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A Priori Error Analysis of Mixed Virtual Element Methods for Optimal Control Problems Governed by Darcy Equation

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Abstract. A mixed virtual element discretization of optimal control problems governed by the Darcy equation with pointwise control constraint is investigated. A discrete scheme uses virtual element approximations of the state equation and a variational discretization of the control variable. A discrete first-order optimality system is obtained by the first-discretize-then-optimize approach. A priori error estimates of the state, adjoint, and control variables are derived. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65N30, 49J20

Key words: A priori error estimate, mixed virtual element method, optimal control problem.

1. Introduction

The aim of this work is to develop mixed virtual element approximations of optimal control problems governed by the Darcy equation. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with the Lipschitz boundary Γ. Besides, we also assume that Γ admits a disjoint partition $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ with open subsets Γ_1 and Γ_2 such that $|\Gamma_1|, |\Gamma_2| \neq 0$.

We consider an optimal control problem of the Darcy flow in a porous medium — viz.

$$\min_{u \in U_{ad}} J(\boldsymbol{p}, y, u) = \frac{1}{2}\int_\Omega (\boldsymbol{p} - \boldsymbol{p}_d)^2 \, d\Omega + \frac{1}{2}\int_\Omega (y - y_d)^2 \, d\Omega + \frac{\gamma}{2}\int_\Omega u^2 \, d\Omega \tag{1.1}$$

subject to the following conditions:

$$\mathrm{div}\boldsymbol{p} = f + u \quad \text{in } \Omega, \tag{1.2a}$$

$$\boldsymbol{p} = -\mathbb{K}\nabla y \quad \text{in } \Omega, \tag{1.2b}$$

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$$
\begin{aligned}
y &= 0 \quad \text{on } \Gamma_1, && (1.2\text{c})\\
\boldsymbol{p}\cdot\mathbf{n} &= 0 \quad \text{on } \Gamma_2, && (1.2\text{d})
\end{aligned}
$$

where $\boldsymbol{p}$ is the velocity, y the pressure, $J(\boldsymbol{p}, y, u)$ a convex cost functional, $\mathbb{K}$ the permeability of the medium, $\gamma > 0$ a regularization parameter, and $f \in L^2(\Omega)$ the source term. For the sake of simplicity, we assume that $\mathbb{K}$ is a constant matrix and let $\|\mathbb{K}^{-1}\|$ be the matrix norm of $\mathbb{K}^{-1}$. The ideal states $\boldsymbol{p}_d$ and y_d respectively belong to spaces $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^2$ and $L^2(\Omega)$, and the admissible control set U_{ad} is defined by

$$
U_{ad} = \left\{ u \in L^2(\Omega) : a \le u(x) \le b \text{ a.e. in } \Omega \right\}.
$$

PDEs-constrained optimal control problems play increasingly important role in physics, biology, medicine, and efficient numerical methods are the key to their successful applications. We note that there are numerous studies devoted to the development of numerical methods and algorithms for such problems. In particular, the works [10–12, 16] deal with finite element methods (FEMs) and [15, 21] with discontinuous Galerkin methods. In various practical applications the cost functional often contains the gradient of the state variable. Therefore, the accuracy of the gradient approximation of the state variable becomes an important issue. This enhances interest to mixed numerical methods, such as the mixed finite element method [6]. In optimal control problems, mixed numerical methods have been studied in various works. Thus for mixed finite element discretizations, a priori and a posteriori error estimates of elliptic distributed optimal control problems are investigated [7, 8], whereas the Dirichlet boundary optimal control problem and pointwise state constrained optimal control problem are discussed in [13, 14].

Recently, virtual element methods (VEMs) have attracted a considerable attention. Compared with FEMs, the VEMs can handle general polygonal and polyhedral grids. Such methods have been introduced in [1] in order to solve the elliptic problems. Mixed VEMs have been considered [5], where VEM discretizations of $H(\text{div})$-conforming vector fields are introduced. Subsequently, mixed VEMs have been applied to general linear second-order elliptic problems [2], Darcy and Brinkman equations [18], three-dimensional elliptic equations of mixed formulation [9], the Laplacian eigenvalue problem [17], and optimal control problem governed by elliptic equations [4, 19, 20].

