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Existence of Solution for a General Class of Strongly Nonlinear Elliptic Problems Having Natural Growth Terms and L^1-Data

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Abstract. This paper is concerned with the existence of solution for a general class of strongly nonlinear elliptic problems associated with the differential inclusion

$$\beta(u) + A(u) + g(x, u, Du) \ni f,$$

where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual, β maximal monotone mapping such that $0 \in \beta(0)$, while $g(x, s, \xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s but it satisfies a sign-condition on s. The right hand side f is assumed to belong to $L^1(\Omega)$.

Key Words: Sobolev spaces, Leray-Lions operator, trunctions, maximal monotone graphe.

AMS Subject Classifications: 35J15, 35J20, 35J60

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. Our aim is to show existence of solutions for the following strongly nonlinear elliptic inclusion

$$(E, f) \quad \begin{cases} \beta(u) + A(u) + g(x, u, Du) \ni f & \text{in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega), \quad g(x, u, Du) \in L^1(\Omega), \end{cases}$$

where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ $(1 < p < \infty)$ defined as $A(u) = -div(a(x, u, Du))$, β maximal montone mapping such that $0 \in \beta(0)$, g is a nonlinear lower term having "natural growth" (of order p) with respect to Du, with

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*Corresponding author. *Email addresses:* `youssef.akdim@usmba.ac.ma` (Y. Akdim), `morad.ouboufettal@usmba.ac.ma` (M. Ouboufettal) respect to u, we do not assume any growth restrictions, but it satisfies a "sign-condition" on s and $f \in L^1(\Omega)$.

It will turn out that, for each solution u, $g(x,u,Du)$ will be in $L^1(\Omega)$, but for each $v \in W_0^{1,p}(\Omega)$, $g(x,v,Dv)$ can be very odd, and does not necesserily belong to $W^{-1,p'}(\Omega)$.

Particular instances of problem (E,f) have been studied for $\beta = 0$, Boccardo, Gallouët and Murat in [6] have proved the existence of at least one solution for the problem. Let us point out that another work in this direction can be found in [4].

Another important work in the L^1-theory for p-Laplacian type equations is [3] where problem

$$\begin{cases} -div(a(x,Du)) + \beta(u) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [1], Y.Akdim and C.Allalou have proved the existence of renormalized solution for an elliptic problem type diffusion-convection in the framework of weighted variable exponent Sobolev spaces

$$(E) \quad \begin{cases} \beta(u) - div(a(x,Du) + F(u)) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We also refer to [10,13], For results on the existence of renormalized solutions of elliptic problems of type (E).

The present paper is organized as follows: in Section 2, we give basic assumptions on a, g, β and f. In Section 3, we study our main result, existence of solution to (E,f) for any L^1-data f. To prove the main result, we will introduce and solve, in Section 4, approximating problems for any L^∞-data f. The proof of main result is given in Section 5. The last section is devoted to an example for illustrating our abstract result.

2 Assumptions

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. Our aim is to show existence of solution to the strongly nonlinear elliptic inclusion problem with Dirichlet boundary conditions

$$(E,f) \quad \begin{cases} \beta(u) + A(u) + g(x,u,Du) \ni f & \text{in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega), & g(x,u,Du) \in L^1(\Omega), \end{cases}$$

with right-hand side $f \in L^1(\Omega)$. A is a non linear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) defined by

$$A(u) = -div(a(x,u,Du)),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function satisfaying the following assumptions:

Assumption (H_1)

$$a(x,s,\xi)\cdot\xi \geq \lambda|\xi|^p, \quad \text{where} \quad \lambda > 0, \tag{2.1a}$$

$$|a(x,s,\xi)| \leq \beta(k(x)+|s|^{p-1}+|\xi|^{p-1}), \quad \text{where} \quad k(x)\in L^{p'}(\Omega),\ k\geq 0,\ \beta>0, \tag{2.1b}$$

$$(a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta) > 0 \quad \text{for} \quad \xi \neq \eta \in \mathbb{R}^N. \tag{2.1c}$$

Moreover, $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is Carathéodory function such that

Assumption (H_2)

$$g(x,s,\xi)s \geq 0, \tag{2.2a}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x)+|\xi|^p), \tag{2.2b}$$

$$\text{there exist } \sigma > 0 \quad \text{and} \quad \gamma > 0 \quad \text{such that} \quad |g(x,s,\xi)| \geq \gamma|\xi|^p, \quad \text{when } |s| \geq \sigma, \tag{2.2c}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(x)$ a positive function wich is in $L^1(\Omega)$.

