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Busemann-Petty Type Problem for the General L_p-Centroid Bodies

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Abstract. Lutwak showed the Busemann-Petty type problem (also called the Shephard type problem) for the centroid bodies. Grinberg and Zhang gave an affirmation and a negative form of the Busemann-Petty type problem for the L_p-centroid bodies. In this paper, we obtain an affirmation form and two negative forms of the Busemann-Petty type problem for the general L_p-centroid bodies.

Key Words: L_p-centroid body, general L_p-centroid body, Busemann-Petty problem, affirmation form, negation form.

AMS Subject Classifications: 52A40, 52A20, 52A39, 52A38

1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in n-dimensional Euclidean space $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies, we write $\mathcal{K}^n_o$ and $\mathcal{K}^n_{os}$, respectively. Let $\mathcal{S}^n_o$ and $\mathcal{S}^n_{os}$ orderly denote the set of star bodies (about the origin) and the set of origin-symmetric star bodies in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the n-dimensional volume of a body K, for the standard unit ball B in $\mathbb{R}^n$, write $\omega_n = V(B)$.

Centroid body was attributed by Blaschke to Dupin (see [6, 18]), its definition was extended by Petty (see [17]). Let K is a compact set, the centroid body, ΓK, of K is an origin-symmetric convex body whose support function is given by (see [6])

$$h_{\Gamma K}(u) = \frac{1}{V(K)} \int_K |u \cdot x| dx \tag{1.1}$$

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for all $u \in S^{n-1}$.

Centroid bodies are very important in Brunn-Minkowski theory. For decades, centroid bodies have attracted increased attention (for example see articles [10,11,17,27] and books [6,18]). In particular, Lutwak [11] showed an affirmation and a negative form of the Busemann-Petty type problems for the centroid bodies as follows:

Theorem 1.1. *For $K \in \mathcal{S}_o^n$, $L \in \mathcal{P}^*$, if $\Gamma K \subseteq \Gamma L$, then*

$$V(K) \leq V(L),$$

and $V(K) = V(L)$ if and only if $K = L$. Here $\mathcal{P}^*$ denotes the set of polars of all projection bodies.

Theorem 1.2. *If $K \in \mathcal{S}_{os}^n \backslash \mathcal{P}^*$ is infinite smooth, then there exists $L \in \mathcal{S}_{os}^n \backslash \mathcal{P}^*$ is infinite smooth, such that $\Gamma K \subset \Gamma L$, but*

$$V(K) > V(L).$$

In 1997, Lutwak and Zhang [15] introduced the notion of L_p-centroid bodies. For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and real $p \geq 1$, the L_p-centroid body, $\Gamma_p K$, of K is an origin-symmetric convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma_p K}(u) =& \frac{1}{c_{n,p} V(K)} \int_K \mid u \cdot x \mid^p dx \\ =& \frac{1}{c_{n,p}(n+p) V(K)} \int_{S^{n-1}} \mid u \cdot v \mid^p \rho_K(v)^{n+p} dv \end{aligned} \tag{1.2}$$

for all $u \in S^{n-1}$. Here

$$c_{n,p} = \omega_{n+p} / \omega_2 \omega_n \omega_{p-1} \tag{1.3}$$

and dv is the standard spherical Lebesgue measure on S^{n-1}. The normalization above is chosen so that for the standard unit ball B in $\mathbb{R}^n$, we have $\Gamma_p B = B$. For the case $p = 1$, by (1.1) and (1.2), we see that $\Gamma_1 K$ is the centroid body ΓK under the normalization of definition (1.2) and $\Gamma_1 K = c_{n,1}^{-1} \Gamma K$ (see [6]).

Further, Lutwak and Zhang [15] established the L_p-centroid affine inequality. Whereafter, associated with the L_p-centroid bodies, Lutwak, Yang and Zhang [14] proved the L_p-Busemann-Petty centroid inequality which is stronger than the L_p-centroid affine inequality. The L_p-centroid bodies mean that the centroid bodies are extended from the Brunn-Minkowski theory to the L_p-Brunn-Minkowski theory. Regarding the studies of the L_p-centroid bodies, also see [1–3,7,21,22,24] and books [6,18]. In particular, Grinberg and Zhang [7] gave the following the Busemann-Petty type problem for the L_p-centroid bodies.

Theorem 1.3. *If $K \in \mathcal{S}_o^n$, $L \in \mathcal{P}_p^*$, then $\Gamma_p K \subseteq \Gamma_p L$ implies*

$$V(K) \leq V(L),$$

and $V(K) = V(L)$ if and only if $K = L$. Here $\mathcal{P}_p^*$ denotes the set of polars of all L_p-projection bodies.