In this paper, we apply a mixed virtual element method to the optimal control problem for the Darcy equation with pointwise control constraints. The state equation is approximated by a mixed virtual element method and the control variable is implicitly discretized. In order to guarantee the computability of the discrete scheme, a piecewise L^2 projection on the discrete state $\boldsymbol{p}_h$ is used in the cost functional. A discrete first-order optimality system is derived by using a first-discretize-then-optimize approach. A priori error estimates for state, adjoint state and control variables are deduced. Two numerical examples are given to illustrate the theoretical results.

The article is organized as follows. In Section 2, we recall auxiliary results related to mixed virtual element methods and the continuous first-order optimality condition in the optimal control problem for the Darcy equation. In Section 3, we consider a virtual element discrete formulation of the problem (1.1)-(1.2) and the discrete form of the first-order optimality condition. Section 4 presents error estimates for auxiliary problems and a priori error estimates for mixed virtual element approximations. The results of numerical tests aimed to verify the theoretical analysis are shown in Section 5. Finally, Section 6 contains our concluding remarks.

Throughout the paper we use the following notation. If E is a bounded domain and s an integer, then $|\cdot|_{s,E}$ and $\|\cdot\|_{s,E}$ refer to the standard $H^s(E)$ semi-norm and norm, respectively. The $L^2(E)$ scalar product and the norm are respectively denoted by $(\cdot,\cdot)_{0,E}$ and $\|\cdot\|_{0,E}$, and the same notation is used for the scalar product and norm in the vector space $\boldsymbol{L}^2(E)$. If E is a Lipschitz-continuous domain Ω, we usually omit the subscript. If $k \geq 0$ is an integer, we denote by $\mathbb{P}_k(E)$ the space of all polynomials on E of the degree at most k and by $\mathbf{P}_k(E)$ the vector space $(\mathbb{P}_k(E))^2$. Besides, the terms C with or without subscripts are generic constants, which do not depend on the mesh size.

2. Preliminary

In this section, we first recall the variational form of the state equation (1.2) and then define two virtual element spaces and projection operators. After that we review the continuous first-order optimality condition for the control problem (1.1)-(1.2).

We introduce the spaces

$$\begin{aligned} &\boldsymbol{H} := \boldsymbol{L}^2(\Omega), \quad Q := L^2(\Omega), \\ &\boldsymbol{V} := \left\{ \boldsymbol{p} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{p}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_2 \right\}, \end{aligned}$$

equipped with the norms

$$\begin{aligned} &\|\boldsymbol{p}\|_{\boldsymbol{H}}^2 = \int_\Omega |\boldsymbol{p}|^2\, d\Omega, \quad \|y\|_Q^2 = \int_\Omega |y|^2\, d\Omega, \\ &\|\boldsymbol{p}\|_{\boldsymbol{V}}^2 = \int_\Omega |\boldsymbol{p}|^2\, d\Omega + \int_\Omega |\mathrm{div}\boldsymbol{p}|^2\, d\Omega. \end{aligned}$$

The variational form of Eq. (1.2) is to find $(\boldsymbol{p}, y) \in \boldsymbol{V}\times Q$ such that

$$\begin{aligned} &a(\boldsymbol{p},\boldsymbol{v}) - b(\boldsymbol{v}, y) = 0 \quad \text{for all} \quad \boldsymbol{v}\in\boldsymbol{V}, \\ &b(\boldsymbol{p}, w) = (f+u, w) \quad \text{for all} \quad w \in Q, \end{aligned} \tag{2.1}$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bilinear forms defined by

$$\begin{aligned} &a(\boldsymbol{p},\boldsymbol{v}) := \int_\Omega \mathbb{K}^{-1}\boldsymbol{p}\cdot\boldsymbol{v}\, d\Omega, \quad \boldsymbol{p},\boldsymbol{v}\in\boldsymbol{V}, \\ &b(\boldsymbol{v}, y) := \int_\Omega \mathrm{div}\boldsymbol{v}\cdot y\, d\Omega, \quad \boldsymbol{v}\in\boldsymbol{V}, \quad y\in Q. \end{aligned}$$