As to the nonlinearity β in the problem (E,f) we assume that $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0 \in \beta(0)$.

3 Notion of solutions and main results

Definition 3.1. *A weak solution to (E,f) is a pair of solution $(u,b) \in W_0^{1,p}(\Omega)\times L^1(\Omega)$ satisfying $b(x)\in\beta(u(x))$ a.e in Ω, $g(x,u,Du)\in L^1(\Omega)$ and*

$$b - div(a(x,u,Du)) + g(x,u,Du) = f \quad in \ \ D'(\Omega).$$

The main existence result is the following theorem:

Theorem 3.1. *Under the Assumptions (H_1)-(H_2) and $f \in L^1(\Omega)$ there exists a solution of (E,f) in the sense of Definition* 3.1.

Remark 3.1. We shall prove the existence of a solution in $W_0^{1,p}(\Omega)$, but it should be emphasized that for $\beta = 0$ and $g = 0$, the existence of u in such a space cannot expected, if $p \leq N$. In [5] the existence of a solution has been proved in $W_0^{1,q}(\Omega)$ for all $q < (N(p-1))/(N-1)$.

4 Result of existence where $f \in L^\infty$-data

To prove Theorem 3.1, we will introduce and solve approximating problems.

The next proposition will give us existence of solution $(u_n,b_n)\in W_0^{1,p}(\Omega)\times L^\infty(\Omega)$ of (E,f_n) for each $n\in\mathbb{N}$, where f_n is a sequence of L^∞-functions which converges strongly to f in $L^1(\Omega)$ and $|f_n| \leq |f|$.

Proposition 4.1. *Under the Assumptions* (H_1)-(H_2) *and* $f \in L^\infty(\Omega)$ *there exists a solution of* (E, f) *in the sens of Definition* 3.1.

Proof. Step 1: The approximation problem. From now on, we will use the standard truncation function T_k, $k \geq 0$, defined for all $s \in \mathbb{R}$ by $T_k(s) = \max\{-k, \min\{s, k\}\}$.

First we introduce the approximate problem

$$(E_\varepsilon, f) \quad \begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - div(a(x, u_\varepsilon, Du_\varepsilon)) + g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = f, \\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{cases}$$

where for each $\varepsilon \in]0; 1]$, $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of β, note that, for any $u \in W_0^{1,p}(\Omega)$ and $0 < \varepsilon \leq 1$ we have

$$\langle \beta_\varepsilon(u), u\rangle \geq 0, \quad |\beta_\varepsilon(u)| \leq \frac{1}{\varepsilon}|u| \quad \text{and} \quad \lim_{\varepsilon \to 0} \beta_\varepsilon(u) = \beta(u),$$

and where

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon|g(x, s, \xi|}$$

satisfies

$$g_\varepsilon(x, s, \xi)s \geq 0, \quad |g_\varepsilon(x, s, \xi)| \leq |g(x, s, \xi)| \quad \text{and} \quad |g_\varepsilon(x, s, \xi)| \leq \frac{1}{\varepsilon}.$$

Since

$$|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_{\varepsilon))}| \leq \frac{1}{\varepsilon^2}$$

and g_ε is bounded for any fixed $\varepsilon > 0$, there exists at least one solution u_ε of (E_ε, f) (cf. [11,12]), i.e., for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'}(\Omega)$ there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon)D\varphi + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\varphi = \langle f, \varphi\rangle \tag{4.1}$$

holds for all $\varphi \in W_0^{1,p}(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

