{2}}\right)\right\|\Big)+C\tau\sum_{k=0}^{P-1}\|\xi^k\|.
\end{aligned} \tag{4.8}
$$

Applying the Gronwall inequality to inequality (4.8), we arrive at the estimate

$$
\|\xi^P\|\le C\left(\|\xi^0\|+\tau\|\mathcal{H}\delta_x^2\xi^0\|+\tau\|\mathcal{A}\delta_x^{\beta}\xi^0\|+\max_{0\le n\le P}\left\|\mathcal{L}\left(F^{n+\frac{1}{2}}-\hat{F}^{n+\frac{1}{2}}\right)\right\|\right),
$$

this completes the proof. □

5 Numerical experiments

Example 5.1. Consider the following problem

$$
\begin{aligned}
&{}_0^C D_t^{\alpha} u(x,t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^\beta}{\partial |x|^\beta} \right) u(x,t) + g(u) + f(x,t), \\
&u(x,0)=0, \quad u_t(x,0)=0, \quad 0<x<1, \\
&u(0,t)=u(L,t)=0, \qquad\quad 0<t\leq 1,
\end{aligned}
$$

where $1<\sigma<\alpha$, and

$$
\begin{aligned}
f(x,t) = &\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\alpha+1)} t^{\sigma-\alpha} x^4 (1-x)^4 - 4t^\sigma x^2 (x-1)^2 (14x^2-14x+3) \\
&+ \frac{t^\sigma}{2\cos\left(\frac{\beta\pi}{2}\right)} h(x,\beta) - t^{2\sigma} x^8 (1-x)^8,
\end{aligned}
$$

where

$$
\begin{aligned}
h(w,\beta) = &\frac{\Gamma(9)}{\Gamma(9-\beta)} \left(w^{8-\beta} + (1-w)^{8-\beta} \right) - 4\frac{\Gamma(8)}{\Gamma(8-\beta)} \left(w^{7-\beta} + (1-w)^{7-\beta} \right) \\
&+ 6\frac{\Gamma(7)}{\Gamma(7-\beta)} \left(w^{6-\beta} + (1-w)^{6-\beta} \right) - 4\frac{\Gamma(6)}{\Gamma(6-\beta)} \left(w^{5-\beta} + (1-w)^{5-\beta} \right) \\
&+ \frac{\Gamma(5)}{\Gamma(5-\beta)} \left(w^{4-\beta} + (1-w)^{4-\beta} \right).
\end{aligned}
$$

The nonlinear function $g(u)=u^2$ and the exact solution $u(x,t)=t^\sigma x^4(1-x)^4$.

Firstly, taking $\sigma=1.3$, $\alpha=1.7$ and $\beta=1.5$, we plot the numerical and exact solutions of the considered problem in Fig. 1. It is observed that numerical and exact results are in excellent agreement. Fig. 2 shows that the errors are small, implying that our numerical solutions can accurately approximate the exact solutions. Secondly, to confirm Theorem 4.1, set a suitable small h, the errors at t_1 and the numerical convergence orders are reported in Table 1. According to the data in Table 1, we conclude that the σ-order accuracy in time is obtained. To verify Remark 4.1, Tables 2 and 3 show the errors and the temporal numerical convergence orders at t_1 and t_N for $\beta=1.5$, $\alpha=1.9$ with various σ. As expected, the numerical convergence order approaches σ at t_1 and is close to 2 at t_N.

Finally, we verify the numerical accuracy in space of the proposed scheme (3.5) and (3.6). The numerical errors and convergence orders in maximum norm are listed in Table 4. According to the data in the Table, we conclude that the fourth-order convergence accuracy in space is verified.

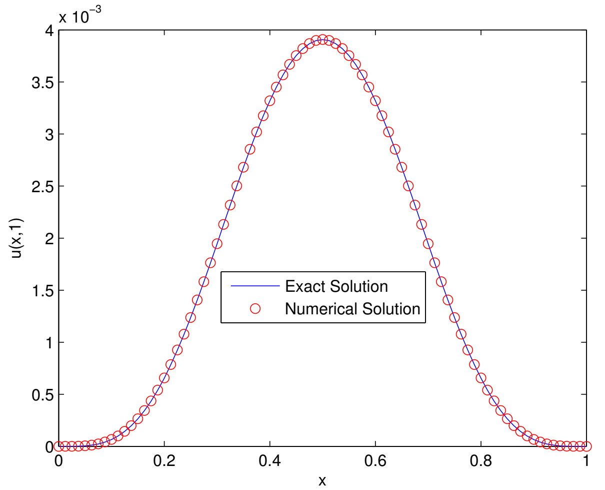

Figure 1: The comparison of numerical solution of linearized compact Schemes (3.5) and (3.6) with the exact solution for $\tau=1/10$, $h=1/80$, $\sigma=1.3$, $\alpha=1.7$, and $\beta=1.5$.