It was shown in [6] that:

- The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous — i.e.

$$|a(\boldsymbol{p},\boldsymbol{v})| \le \|a\|\|\boldsymbol{p}\|_{\boldsymbol{H}}\|\boldsymbol{v}\|_{\boldsymbol{H}} \quad \text{for all} \quad \boldsymbol{p},\boldsymbol{v}\in\boldsymbol{H},$$
$$|b(\boldsymbol{v},y)| \le \|\mathrm{b}\|\|\boldsymbol{v}\|_V\|y\|_Q \quad \text{for all} \quad \boldsymbol{v}\in\boldsymbol{V}, \quad y\in Q,$$

  where $\|a\|$ depends on $\mathbb{K}$ and $\|b\|$ is actually equal to 1.

- For the bilinear form $a(\cdot,\cdot)$, there exists a constant $\alpha>0$ depending on $\mathbb{K}$ such that

$$a(\boldsymbol{p},\boldsymbol{p}) \ge \alpha\|\boldsymbol{p}\|^2_{\boldsymbol{H}} \quad \text{for all} \quad \boldsymbol{p}\in\boldsymbol{H}. \tag{2.2}$$

- The bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition — i.e. there exists a constant $\beta>0$ depending on Ω such that

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(\boldsymbol{v},y)}{\|\boldsymbol{v}\|_V} \ge \beta\|y\|_Q \quad \text{for all} \quad y\in Q.$$

Hence, the existence and uniqueness of a weak solution of Eq. (2.1) is the consequence of the Babůska-Brezzi theory.

To define a virtual element space we consider a mesh partition $\{\mathscr{T}_h\}$. The sequence $\{\mathscr{T}_h\}$ is the decomposition of Ω into open non-overlapping polygonal elements E. For each $E\in\mathscr{T}_h$, let h_E be the diameter of the element E, $h=\sup_{E\in\mathscr{T}_h} h_E$ the maximum size of the elements E, and ρ_e the length of the edge e of E. Furthermore, we assume that the mesh satisfies the following conditions.

Condition 2.1. There exists a positive constant $C_{\mathscr{T}}$ such that:

1. Every edge e of E satisfies the estimate $\rho_e \geqslant C_{\mathscr{T}} h_E$.
2. E is star-shaped with respect to every point of a ball of radius $C_{\mathscr{T}} h_E$.

According to [5], the virtual element spaces are introduced as follows:

$$\boldsymbol{V}_h := \left\{\boldsymbol{p}_h\in\boldsymbol{V} : \boldsymbol{p}_h|_E \in \boldsymbol{V}_h^E \ \text{ for all } \ E\in\mathscr{T}_h\right\},$$
$$Q_h := \left\{y_h\in Q : y_h|_E\in\mathbb{P}_{k-1}(E) \ \text{ for all } \ E\in\mathscr{T}_h\right\},$$

where

$$\boldsymbol{V}_h^E := \big\{\boldsymbol{p}_h\in\boldsymbol{H}(\mathrm{div};E)\cap\boldsymbol{H}(\mathrm{rot};E) : (\boldsymbol{p}_h\cdot\mathbf{n})|_e\in\mathbb{P}_k(e)$$
$$\text{for all} \quad e\in\partial E, (\mathrm{div}\boldsymbol{p}_h)|_E\in\mathbb{P}_{k-1}(E), (\mathrm{rot}\boldsymbol{p}_h)|_E\in\mathbb{P}_{k-1}(E)\big\}.$$