Step 2: The priori estimats. Taking u_ε as a test function in (4.1), we obtain

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))u_\varepsilon + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon)Du_\varepsilon + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon = \int_\Omega f u_\varepsilon \tag{4.2}$$

as the first term on the left hand side is nonnegative and since g_ε verifies the sign condition, by (2.1a) we have

$$\lambda||u_\varepsilon||^p_{W_0^{1,p}(\Omega)} \leq C||f||_{L^\infty(\Omega)}||u_\varepsilon||_{W_0^{1,p}(\Omega)},$$

where C is a positive constant coming from the Hölder and Poincaré inequalities, then

$$||u_\varepsilon||_{W_0^{1,p}(\Omega)} \leq C_1. \tag{4.3}$$

Moreover, from (4.2) and (4.3), we infer that

$$0 \leq \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon \leq C_2. \tag{4.4}$$

For $\delta > 0$, we define $H_\delta^+ : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$H_\delta^+(r) = \begin{cases} 1, & \text{if } \ r > \delta, \\ \dfrac{r}{\delta}, & \text{if } \ 0 \leq r \leq \delta, \\ 0, & \text{if } \ r < 0. \end{cases}$$

Clearly, H_δ^+ is an approximation of $sign_0^+$. We use the test function

$$\varphi = H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)$$

in (4.1). Since β_ε monotone increasing with $\beta_\varepsilon(0) = 0$. Also by (2.1a)

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)(H_\delta^+)'(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)\beta_\varepsilon'(T_{\frac{1}{\varepsilon}}(u_\varepsilon))Du_\varepsilon \geq 0$$

and since g_ε verifies the sign condition

$$\int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k) \geq 0.$$

Consequently, we have

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k) \leq \int_\Omega (f - k)H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k).$$

Taking $\delta \longrightarrow 0$ yields

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)^+ \leq \int_\Omega (f - k)^+. \tag{4.5}$$

Similarly, one can show

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + k)^- \leq \int_\Omega (f + k)^-. \tag{4.6}$$

Combining (4.5) and (4.6) gives

$$\int_\Omega (|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| - k)^+ \leq \int_\Omega (|f| - k)^+. \tag{4.7}$$

Choosing $k > ||f||_\infty$, we obtain

$$||\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))||_\infty \leq ||f||_\infty. \tag{4.8}$$

Step 3: Basic convergence results. By (4.8), there exist $b \in L^\infty(\Omega)$ such that

$$\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} b \quad \text{in} \quad L^\infty(\Omega). \tag{4.9}$$

Since u_ε remains bounded in $W_0^{1,p}(\Omega)$, we can extract a subsequence, still denoted by u_ε, such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \quad &&\text{weakly in} \quad W_0^{1,p}(\Omega), \\ &u_\varepsilon \to u \quad &&\text{a.e in} \ \ \Omega. \end{aligned}$$

We already know that for any fixed $k \in \mathbb{R}^{*+}$

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^{1,p}(\Omega).$$

Our objective is to prove that

$$T_k(u_\varepsilon) \to T_k(u) \quad \text{strongly in} \ \ W_0^{1,p}(\Omega).$$

We shall use in (4.1) the test function

$$v_\varepsilon = \varphi(z_\varepsilon),$$

where

$$z_\varepsilon = T_k(u_\varepsilon) - T_k(u) \quad \text{and} \quad \varphi(s) = se^{\lambda s^2}.$$

We get

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon + \int_\Omega a(x,u_\varepsilon,Du_\varepsilon)Dv_\varepsilon + \int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)v_\varepsilon = \int_\Omega fv_\varepsilon. \tag{4.10}$$

From now on, we denote by $\eta^1(\varepsilon), \eta^2(\varepsilon), \cdots$, various sequences of real numbers which converge to zero when ε tends to zero.