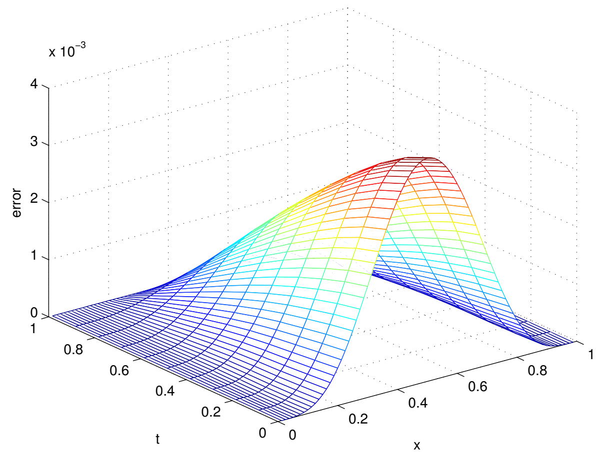

Figure 2: The error surface between numerical solutions and exact solutions for $\tau=1/10$, $h=1/80$, $\sigma=1.3$, $\alpha=1.7$, and $\beta=1.5$.

6 Conclusions

In this paper, the classical central difference formula and the fourth-order compact difference methods are used to discritize the spatial derivatives, while the Crank-Nicolson technique, the midpoint formula and the second-order convolution formula are used for temporal discretizations. Then, the linearized compact finite difference Schemes (3.5) and

Table 1: The errors at t_1 and temporal numerical convergence orders with fixed $h=0.001$, $\sigma=1.5$, and $\beta=1.4$.

τ	$\alpha=1.6$		$\alpha=1.75$		$\alpha=1.9$	
	error	order	error	order	error	order
1/20	1.2617×10^{-6}		1.1083×10^{-6}		1.1001×10^{-6}	
1/40	4.0007×10^{-7}	1.6571	4.3309×10^{-7}	1.3557	4.6194×10^{-7}	1.2518
1/80	1.6390×10^{-7}	1.2875	1.7151×10^{-7}	1.3364	1.7601×10^{-7}	1.3920
1/160	6.1895×10^{-8}	1.4049	6.3013×10^{-8}	1.4445	6.3573×10^{-8}	1.4692
1/320	2.2394×10^{-8}	1.4667	2.2542×10^{-8}	1.4830	2.2607×10^{-8}	1.4916

Table 2: The errors at t_1 and temporal numerical convergence orders with fixed $h=0.001$, $\beta=1.5$, and $\alpha=1.9$.

τ	$\sigma=1.6$		$\sigma=1.7$		$\sigma=1.8$	
	error	order	error	order	error	order
1/20	7.4562×10^{-7}		4.7407×10^{-7}		2.9809×10^{-7}	
1/40	2.8814×10^{-7}	1.3717	1.6273×10^{-7}	1.5426	7.8316×10^{-8}	1.9284
1/80	1.0359×10^{-7}	1.4758	5.5459×10^{-8}	1.5530	2.5645×10^{-8}	1.6106
1/160	3.5034×10^{-8}	1.5641	1.7591×10^{-8}	1.6566	7.6754×10^{-9}	1.7404
1/320	1.1636×10^{-8}	1.5902	5.4590×10^{-9}	1.6881	2.2294×10^{-9}	1.7836

Table 3: The errors at t_N and temporal numerical convergence orders with fixed $h=0.001$, $\beta=1.5$, and $\alpha=1.9$.

τ	$\sigma=1.6$		$\sigma=1.7$		$\sigma=1.8$	
	error	order	error	order	error	order
1/10	1.5983×10^{-5}		1.9179×10^{-5}		2.2293×10^{-5}	
1/20	3.2939×10^{-6}	2.2787	4.1227×10^{-6}	2.2179	4.9774×10^{-6}	2.1631
1/40	8.1363×10^{-7}	2.0174	9.8729×10^{-7}	2.0621	1.1865×10^{-6}	2.0686
1/80	2.1707×10^{-7}	1.9062	2.5101×10^{-7}	1.9757	2.9583×10^{-7}	2.0039
1/160	6.1055×10^{-8}	1.8300	6.6207×10^{-8}	1.9227	7.5727×10^{-8}	1.9659
1/320	1.7949×10^{-8}	1.7662	1.8062×10^{-9}	1.8740	1.9888×10^{-8}	1.9289

Table 4: The errors and spatial numerical convergence orders with fixed $\tau=0.01$, $\sigma=1.4$, and $\alpha=1.9$.

h	$\beta=1.3$		$\beta=1.5$		$\beta=1.8$	
	error	order	error	order	error	order
1/5	1.0536×10^{-4}		1.0610×10^{-4}		1.1498×10^{-4}	
1/10	6.9971×10^{-6}	3.9124	6.6788×10^{-6}	3.9897	7.4611×10^{-6}	3.9458
1/20	4.3327×10^{-7}	4.0134	4.0577×10^{-7}	4.0409	4.5665×10^{-7}	4.0302

(3.6) are presented for time-space fractional nonlinear diffusion-wave equations with homogeneous initial boundary value conditions based on their equivalent fractional partial integro-differential equations. The unconditional stability and the convergence of the proposed schemes are proved by energy method. Furthermore, the numerical experiments are presented to support our theoretical results.

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