The degrees of freedom (d.o.f.) of $\boldsymbol{V}_h$ are defined as follows:

$$\text{Type 1.} \quad \int_e \boldsymbol{p}_h\cdot\mathbf{n}\,q\,ds \quad \text{for all} \quad q\in\mathscr{B}_k(e), \qquad e\in\partial E,$$
$$\text{Type 2.} \quad \int_E \boldsymbol{p}_h\cdot\nabla q\,d\Omega \quad \text{for all} \quad q\in\mathscr{B}_{k-1}(E)\setminus\{1\}, \quad E\in\mathscr{T}_h,$$
$$\text{Type 3.} \quad \int_E \mathrm{rot}\boldsymbol{p}_h\cdot q\,d\Omega \quad \text{for all} \quad q\in\mathscr{B}_{k-1}(E), \qquad E\in\mathscr{T}_h,$$

where

$$\mathscr{B}_k(e) = \left\{ \left(\frac{x - x_e}{h_e} \right)^j, \ 0 \le j \le k \right\}$$

is the set of normalized monomials on ∂E with the midpoint x_e and the length h_e, and

$$\mathscr{B}_{k-1}(E) = \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_E}{h_E} \right)^{\boldsymbol{s}}, \ 0 \le |\boldsymbol{s}| \le k-1 \right\}$$

is the set of $k(k+1)/2$ normalized monomials on E with the barycenter x_E and the non-negative multi-index $\boldsymbol{s} = (s_1, s_2)$, $|\boldsymbol{s}| = s_1 + s_2$, and $\boldsymbol{x}^{\boldsymbol{s}} = x_1^{s_1} x_2^{s_2}$.

According to [3], Type 3 d.o.f. can be replaced by

$$\int_E \boldsymbol{p}_h \cdot \boldsymbol{g} \, d\Omega \quad \text{for all} \quad \boldsymbol{g} \in \mathscr{G}_k^{\perp}, \quad E \in \mathscr{T}_h,$$

where $\mathscr{G}_k^{\perp}$ is the orthogonal complement of $\mathscr{G}_k = \nabla \mathbb{P}_{k+1}$ in $\mathbf{P}_k$. The new d.o.f. is mainly used in order to compute an L^2- projection in numerical examples. The L^2- projection are much more difficult to determine if using the original degrees of freedom. In the present paper, we consider lower order case $k = 1$.

To derive the error estimates, we also need an interpolant from [5]. More exactly, let $(\boldsymbol{p}_I, y_I) \in V_h \times Q_h$ the interpolant of $(\boldsymbol{p}, y)$ defined by

$$\int_E (y - y_I) q_0 \, d\Omega = 0 \quad \text{for all} \quad q_0 \in \mathbb{P}_0(E).$$

Note that we will write locally $y_I = \Pi^0 y$, where Π^0 is the L^2- projection operator on $\mathbb{P}_0(E)$ and globally $y_I = \Pi_{Q_h} y$, where Π_{Q_h} is the L^2- projection on Q_h.

For each $\boldsymbol{p} \in \boldsymbol{V}$ such that $\boldsymbol{p} \in \boldsymbol{H}(\text{div}; E) \cap \boldsymbol{L}^r(E)$ for an $r > 2$ and $\text{rot}\boldsymbol{p} \in L^1(E)$, we can define its interpolant $\boldsymbol{p}_I \in \boldsymbol{V}_h$ by

$$\begin{aligned}
&\int_e (\boldsymbol{p} - \boldsymbol{p}_I) \cdot \mathbf{n} q \, ds && \text{for all} \quad q \in \mathscr{B}_k(e), && e \in \partial E, \\
&\int_E (\boldsymbol{p} - \boldsymbol{p}_I) \cdot \nabla q \, d\Omega && \text{for all} \quad q \in \mathscr{B}_{k-1}(E) \setminus \{1\}, && E \in \mathscr{T}_h, \\
&\int_E \text{rot}(\boldsymbol{p} - \boldsymbol{p}_I) \cdot q \, d\Omega && \text{for all} \quad q \in \mathscr{B}_{k-1}(E), && E \in \mathscr{T}_h.
\end{aligned}$$

Obviously, if $q_0 \in \mathbb{P}_0(E)$, then the integration by parts gives

$$\int_E \text{div}(\boldsymbol{p} - \boldsymbol{p}_I) q_0 \, d\Omega = \int_{\partial E} (\boldsymbol{p} - \boldsymbol{p}_I) \cdot \mathbf{n} q_0 \, ds = 0,$$

and since $\text{div}\boldsymbol{p}_I \in \mathbb{P}_0(E)$, this implies the commutative property

$$\text{div}\boldsymbol{p}_I = \Pi^0(\text{div}\boldsymbol{p}).$$

Let us recall the following error estimates.