Since v_ε converges to zero weakly$*$ in $L^\infty(\Omega)$, we have

$$\int_\Omega fv_\varepsilon \to 0,$$

this implies that

$$\eta^1(\varepsilon) = \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon + \int_\Omega a(x,u_\varepsilon,Du_\varepsilon)Dv_\varepsilon + \int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)v_\varepsilon \to 0.$$

Note that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon = \int_{\{|u_\varepsilon|\le k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon + \int_{\{|u_\varepsilon|>k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon,$$

the second term on the right hand is nonnegative. Also $\chi_{\{|u_\varepsilon|\le k\}}\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$ is uniformly bounded, together with the Lebesgue Dominated Convergence Theorem provides that

$$\int_{\{|u_\varepsilon|\le k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon \to 0.$$

This implies that

$$\int_\Omega a(x,u_\varepsilon,Du_\varepsilon)Dv_\varepsilon + \int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)v_\varepsilon \le \eta^2(\varepsilon).$$

Using same arguments in [6], we obatin

$$0 \le \int_\Omega [a(x,T_k(u_\varepsilon),DT_k(u_\varepsilon)) - a(x,T_k(u_\varepsilon),DT_k(u))]D(T_k(u_\varepsilon)-T_k(u)) \le \eta^3(\varepsilon).$$

Finally, a result in [7] (see also [9]) implies

$$T_k(u_\varepsilon) \to T_k(u) \quad \text{strongly in} \quad W_0^{1,p}(\Omega). \tag{4.11}$$

Step 4: Passing to the limit. In vertue of (4.11), we have for a subsequence

$$Du_\varepsilon \to Du \quad \text{a.e in} \quad \Omega,$$

which with

$$u_\varepsilon \to u \quad \text{a.e in} \quad \Omega,$$

yields, since $a(x,u_\varepsilon,Du_\varepsilon)$ is bounded in $(L^{p'}(\Omega))^N$

$$a(x,u_\varepsilon,Du_\varepsilon) \rightharpoonup a(x,u,Du) \quad \text{weakly in} \quad (L^{p'}(\Omega))^N, \tag{4.12}$$

as well as

$$g_\varepsilon(x,u_\varepsilon,Du_\varepsilon) \to g(x,u,Du) \quad \text{a.e in} \quad \Omega. \tag{4.13}$$

We now use the classical trick in order to prove that $g_\varepsilon(x,u_\varepsilon,\, Du_\varepsilon)$ is uniformly equi-integrable.

For any measurable subset E of Ω and for any $m \in \mathbb{R}^+$, we have

$$\begin{aligned}\int_E |g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)| &= \int_{E\cap\{|u_\varepsilon|\le m\}} |g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)| + \int_{E\cap\{|u_\varepsilon|>m\}} |g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)| \\ &\le \int_E |g_\varepsilon(x,T_m(u_\varepsilon),DT_m(u_\varepsilon))| + \frac{1}{m}\int_E g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon.\end{aligned}$$

Using (2.2b) and (4.4), we obtain

$$\int_E |g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)| \le b(m)\int_E (c(x)+|DT_m(u_\varepsilon)|^p)+\frac{C_2}{m},$$

since the sequence $(DT_m(u_\varepsilon))$ converge strongly in $(L^p(\Omega))^N$ the above inequality implies the equi-integrability of $g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$.

In view of (4.13), we thus have

$$g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\to g(x,u,Du) \quad \text{strongly in} \quad L^1(\Omega). \tag{4.14}$$

From (4.12), (4.14) and (4.9), we can pass to the limit in (4.1)

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi+\int_\Omega a(x,u_\varepsilon,Du_\varepsilon)D\varphi+\int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi=\int_\Omega f\varphi,$$

we obtain

$$\begin{aligned}&\int_\Omega b\varphi+\int_\Omega a(x,u,Du)D\varphi+\int_\Omega g(x,u,Du)\varphi\\ =&\int_\Omega f\varphi \quad \text{for any} \quad \varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).\end{aligned} \tag{4.15}$$

Moreover, since $g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon\ge 0$ a.e in Ω and $g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon\to g(x,u,Du)u$ a.e in Ω and

$$0\le\int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon\le C,$$

by Fatou's lemma, we have

$$g(x,u,Du)u\in L^1(\Omega).$$

Step 5: Subdifferential argument. Since β is a maximal monotone graph, there exists a convex, l.s.c and proper function

$$j:\mathbb{R}\to[0,\infty] \quad \text{such that} \quad \beta(r)=\partial j(r) \quad \text{for all} \quad r\in\mathbb{R}.$$