Theorem 1.4. *If $K\in\mathcal{F}^2_{os}\backslash\mathcal{L}_p$, then there exists $L\in\mathcal{K}^n_{os}$ such that $\Gamma_pK\subset\Gamma_pL$, but*

$$V(K)>V(L).$$

Here $\mathcal{F}^2_{os}$ denotes the set of origin-symmetric convex bodies whose support functions are of C^2 and have positive continuous curvature functions, and $\mathcal{L}_p$ denotes the set of L_p-balls (see [7]).

In 2005, Ludwig [9] introduced a function $\varphi_\tau:\mathbb{R}\to[0,+\infty)$ by

$$\varphi_\tau(t)=|t|+\tau t \tag{1.4}$$

with a parameter $\tau\in[-1,1]$. From (1.4), Ludwig [9] introduced the notions of general L_p-projection bodies. Whereafter, Haberl and Schuster [8] derived a general L_p-projection body is the L_p-Minkowski combination of two asymmetric L_p-projection bodies, and established the general L_p-Petty projection inequality and the general L_p-Busemann-Petty centroid inequality.

Recently, motivated by Ludwig, Haberl and Schuster's work, Feng, Wang and Lu [5] defined the general L_p-centroid bodies as follows: For $K\in\mathcal{S}^n_o$, $p\geq 1$ and $\tau\in[-1,1]$, the general L_p-centroid body, Γ^τ_pK, of K is the convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma^\tau_pK}(u)=&\frac{2}{c_{n,p}(\tau)V(K)}\int_K\varphi_\tau(u\cdot x)^pdx\\ =&\frac{2}{c_{n,p}(\tau)(n+p)V(K)}\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p\rho_K(v)^{n+p}dv, \end{aligned}\tag{1.5}$$

where

$$c_{n,p}(\tau)=c_{n,p}[(1+\tau)^p+(1-\tau)^p]$$

and $c_{n,p}$ satisfies (1.3). The normalization is chosen such that $\Gamma^\tau_pB=B$ for every $\tau\in[-1,1]$. Obviously, if $\tau=0$ then $\Gamma^\tau_pK=\Gamma_pK$. Further, let $\tau=1$ in (1.5), they [5] defined the asymmetric L_p-centroid body, Γ^+_pK, of $K\in\mathcal{S}^n_o$ by

$$\begin{aligned} h^p_{\Gamma^+_pK}(u)=&\frac{2}{c_{n,p}V(K)}\int_K(u\cdot x)^p_+dx\\ =&\frac{2}{c_{n,p}(n+p)V(K)}\int_{S^{n-1}}(u\cdot v)^p_+\rho_K(v)^{n+p}dv, \end{aligned}\tag{1.6}$$

where $(u\cdot x)_+=\max\{u\cdot x,0\}$. Besides, they [5] also defined $\Gamma^-_pK=\Gamma^+_p(-K)$.

According to the definitions of $\Gamma^\pm_pK$ and (1.5), it is easy to verity that for $K\in\mathcal{S}^n_o$, $p\geq 1$, $\tau\in[-1,1]$ and $u\in S^{n-1}$,

$$h(\Gamma^\tau_pK,u)^p=f_1(\tau)h(\Gamma^+_pK,u)^p+f_2(\tau)h(\Gamma^-_pK,u)^p, \tag{1.7}$$

where

$$f_1(\tau)=\frac{(1+\tau)^p}{(1+\tau)^p+(1-\tau)^p},\qquad f_2(\tau)=\frac{(1-\tau)^p}{(1+\tau)^p+(1-\tau)^p}. \tag{1.8}$$

From (1.8), we easily know that

$$f_1(-\tau)=f_2(\tau),\qquad f_2(-\tau)=f_1(\tau), \tag{1.9a}$$
$$f_1(\tau)+f_2(\tau)=1. \tag{1.9b}$$

Let $\tau=0$ in (1.7), and combine with (1.8), we have for $u\in S^{n-1}$,

$$h(\Gamma_pK,u)^p=\frac{1}{2}h(\Gamma_p^+K,u)^p+\frac{1}{2}h(\Gamma_p^-K,u)^p. \tag{1.10}$$

If $\tau=\pm1$ in (1.7) and use (1.8), then

$$\Gamma_p^{+1}K=\Gamma_p^+K,\qquad \Gamma_p^{-1}K=\Gamma_p^-K.$$

For the research results of general L_p-centroid bodies, we can find in [5,16,23]. In this paper, we research the Busemann-Petty type problem for the general L_p-centroid bodies. Our works belong to part of a new and rapidly evolving asymmetric L_p Brunn-Minkowski theory.