Lemma 2.1 (cf. Brezzi *et al.* [5]). *For each $E \in \mathcal{T}_h$, there exists a constant $C > 0$ depending only on the $C_{\mathcal{T}}$ such that*

$$\|\boldsymbol{p} - \boldsymbol{p}_I\|_{0,E} \le Ch_E^s |\boldsymbol{p}|_{s,E}, \quad 1 \le s \le 2,$$
$$\|y - y_I\|_{0,E} \le Ch_E^s |y|_{s,E}, \quad 0 \le s \le 1.$$

Definition 2.1. *The L^2-orthogonal projector $\boldsymbol{\Pi}_1^0 : \boldsymbol{V}_h^E \to \mathbf{P}_1(E)$ is defined by*

$$\left(\boldsymbol{\Pi}_1^0 \boldsymbol{v}_h - \boldsymbol{v}_h, \boldsymbol{q}\right)_{0,E} = 0 \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h^E, \quad \boldsymbol{q} \in \mathbf{P}_1(E).$$

The following approximation property holds for the above projection operator.

Lemma 2.2 (cf. Meng *et al.* [17]). *There exists a constant C depending only on the $C_{\mathcal{T}}$ such that for every $E \in \mathcal{T}_h$ and for each smooth enough function $\boldsymbol{w}$ on E the following estimate holds:*

$$\left\|\boldsymbol{w} - \boldsymbol{\Pi}_1^0 \boldsymbol{w}\right\|_{0,E} \le Ch_E^s |\boldsymbol{w}|_{s,E}, \quad 0 \le s \le 2.$$

Finally, for the optimal control problem (1.1)-(1.2), we consider the continuous first-order optimality condition

$$\begin{aligned} &a(\boldsymbol{p}, \boldsymbol{v}) - b(\boldsymbol{v}, y) = 0 && \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}, \\ &b(\boldsymbol{p}, w) = (f + u, w) && \text{for all} \quad w \in Q, \end{aligned} \tag{2.3}$$

$$a(\boldsymbol{q}, \boldsymbol{v}) + b(\boldsymbol{v}, z) = (\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}, \tag{2.4a}$$
$$b(\boldsymbol{q}, w) = -(y - y_d, w) \quad \text{for all} \quad w \in Q, \tag{2.4b}$$

$$(\gamma u + z, \tilde{u} - u) \ge 0 \quad \text{for all} \quad \tilde{u} \in U_{ad}, \tag{2.5}$$

where $\boldsymbol{q}$ and z are the adjoint state variables — cf. [7]. Let

$$\mathrm{P}_{U_{ad}}(u) = \max\left\{a, \min\{u, b\}\right\}$$

represent the pointwise projection onto the control set U_{ad}. Therefore, Eq. (2.5) can be simplified as

$$u = \mathrm{P}_{U_{ad}}\left(-\frac{1}{\gamma} z\right).$$

3. Virtual Element Approximation

3.1. Virtual element discrete scheme for state equation

The discrete scheme of the state equation is to find $(\boldsymbol{p}_h(u), y_h(u)) \in \boldsymbol{V}_h \times Q_h$ such that

$$\begin{aligned} &a_h\big(\boldsymbol{p}_h(u), \boldsymbol{v}_h\big) - b\big(\boldsymbol{v}_h, y_h(u)\big) = 0 && \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ &b\big(\boldsymbol{p}_h(u), w_h\big) = (f + u, w_h) && \text{for all} \quad w_h \in Q_h, \end{aligned} \tag{3.1}$$

where

$$
\begin{aligned}
a_h\big(\boldsymbol{p}_h(u),\boldsymbol{v}_h\big) &:= \sum_{E\in\mathcal{T}_h} a_h^E\big(\boldsymbol{p}_h(u),\boldsymbol{v}_h\big)\\
&= \sum_{E\in\mathcal{T}_h} \Big(a^E\big(\widehat{\boldsymbol{\Pi}}\boldsymbol{p}_h(u),\widehat{\boldsymbol{\Pi}}\boldsymbol{v}_h\big) + \|\mathbb{K}^{-1}\|S^E\big(\boldsymbol{p}_h(u)-\widehat{\boldsymbol{\Pi}}\boldsymbol{p}_h(u),\boldsymbol{v}_h-\widehat{\boldsymbol{\Pi}}\boldsymbol{v}_h\big)\Big).
\end{aligned}
$$

Here the projector $\widehat{\boldsymbol{\Pi}}$ is defined as follows.