According to [8], for $0<\varepsilon\le 1$, $j_\varepsilon:\mathbb{R}\to\mathbb{R}$ defined by

$$j_\varepsilon(r)=\int_0^r \beta_\varepsilon(s)ds$$

has the following properties as in [13]

i) For any $0<\varepsilon\le 1$, j_ε is convex and differentiable for all $r\in\mathbb{R}$, such that $j'_\varepsilon(r)=\beta_\varepsilon(r)$ for all $r\in\mathbb{R}$ and any $0<\varepsilon\le 1$

ii) $j_\varepsilon(r)\to j(r)$ for all $r\in\mathbb{R}$ as $\varepsilon\to 0$.

From i) it follows that for any $0 < \varepsilon \le 1$

$$j_\varepsilon(r) \ge j_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + (r - T_{\frac{1}{\varepsilon}}(u_\varepsilon))\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \tag{4.16}$$

holds for all $r \in \mathbb{R}$ and almost everywhere in Ω.

Let $E \subset \Omega$ be an arbitrary measurable set and χ_E its characteristic function. Let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_l(r) = \min(1, (l + 1 - |r|)^+)$$

for each $r \in \mathbb{R}$.

We fix $\varepsilon_0 > 0$, multiplying (4.16) by $h_l(u_\varepsilon)\chi_E$, integrating over Ω and using ii), we obtain

$$j(r)\int_E h_l(u_\varepsilon) \ge \int_E j_{\varepsilon_0}(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \tag{4.17}$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon < \min(\varepsilon_0, \frac{1}{l})$. As $\varepsilon \to 0$, taking into account that E arbitrary, we obtain from (4.17)

$$j(r)h_l(u) \ge j_{\varepsilon_0}(T_{l+1}(u))h_l(u) + bh_l(u)(r - T_{l+1}(u)) \tag{4.18}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω.

Passing to the limit with $l \to \infty$ and then with $\varepsilon_0 \to 0$ in (4.18) finally yields

$$j(r) \ge j(u(x)) + b(x)(r - u(x)) \tag{4.19}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω, hence $u \in D(\beta)$ and $b \in \beta(u)$ for almost everywhere in Ω.

With this last step the proof of Proposition 4.1 is concluded. □

5 Proof of Theorem 3.1

The proof of Theorem 3.1 will be divided into several steps.

5.1 The approximation problem

Consider the sequence of approximate equations

$$(E, f_n) \quad \begin{cases} \beta(u_n) - div(a(x, u_n, Du_n)) + g(x, u_n, Du_n) \ni f_n, \\ u_n \in W_0^{1,p}(\Omega), \end{cases}$$

where f_n is a sequence of L^∞-functions which converges strongly to f in $L^1(\Omega)$ and $|f_n| \le |f|$.

From Proposition 4.1, there exists a solution $(u_n, b_n) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ of (E, f_n) such that

$$\int_\Omega b_n\varphi + \int_\Omega a(x, u_n, Du_n)D\varphi + \int_\Omega g(x, u_n, Du_n)\varphi = \int_\Omega f_n\varphi \tag{5.1}$$

holds for all $n \in \mathbb{N}$ and $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

5.2 The priori estimats

Lemma 5.1. *For $n \in \mathbb{N}$ let $(u_n, b_n) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ be a solution of (E, f_n). Then, there exists a constant C, not depending on n, such that*

$$||u_n||_{W_0^{1,p}(\Omega)} \leq C, \tag{5.2}$$

and

$$||b_n||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}. \tag{5.3}$$

Proof. Taking $T_k(u_n)$ as a test function in (5.1), we obtain

$$\begin{aligned}&\int_\Omega b_n T_k(u_n) + \int_\Omega a(x, u_n, DT_k(u_n))DT_k(u_n) + \int_\Omega g(x, u_n, Du_n)T_k(u_n)\\ =&\int_\Omega f_n T_k(u_n)\end{aligned} \tag{5.4}$$

as the first term on the left hand side is nonnegative and since g verifies the sign condition, by (2.1a) we have

$$\lambda \int_\Omega |DT_k(u_n)|^p \leq \int_\Omega f_n T_k(u_n) \leq k||f||_{L^1(\Omega)}. \tag{5.5}$$