Let $\mathcal{P}_p^{\tau,*}$ denote the set of polars of all general L_p-projection bodies. We first prove an affirmation form of the Busemann-Petty type problem for the general L_p-centroid bodies.

Theorem 1.5. *If $K\in\mathcal{S}_o^n$, $p\geq1$, $L\in\mathcal{P}_p^{\tau,*}$ and $\tau\in[-1,1]$, then $\Gamma_p^\tau K\subseteq\Gamma_p^\tau L$ implies*

$$V(K)\leq V(L),$$

and $V(K)=V(L)$ if and only if $K=L$.

Obviously, if $\tau=0$, then Theorem 1.5 gives Theorem 1.3. Further, we give a negation form of the Busemann-Petty type problem for the general L_p-centroid bodies.

Theorem 1.6. *If $L\in\mathcal{S}_o^n\backslash\mathcal{S}_{os}^n$ and $p\geq1$, then for any $\tau\in(-1,1)$, there exists $K\in\mathcal{S}_o^n$ (for $\tau=0$, $K\in\mathcal{S}_{os}^n$; for $\tau\neq0$, $K\in\mathcal{S}_o^n$) such that $\Gamma_p^\tau K\subset\Gamma_p^\tau L$, but*

$$V(K)>V(L).$$

Let $\tau=0$ in Theorem 1.6, we easily obtain the following.

Corollary 1.1. *If $L\in\mathcal{S}_o^n\backslash\mathcal{S}_{os}^n$ and $p\geq1$, then there exists $K\in\mathcal{S}_{os}^n$ such that $\Gamma_pK\subset\Gamma_pL$, but*

$$V(K)>V(L).$$

Corollary 1.1 shows a negation form of the Busemann-Petty type problem for the L_p-centroid bodies. Actually, we extend the scope of negation solutions in Corollary 1.1 from $K\in\mathcal{S}_{os}^n$ to $K\in\mathcal{S}_o^n$ as follows:

Theorem 1.7. *If $L \in \mathcal{S}_o^n \backslash \mathcal{S}_{os}^n$ and $p \geq 1$, then there exists $K \in \mathcal{S}_o^n$ such that $\Gamma_p K \subset \Gamma_p L$, but*

$$V(K) > V(L).$$

Finally, we give another negation form of the Busemann-Petty type problem for the L_p-centroid bodies, it is the L_p-analogues of Theorem 1.2.

Theorem 1.8. *For $p \geq 1$. If $K \in \mathcal{S}_{os}^n \backslash \mathcal{P}_p^*$ is infinite smooth and p is not an even integer, then there exists $L \in \mathcal{S}_{os}^n \backslash \mathcal{P}_p^*$ is infinite smooth, such that $\Gamma_p K \subset \Gamma_p L$, but*

$$V(K) > V(L).$$

2 Some notions

2.1 Support function, radial function and polar body

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \longrightarrow (-\infty, +\infty)$, is defined by (see [6])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y. From the definition of support function, we easily know that for $c > 0$, $h(cK, \cdot) = ch(K, \cdot)$, and $h(K, \cdot) = h(L, \cdot)$ if and only if $K = L$.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0, +\infty)$, is defined by (see [6])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If E is a nonempty set in $\mathbb{R}^n$, the polar set of E, E^*, is defined by (see [6])

$$E^* = \{x : x \cdot y \leq 1, \ y \in E\}, \qquad x \in \mathbb{R}^n. \tag{2.1}$$

From (2.1), we easily know that (see [6]) for $K \in \mathcal{K}_o^n$,

$$h(K, \cdot) = \frac{1}{\rho(K^*, \cdot)}. \tag{2.2}$$

2.2 L_p-mixed volumes and L_p-dual mixed volumes

In 1993, Lutwak [12] defined the L_p-mixed volumes as follows: For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of K and L is given by

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K, u). \tag{2.3}$$

The measure $S_p(K,\cdot)$ is called the L_p-surface area measure.

Whereafter, Lutwak [13] introduced the L_p-dual mixed volumes: For $K,L\in\mathcal{S}_o^n$ and $p\geq 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K,L)$, of K and L is given by

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n+p}(u)\rho_L^{-p}(u)du. \tag{2.4}$$

From (2.4), it follows immediately that for each $K\in\mathcal{S}_o^n$ and $p\geq 1$,

$$\widetilde{V}_{-p}(K,K)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n}(u)du=V(K). \tag{2.5}$$

The L_p-dual Minkowski inequality can be stated that (see [13]): if $K,L\in\mathcal{S}_o^n$ and $p\geq 1$, then

$$\widetilde{V}_{-p}(K,L)\geq V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}}, \tag{2.6}$$

with equality if and only if K and L are dilates.