On the other hand, we have

$$k \int_{\{|u_n|>k\}} |g(x, u_n, Du_n)| \leq \int_\Omega |f_n||T_k(u_n)| \leq k||f||_{L^1(\Omega)}. \tag{5.6}$$

Hence from (2.2c), (5.5), (5.6) and for $k > \sigma$, we obtain

$$\begin{aligned}\int_\Omega |D(u_n)|^p &= \int_{\{|u_n|>k\}} |D(u_n)|^p + \int_\Omega |DT_k(u_n)|^p\\ &\leq \frac{1}{\gamma} \int_{\{|u_n|>k\}} |g(x, u_n, Du_n)| + \frac{k}{\lambda}||f||_{L^1(\Omega)}\\ &\leq \Big(\frac{1}{\gamma} + \frac{k}{\lambda}\Big)||f||_{L^1(\Omega)},\end{aligned}$$

then

$$||u_n||_{W_0^{1,p}(\Omega)} \leq C.$$

We neglect in (5.4) the positive terms

$$a(x, u_n, DT_k(u_n))DT_k(u_n), \quad g(x, u_n, Du_n)T_k(u_n),$$

and keep

$$\int_\Omega b_n T_k(u_n) \leq \int_\Omega f_n T_k(u_n) \leq k||f||_{L^1(\Omega)},$$

since $b_n \in \beta(u_n)$ a.e in Ω

$$\int_{\{|u_n|>k\}} |b_n| \leq ||f||_{L^1(\Omega)},$$

passing the limit as $k \downarrow 0$ and using the Fatou Lemma, we find

$$\int_\Omega |b_n| \leq ||f||_{L^1(\Omega)}.$$

Thus, we complete the proof. □

5.3 Basic convergence results

Lemma 5.2. *For $n \in \mathbb{N}$ let $(u_n, b_n) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega)$ be a solution of (E, f_n). We have*

$$b_n \rightharpoonup b \quad \textit{weakly in } \ L^1(\Omega), \tag{5.7a}$$

$$u_n \rightharpoonup u \quad \textit{weakly in } \ W_0^{1,p}(\Omega), \tag{5.7b}$$

$$u_n \to u \quad \textit{a.e in } \ \Omega, \tag{5.7c}$$

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in } \ W_0^{1,p}(\Omega). \tag{5.7d}$$

Proof. Let $(u_n^\varepsilon, b_n^\varepsilon)$ be a solution for the problem

$$\begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_n^\varepsilon)) - div(a(x, u_n^\varepsilon, Du_n^\varepsilon)) + g_\varepsilon(x, u_n^\varepsilon, Du_n^\varepsilon) = f_n, \\ u_\varepsilon \in W_0^{1,p}(\Omega). \end{cases}$$

By (4.7) we have

$$\int_\Omega (|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_n^\varepsilon))| - k)^+ \leq \int_\Omega (|f_n| - k)^+.$$

Using the fact that $\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_n^\varepsilon)) \overset{*}{\rightharpoonup} b_n$ in $L^\infty(\Omega)$ we get

$$\int_\Omega (|b_n| - k)^+ \leq \int_\Omega (|f_n| - k)^+. \tag{5.8}$$

The sequence b_n is weakly sequentially compact in $L^1(\Omega)$.

This follows from the following criterion for weak sequential compactness of subset F of $L^1(\mu)$ where μ is a finite measure

$$\lim_{k\to\infty} \sup_F \int_\Omega (|f| - k)^+ d\mu = 0.$$

Indeed, this condition is easily seen to be equivalent to the uniform integrability of the family F (that is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_A |f| d\mu < \varepsilon$ if $\mu(A) < \delta$) and this implies the weak sequential compactness.

Since we have (5.8) and f_n convergent in $L^1(\Omega)$ by assumption, implies that b_n is weakly precompact in $L^1(\Omega)$. Then

$$b_n \rightharpoonup b \quad \text{weakly in } \ L^1(\Omega).$$

From (5.2) we deduce that for a subsequence still indexed by n, (5.7c) hold as $n \to \infty$, where u is a measurable function defined on Ω.

We already know that for any fixed $k \in \mathbb{R}^{*+}$

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } \ W_0^{1,p}(\Omega).$$

Our objective is to prove that

$$T_k(u_n \to T_k(u) \quad \text{strongly in } \ W_0^{1,p}(\Omega).$$

We shall use in (5.1) the test function

$$v_n = \varphi(z_n),$$

where

$$z_n = T_k(u_n) - T_k(u) \quad \text{and} \quad \varphi(s) = se^{\lambda s^2}.$$

We get

$$\int_\Omega b_n v_n + \int_\Omega a(x,u_n,Du_n)Dv_n + \int_\Omega g(x,u_n,Du_n)v_n = \int_\Omega f_n v_n. \tag{5.9}$$

From now on, we denote by $\varepsilon_1(n), \varepsilon_2(n), \cdots$, various sequences of real numbers which converge to zero when $n \to \infty$.

Since v_n converges to zero weakly$*$ in $L^\infty(\Omega)$, f_n converges strongly to f in $L^1(\Omega)$,

$$\int_\Omega f_n v_n \to 0.$$

We have

$$\int_\Omega b_n v_n = \int_{\{|u_n|\le k\}} b_n v_n + \int_{\{|u_n|>k\}} b_n v_n.$$

Since $b_n \in \beta(u_n)$, the second term on the right hand is nonnegative. Also $\chi_{\{|u_n|\le k\}} b_n$ is uniformly bounded, together with the Lebesgue Dominated Convergence Theorem provides that

$$\int_{\{|u_n|\le k\}} b_n v_n \to 0.$$

This implies that

$$\int_\Omega a(x,u_n,Du_n)Dv_n + \int_\Omega g(x,u_n,Du_n)v_n \le \varepsilon_1(n).$$

Using same arguments in [6], we obtain

$$0 \le \int_\Omega [a(x,T_k(u_n),DT_k(u_n)) - a(x,T_k(u_n),DT_k(u))]D(T_k(u_n)-T_k(u))$$
$$\le \varepsilon_2(n),$$

then

$$\int_\Omega [a(x,T_k(u_n),DT_k(u_n)) - a(x,T_k(u_n),DT_k(u))]D(T_k(u_n)-T_k(u))$$
$$\longrightarrow \ 0 \quad \text{as} \quad n\to\infty.$$

Finally, a result in [7] (see also [9]) implies

$$T_k(u_n)\to T_k(u) \quad \text{strongly in} \quad W_0^{1,p}(\Omega).$$

Thus, we complete the proof. □

5.4 Passing to the limit

In vertue of (5.7d), we have for a subsequence

$$Du_n \to Du \quad \text{a.e in} \quad \Omega,$$

which with

$$u_n \to u \quad \text{a.e in} \quad \Omega$$

yields, since $a(x,u_n,Du_n)$ is bounded in $(L^{p'}(\Omega))^N$

$$a(x,u_n,Du_n) \rightharpoonup a(x,u,Du) \quad \text{weakly in} \quad (L^{p'}(\Omega))^N \tag{5.10}$$

as well as

$$g(x,u_n,Du_n) \to g(x,u,Du) \quad \text{a.e in} \quad \Omega. \tag{5.11}$$

We now use the classical trick in order to prove that $g(x,u_n,Du_n)$ is uniformly equi-integrable. For any measurable subset E of Ω and for any $m\in\mathbb{R}^+$, we have

$$\begin{aligned}\int_E |g(x,u_n,Du_n)| &= \int_{E\cap\{|u_n|\le m\}} |g(x,u_n,Du_n)| + \int_{E\cap\{|u_n|>m\}} |g(x,u_n,Du_n)|\\ &\le \int_E |g(x,T_m(u_n),DT_m(u_n))| + \int_{\{|u_n|>m\}} |g(x,u_n,Du_n)|.\end{aligned}$$