2.3 General L_p-harmonic Blaschke bodies

For $K,L\in\mathcal{S}_o^n$, $p\geq 1$, $\lambda,\mu\geq 0$ (not both zero), the L_p-harmonic Blaschke combination, $\lambda\circ K\tilde{+}_p\mu\circ L$, of K and L is given by (see [3])

$$\frac{\rho(\lambda\circ K\tilde{+}_p\mu\circ L,\cdot)^{n+p}}{V(\lambda\circ K\tilde{+}_p\mu\circ L)}=\lambda\frac{\rho(K,\cdot)^{n+p}}{V(K)}+\mu\frac{\rho(L,\cdot)^{n+p}}{V(L)}. \tag{2.7a}$$

Let $\lambda=\mu=1/2$ and $L=-K$ in (2.7a), then the L_p-harmonic Blaschke body, $\widetilde{\nabla}_pK$, of $K\in\mathcal{S}_o^n$ is written by

$$\widetilde{\nabla}_pK=\frac{1}{2}\circ K\tilde{+}_p\frac{1}{2}\circ(-K).$$

Feng and Wang [4] defined the general L_p-harmonic Blaschke bodies as follows: For $K\in\mathcal{S}_o^n$, $p\geq 1$ and $\tau\in[-1,1]$, the general L_p-harmonic Blaschke body,

$$\widetilde{\nabla}_p^\tau K=f_1(\tau)\circ K\tilde{+}_pf_2(\tau)\circ(-K)$$

of K is defined by

$$\frac{\rho(\widetilde{\nabla}_p^\tau K,\cdot)^{n+p}}{V(\widetilde{\nabla}_p^\tau K)}=f_1(\tau)\frac{\rho(K,\cdot)^{n+p}}{V(K)}+f_2(\tau)\frac{\rho(-K,\cdot)^{n+p}}{V(-K)}. \tag{2.7b}$$

Here $f_1(\tau)$ and $f_2(\tau)$ satisfy (1.8). Obviously, if $\tau=0$, then

$$\widetilde{\nabla}_p^\tau K=\widetilde{\nabla}_pK.$$

In addition, if $\tau=\pm1$ we write

$$\widetilde{\nabla}_p^\tau K=\widetilde{\nabla}_p^\pm K,$$

then

$$\widetilde{\nabla}_p^+K=K,\qquad \widetilde{\nabla}_p^-K=-K.$$

2.4 General L_p-projection bodies and L_p-cosine transformations

In 2005, Ludwig [9] introduced the notion of general L_p-projection body as follows: for $K \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$, the general L_p-projection body, $\Pi_p^\tau K \in \mathcal{K}_o^n$, of K whose support function is given by

$$h^p_{\Pi_p^\tau K}(u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p dS_p(K,v), \tag{2.8}$$

where $S_p(K,\cdot)$ is the L_p-surface area measure of K, $\varphi_\tau(\cdot)$ is given by (1.4),

$$\alpha_{n,p}(\tau) = \frac{2}{(n+p)c_{n,p}\omega_n[(1+\tau)^p + (1-\tau)^p]} \tag{2.9}$$

and $c_{n,p}$ satisfies (1.3). For the general L_p-projection bodies, some works have made in [19, 20,25,26].

If $\tau = 0$, then (2.8) and (2.9) yield the following L_p-projection body $\Pi_p K$ of K, i.e.,

$$h^p_{\Pi_p K}(u) = \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |u \cdot v|^p dS_p(K,v), \tag{2.10}$$

which is defined by Lutwak, Yang and Zhang (see [14]).

If $K \in \mathcal{K}_o^n$ has L_p-curvature function $f_p(K,v) : S^{n-1} \to \mathbb{R}$, then we have (see [13])

$$dS_p(K,v) = f_p(K,v)dv,$$

where dv is the standard spherical Lebesgue measure on S^{n-1}. From this, if $K \in \mathcal{K}_o^n$ has L_p-curvature function, then (2.10) can be written as

$$h^p_{\Pi_p K}(u) = \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |u \cdot v|^p f_p(K,v)dv. \tag{2.11}$$

Let $C(S^{n-1})$ denote the set of all continuous functions on S^{n-1}. For $p \geq 1$ and function $\varphi \in C(S^{n-1})$, the L_p-cosine transformation, $C_p\varphi$, of φ is defined by (see [6])

$$C_p\varphi(u) = \int_{S^{n-1}} |u \cdot v|^p \varphi(v)dv, \quad u \in S^{n-1}. \tag{2.12}$$

For the L_p-cosine transformation, also see [6,14].