Using (2.2b), we obtain

$$\int_E |g(x,u_n,Du_n)| \le b(m)\int_E (c(x)+|DT_m(u_n)|^p) + \int_{\{|u_n|>m\}} |g(x,u_n,Du_n)|. \tag{5.12}$$

For fixed m, the first integral of the right hand side of (5.12) is small uniformly in n when the measure of E is small (due to $DT_m(u_n)$ converges strongly in $L^p(\Omega)$).

We now discuss the behaviour of the second integral of the right hand side of (5.12). We use in (5.1) the test function $S_m(u_n)$, where for $m>1$

$$\begin{cases} S_m(s)=0, & \text{if} \quad |s|\leq m-1, \\ S_m(s)=\dfrac{|s|}{s}, & \text{if} \quad |s|\geq m, \\ S'_m(s)=1, & \text{if} \quad m-1\leq |s|\leq m. \end{cases}$$

This yields

$$\int_\Omega b_n S_m(u_n)+\int_\Omega a(x,u_n,Du_n)Du_n S'_m(u_n)+\int_\Omega g(x,u_n,Du_n)S_m(u_n)=\int_\Omega f_n S_m(u_n).$$

Which implies

$$\int_{\{|u_n|>m\}}|g(x,u_n,Du_n)|\leq \int_{\{|u_n|>m-1\}}|f_n|$$

and thus

$$\limsup_{n\to\infty}\int_{\{|u_n|>m\}}|g(x,u_n,Du_n)|\leq \int_{\{|u|>m-1\}}|f|.$$

We have proved that the seconde terme of the right hand side of (5.12) is small, uniformly in n and in E, when m is sufficiently large.

This completes the proof of the uniforme equi-integrability of $g(x,u_n,Du_n)$.

In view of (5.11), we thus have

$$g(x,u_n,Du_n)\to g(x,u,Du) \quad \text{strongly in} \quad L^1(\Omega). \tag{5.13}$$

From (5.10), (5.13), we can pass to the limit in (4.1)

$$\int_\Omega b_n\varphi+\int_\Omega a(x,u_n,Du_n)D\varphi+\int_\Omega g(x,u_n,Du_n)\varphi=\int_\Omega f_n\varphi,$$

and we obtain

$$\begin{aligned} &\int_\Omega b\varphi+\int_\Omega a(x,u,Du)D\varphi+\int_\Omega g(x,u,Du)\varphi \\ =&\int_\Omega f\varphi \quad \text{for any} \quad \varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \end{aligned} \tag{5.14}$$

With this last step the proof of Theorem 3.1 is concluded. □

6 Example

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$. Let us consider the Carathéodory functions

$$a(x,s,\xi) = |\xi|^{p-2}\xi,$$
$$g(x,s,\xi) = \rho s|s|^r|\xi|^p, \quad \rho > 0, \quad r > 0,$$

and β the maximal monotone graph defined by

$$\beta(s) = (s-1)^+ - (s+1)^-.$$

It is easy to show that the Carathéodory function $a(x,s,\xi)$ satisfies the growth condition (2.1b), the coercivity (2.1a) and the strict monotonie condition (2.1c). Also the Carathéodory function $g(x,s,\xi)$ satisfies the conditions (2.2a), (2.2b) and (2.2c) with $|s| > \sigma = 1$ and $\gamma = \rho$.

Finally, the hypotheses of Theorem 3.1 are satisfied, therefore for all $f \in L^1(\Omega)$ the following problem

$$(E,f) \qquad \begin{cases} \beta(u) - \Delta_p(u) + g(x,u,Du) \ni f & \text{in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega), \quad g(x,u,Du) \in L^1(\Omega), \end{cases}$$

has at least one solution.

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