From (2.11) and (2.12), we easily see that for $K \in \mathcal{K}_o^n$ has L_p-curvature function and all $u \in S^{n-1}$,

$$h^p_{\Pi_p K}(u) = \frac{1}{(n+p)c_{n,p}\omega_n} C_p f_p(K,u). \tag{2.13}$$

In addition, according to (2.12) and (1.2), we have that for all $u \in S^{n-1}$,

$$h^p_{\Gamma_p K}(u) = \frac{1}{(n+p)c_{n,p}V(K)} C_p \rho_K^{n+p}(u). \tag{2.14}$$

If $F,G\in C(S^{n-1})$, write

$$(F,G)=\frac{1}{n}\int_{S^{n-1}}F(u)G(u)du, \tag{2.15}$$

then by (2.12), we have

$$(C_pf,g)=(f,C_pg)=\frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}}|u\cdot v|^pf(v)g(u)dudv. \tag{2.16}$$

For the L_p-cosine transformation C_p, we know the following fact (see [6]).

Theorem 2.1. *If $p\geq 1$, then $C_p:C_e(S^{n-1})\to C_e(S^{n-1})$ is injective if and only if p is not an even integer. Here $C_e(S^{n-1})$ denotes the set of all even continuous functions on S^{n-1}.*

3 Busemann-Petty type problem for the general L_p-centroid bodies

In the section, we will research Busemann-Petty type problem for the general L_p-centroid bodies. Associated with the general L_p-projection bodies and general L_p-centroid bodies, Feng, Wang and Lu [5] gave that

Lemma 3.1. *If $K\in\mathcal{K}_o^n$, $L\in\mathcal{S}_o^n$, $p\geq 1$ and $\tau\in[-1,1]$, then*

$$\frac{V_p(K,\Gamma_p^\tau L)}{\omega_n}=\frac{\widetilde{V}_{-p}(L,\Pi_p^{\tau,*}K)}{V(L)}. \tag{3.1}$$

Here $\Pi_p^{\tau,*}K$ denotes the polar of general L_p-projection body $\Pi_p^\tau K$.

According to Lemma 3.1, we give an extension of Theorem 1.5 as follows:

Theorem 3.1. *For $K,L\in\mathcal{S}_o^n$, $p\geq 1$ and $\tau\in[-1,1]$, if $\Gamma_p^\tau K\subseteq\Gamma_p^\tau L$, then for any $Q\in\mathcal{P}_p^{\tau,*}$,*

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)}\leq\frac{\widetilde{V}_{-p}(L,Q)}{V(L)}, \tag{3.2}$$

with equality in (3.2) *if and only if $K=L$.*

Proof. Since $Q\in\mathcal{P}_p^{\tau,*}$, thus there exists $R\in\mathcal{K}_o^n$ such that $Q=\Pi_p^{\tau,*}R$, by (2.3) and (3.1), we get

$$\begin{aligned}\frac{\widetilde{V}_{-p}(L,Q)/V(L)}{\widetilde{V}_{-p}(K,Q)/V(K)}&=\frac{\widetilde{V}_{-p}(L,\Pi_p^{\tau,*}R)/V(L)}{\widetilde{V}_{-p}(K,\Pi_p^{\tau,*}R)/V(K)}=\frac{V_p(R,\Gamma_p^\tau L)}{V_p(R,\Gamma_p^\tau K)}\\&=\frac{\int_{S^{n-1}}h(\Gamma_p^\tau L,u)^pdS_p(R,u)}{\int_{S^{n-1}}h(\Gamma_p^\tau K,u)^pdS_p(R,u)}.\end{aligned}$$

From this, if $\Gamma_p^\tau K\subseteq\Gamma_p^\tau L$, then (3.2) is obtained.

Obviously, by L_p-dual Minkowski inequality (2.6), we know that equality holds in (3.2) if and only if $K=L$. ☐

Note that the case $\tau = 0$ of Theorem 3.1 was given by Grinberg and Zhang [7].

Proof of Theorem 1.5. Since $L \in \mathcal{P}_p^{\tau,*}$, thus taking $Q = L$ in Theorem 3.1, and combining with (2.5) and inequality (2.6), we get

$$V(K) \geq \widetilde{V}_{-p}(K,L) \geq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}},$$

i.e., $V(K) \leq V(L)$.

According to the equality condition of (3.1), we see that $V(K) = V(L)$ if and only if $K = L$. □

The proof of Theorem 1.6 requires the following two lemmas.

Lemma 3.2. *If $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$, then*

$$V(\widetilde{\nabla}_p^\tau K) \geq V(K). \tag{3.3}$$

For $\tau \in (-1,1)$, equality holds if and only if K is origin-symmetric. For $\tau = \pm 1$, (3.3) *becomes an equality.*

Proof. From (2.7b) and (2.4), we have that for any $Q \in \mathcal{S}_o^n$,

$$\frac{\widetilde{V}_{-p}(\widetilde{\nabla}_p^\tau K, Q)}{V(\widetilde{\nabla}_p^\tau K)} = f_1(\tau)\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} + f_2(\tau)\frac{\widetilde{V}_{-p}(-K,Q)}{V(-K)}.$$

This together with inequality (2.6) and equality (1.9b) yields

$$\begin{aligned}\frac{\widetilde{V}_{-p}(\widetilde{\nabla}_p^\tau K, Q)}{V(\widetilde{\nabla}_p^\tau K)} &\geq f_1(\tau)V(K)^{\frac{p}{n}}V(Q)^{-\frac{p}{n}} + f_2(\tau)V(K)^{\frac{p}{n}}V(Q)^{-\frac{p}{n}} \\ &= V(K)^{\frac{p}{n}}V(Q)^{-\frac{p}{n}}.\end{aligned}$$

Let $Q = \widetilde{\nabla}_p^\tau K$ in above inequality and use (2.5), we obtain

$$V(\widetilde{\nabla}_p^\tau K) \geq V(K).$$

For $\tau \in (-1,1)$, according to the equality condition of inequality (2.6), we see that equality holds in (3.3) if and only if K and $\widetilde{\nabla}_p^\tau K$, $-K$ and $\widetilde{\nabla}_p^\tau K$ both are dilates, i.e., K and $-K$ are dilates. This means that K is origin-symmetric. For $\tau = \pm 1$, by $\widetilde{\nabla}_p^+ K = K$ and $\widetilde{\nabla}_p^- K = -K$, we know that (3.3) becomes an equality. □

Lemma 3.3. *If $K \in \mathcal{S}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$, then*

$$\Gamma_p^+ \widetilde{\nabla}_p^\tau K = \Gamma_p^\tau K, \tag{3.4a}$$

$$\Gamma_p^- \widetilde{\nabla}_p^\tau K = \Gamma_p^{-\tau} K. \tag{3.4b}$$

Proof. By (1.6), (1.9b) and (2.7b), we have that for all $u \in S^{n-1}$,

$$\begin{aligned}
h^p_{\Gamma_p^+\widetilde{\nabla}_p^\tau K}(u) =&\frac{2}{c_{n,p}(n+p)V(\widetilde{\nabla}_p^\tau K)}\int_{S^{n-1}}(u\cdot v)_+^p\rho_{\widetilde{\nabla}_p^\tau K}(v)^{n+p}dv\\
=&\frac{2}{c_{n,p}(n+p)}\int_{S^{n-1}}(u\cdot v)_+^p\left[f_1(\tau)\frac{\rho_K(v)^{n+p}}{V(K)}+f_2(\tau)\frac{\rho_{-K}(v)^{n+p}}{V(-K)}\right]dv\\
=&f_1(\tau)h^p_{\Gamma_p^+K}(u)+f_2(\tau)h^p_{\Gamma_p^+(-K)}(u)\\
=&f_1(\tau)h^p_{\Gamma_p^+K}(u)+f_2(\tau)h^p_{\Gamma_p^-K}(u)=h^p_{\Gamma_p^\tau K}(u).
\end{aligned}$$

This immediately gives (3.4a). Similarly, we know that for all $u \in S^{n-1}$,

$$h^p_{\Gamma_p^-\widetilde{\nabla}_p^\tau K}(u)=h^p_{\Gamma_p^{-\tau}K}(u).$$

This yields (3.4b). $\square$

Proof of Theorem 1.6. Since L is not origin-symmetric and $\tau \in (-1,1)$, thus by Lemma 3.2, we know $V(\widetilde{\nabla}_p^\tau L) > V(L)$. From this, choose $0<\varepsilon<1$ such that $K=(1-\varepsilon)\widetilde{\nabla}_p^\tau L$ (for $\tau=0$, $K\in\mathcal{S}^n_{os}$; for $\tau\neq 0$, $K\in\mathcal{S}^n_o$) satisfies

$$V(K)=V((1-\varepsilon)\widetilde{\nabla}_p^\tau L)>V(L).$$

But by (3.4a), (3.4b) and notice that $\Gamma_p^\pm cK = c\Gamma_p^\pm K$ $(c>0)$, we orderly have

$$\begin{aligned}
\Gamma_p^+K&=\Gamma_p^+(1-\varepsilon)\widetilde{\nabla}_p^\tau L=(1-\varepsilon)\Gamma_p^+\widetilde{\nabla}_p^\tau L=(1-\varepsilon)\Gamma_p^\tau L\subset\Gamma_p^\tau L,\\
\Gamma_p^-K&=\Gamma_p^-(1-\varepsilon)\widetilde{\nabla}_p^\tau L=(1-\varepsilon)\Gamma_p^-\widetilde{\nabla}_p^\tau L=(1-\varepsilon)\Gamma_p^{-\tau} L\subset\Gamma_p^{-\tau} L.
\end{aligned}$$

Notice that $\tau\in(-1,1)$ is equivalent to $-\tau\in(-1,1)$, this means that $\Gamma_p^+K\subset\Gamma_p^\tau L$ and $\Gamma_p^-K\subset\Gamma_p^{-\tau}L$ imply $\Gamma_p^+K\subset\Gamma_p^\tau L$ and $\Gamma_p^-K\subset\Gamma_p^\tau L$ for any $\tau\in(-1,1)$, respectively. Hence, together with (1.7) and (1.9b), we easily obtain that for all $u\in S^{n-1}$,

$$\begin{aligned}
h(\Gamma_p^\tau K,u)^p=&f_1(\tau)h(\Gamma_p^+K,u)^p+f_2(\tau)h(\Gamma_p^-K,u)^p\\
<&f_1(\tau)h(\Gamma_p^\tau L,u)^p+f_2(\tau)h(\Gamma_p^\tau L,u)^p\\
=&h(\Gamma_p^\tau L,u)^p,
\end{aligned}$$

i.e.,

$$\Gamma_p^\tau K\subset\Gamma_p^\tau L.$$

This completes the proof. $\square$

In order to prove Theorem 1.7, we require the following a lemma.

Lemma 3.4. *If $K\in\mathcal{S}^n_o$, $p\geq 1$, $\tau\in[-1,1]$, then*

$$\Gamma_p\widetilde{\nabla}_p^\tau K=\Gamma_pK. \tag{3.5}$$

Proof. From (1.7), (1.9a), (1.9b) and (1.10), we obtain that for $K \in \mathcal{S}_o^n$ and all $u \in S^{n-1}$,

$$\begin{aligned}
&\frac{1}{2}h^p_{\Gamma_p^\tau K}(u) + \frac{1}{2}h^p_{\Gamma_p^{-\tau} K}(u)\\
=&\frac{1}{2}\left[f_1(\tau)h^p_{\Gamma_p^+ K}(u) + f_2(\tau)h^p_{\Gamma_p^- K}(u)\right] + \frac{1}{2}\left[f_1(-\tau)h^p_{\Gamma_p^+ K}(u) + f_2(-\tau)h^p_{\Gamma_p^- K}(u)\right]\\
=&\frac{1}{2}\left[f_1(\tau)h^p_{\Gamma_p^+ K}(u) + f_2(\tau)h^p_{\Gamma_p^- K}(u)\right] + \frac{1}{2}\left[f_2(\tau)h^p_{\Gamma_p^+ K}(u) + f_1(\tau)h^p_{\Gamma_p^- K}(u)\right]\\
=&\frac{1}{2}h^p_{\Gamma_p^+ K}(u) + \frac{1}{2}h^p_{\Gamma_p^- K}(u) = h^p_{\Gamma_p K}(u).
\end{aligned} \tag{3.6}$$

Thus, by (1.10), (3.4a), (3.4b) and (3.6), we have that for all $u \in S^{n-1}$,

$$\begin{aligned}
h^p_{\Gamma_p \widetilde{\nabla}_p^\tau K}(u) =&\frac{1}{2}h^p_{\Gamma_p^+ \widetilde{\nabla}_p^\tau K}(u) + \frac{1}{2}h^p_{\Gamma_p^- \widetilde{\nabla}_p^\tau K}(u)\\
=&\frac{1}{2}h^p_{\Gamma_p^\tau K}(u) + \frac{1}{2}h^p_{\Gamma_p^{-\tau} K}(u) = h^p_{\Gamma_p K}(u).
\end{aligned}$$

So (3.5) is obtained. □

Proof of Theorem 1.7. Since L is not origin-symmetric and $\tau \in (-1,1)$, thus by Lemma 3.2, we know $V(\widetilde{\nabla}_p^\tau L) > V(L)$. From this, choose $0 < \varepsilon < 1$ such that

$$V((1-\varepsilon)\widetilde{\nabla}_p^\tau L) > V(L).$$

Let $K = (1-\varepsilon)\widetilde{\nabla}_p^\tau L$, then $K \in \mathcal{S}_o^n$ and $V(K) > V(L)$. But by (3.5) and notice that $\Gamma_p cK = c\Gamma_p K$ $(c > 0)$, we have

$$\Gamma_p K = \Gamma_p (1-\varepsilon)\widetilde{\nabla}_p^\tau L = (1-\varepsilon)\Gamma_p \widetilde{\nabla}_p^\tau L = (1-\varepsilon)\Gamma_p L \subset \Gamma_p L.$$

This completes the proof. □

Finally, we give the proof of Theorem 1.8.

Proof of Theorem 1.8. Let $C_e^\infty(S^{n-1})$ denote the set of all even and infinite smooth functions on S^{n-1}. Because of $K \in \mathcal{S}_{os}^n \backslash \mathcal{P}_p^*$ is infinite smooth, thus $\rho_K \in C_e^\infty(S^{n-1})$. By Theorem 2.1, we know that there exists $\varphi \in C_e^\infty(S^{n-1})$ when $p \geq 1$ and p is not even integer, such that $\rho_K^{-p} = C_p\varphi$. Since L is not the polar of L_p-projection body, hence function $\varphi < 0$. Otherwise, if $\varphi \geq 0$ and notice $\varphi \in C_e^\infty(S^{n-1})$, it follows from Minkowski's existence theorem that there exists a body $Q \in \mathcal{K}_{os}^n$ has L_p-curvature function such that

$$\varphi = [c_{n,p}(n+p)\omega_n]^{-1} f_p(Q,u) \quad \text{for } u \in S^{n-1}.$$

From this, we know that

$$C_p\varphi = [c_{n,p}(n+p)\omega_n]^{-1} C_p f_p(Q,u),$$

this together with (2.13) yields

$$\rho_K^{-p} = h^p_{\Pi_p Q},$$

this and (2.2) give $K=\Pi_p^* Q$. But $K \notin \mathcal{P}_p^*$, this leads to contradiction.

Therefore, choose $F \in C_e^\infty(S^{n-1})$ and is not identically zero, such that $F \le 0$ when $\varphi < 0$; $F = 0$ when $\varphi \ge 0$. From this, we have

$$(F,\varphi) = \frac{1}{n}\int_{S^{n-1}} F(v)\varphi(v)dv > 0. \tag{3.7}$$

And according to $F \in C_e^\infty(S^{n-1})$ and notice p is not an even integer, then by Theorem 2.1, we know that there exists $g \in C_e^\infty(S^{n-1})$, such that $F = C_p g$. Because of $\rho_K > 0$ ($K \in \mathcal{S}^n_{os}$), thus there exists $\varepsilon > 0$, such that

$$[(n+p)c_{n,p}V(K)]^{-1}\rho_K^{n+p} - \varepsilon g > 0.$$

Notice that

$$[(n+p)c_{n,p}V(K)]^{-1}\rho_K^{n+p} - \varepsilon g \in C_e^\infty(S^{n-1}),$$

then there exist $\mu > 0$ and $L \in \mathcal{S}^n_{os}$ is infinite smooth, such that

$$\mu\rho_L^{n+p} = [(n+p)c_{n,p}V(K)]^{-1}\rho_K^{n+p} - \varepsilon g.$$

This yields

$$\mu(n+p)c_{n,p}V(L)\frac{C_p\rho_L^{n+p}}{(n+p)c_{n,p}V(L)} = \frac{C_p\rho_K^{n+p}}{(n+p)c_{n,p}V(K)} - \varepsilon C_p g.$$

Thus, let $\mu(n+p)c_{n,p}V(L) = 1$ and together with (2.14), we obtain

$$h^p_{\Gamma_p L} = h^p_{\Gamma_p K} - \varepsilon F.$$

Since $F \le 0$ and $p \ge 1$, it follows that $\Gamma_p K \subset \Gamma_p L$. But by (2.4), (2.5), (2.15), (2.16) and (3.4a), we have

$$\begin{aligned}
&V(K) - \widetilde{V}_{-p}(L,K) = \widetilde{V}_{-p}(K,K) - \widetilde{V}_{-p}(L,K)\\
=&(\rho_K^{n+p},\rho_K^{-p}) - (\rho_L^{n+p},\rho_K^{-p}) = (\rho_K^{n+p} - \rho_L^{n+p},\rho_K^{-p})\\
=&(\rho_K^{n+p} - \rho_L^{n+p}, C_p\varphi) = (C_p\rho_K^{n+p} - C_p\rho_L^{n+p},\varphi)\\
=&(h^p_{\Gamma_p K} - h^p_{\Gamma_p L},\varphi) = (\varepsilon F,\varphi) = \varepsilon(F,\varphi) > 0.
\end{aligned}$$

This and inequality (2.6) yield

$$V(K) > \widetilde{V}_{-p}(L,K) \ge V(L)^{\frac{n+p}{n}} V(K)^{-\frac{p}{n}},$$

i.e., $V(K) > V(L)$. Clearly, by Theorem 1.3, we see $L \notin \mathcal{P}_p^*$. $\square$

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