Definition 3.1. *For each $E\in\mathcal{T}_h$, the projector $\widehat{\boldsymbol{\Pi}}:\boldsymbol{V}_h^E\to\widehat{\boldsymbol{V}}^E$ is defined by*

$$
a^E\big(\boldsymbol{v}_h-\widehat{\boldsymbol{\Pi}}\boldsymbol{v}_h,\widehat{\boldsymbol{w}}_h\big)=0 \quad \text{for all} \quad \boldsymbol{v}_h\in\boldsymbol{V}_h^E, \quad \widehat{\boldsymbol{w}}_h\in\widehat{\boldsymbol{V}}^E,
$$

where a^E is the local version of the bilinear form a and

$$
\widehat{\boldsymbol{V}}^E := \left\{\widehat{\boldsymbol{v}}_h\in\boldsymbol{V}_h^E : \widehat{\boldsymbol{v}}_h=\mathbb{K}\nabla\widehat{q},\ \widehat{q}\in\mathbb{P}_2(E)\right\}.
$$

It was shown in [5] that the explicit computation of the projection $\widehat{\boldsymbol{\Pi}}\boldsymbol{v}_h$ involves only the degrees of freedom of $\boldsymbol{v}_h$.

The stabilization term $S^E(\boldsymbol{p}_h,\boldsymbol{v}_h)$ can be any symmetric positive definite bilinear form and there exist two positive constants c_0 and c_1 such that

$$
c_0a^E(\boldsymbol{v}_h,\boldsymbol{v}_h)\le\|\mathbb{K}^{-1}\|S^E(\boldsymbol{v}_h,\boldsymbol{v}_h)\le c_1a^E(\boldsymbol{v}_h,\boldsymbol{v}_h) \quad \text{for all} \quad \boldsymbol{v}_h\in\boldsymbol{V}_h.
$$

The stabilization term S^E can be also defined via the r-th local degree of freedom — viz.

$$
S^E\left(\boldsymbol{p}_h(u)-\widehat{\boldsymbol{\Pi}}\boldsymbol{p}_h(u),\boldsymbol{v}_h-\widehat{\boldsymbol{\Pi}}\boldsymbol{v}_h\right)=\sum_{r=1}^{N^E}\mathrm{dof}_\mathrm{r}\left(\boldsymbol{p}_h(u)-\widehat{\boldsymbol{\Pi}}\boldsymbol{p}_h(u)\right)\mathrm{dof}_\mathrm{r}\left(\boldsymbol{v}_h-\widehat{\boldsymbol{\Pi}}\boldsymbol{v}_h\right),
$$

where N^E are the dimension of $\boldsymbol{V}_h^E$, cf. [1].

Proposition 3.1 (cf. Brezzi *et al.* [5])**.** *For all $\widehat{\boldsymbol{w}}_h\in\widehat{\boldsymbol{V}}^E$ and $\boldsymbol{v}_h\in\boldsymbol{V}_h^E$, the bilinear form a_h^E has the following properties:*

1. *Consistency*

$$
a_h^E(\widehat{\boldsymbol{w}}_h,\boldsymbol{v}_h)=a^E(\widehat{\boldsymbol{w}}_h,\boldsymbol{v}_h). \tag{3.2}
$$

2. *Stability. There are $\alpha^*,\alpha_*>0$ such that*

$$
\alpha_*a^E(\boldsymbol{v}_h,\boldsymbol{v}_h)\le a_h^E(\boldsymbol{v}_h,\boldsymbol{v}_h)\le\alpha^*a^E(\boldsymbol{v}_h,\boldsymbol{v}_h). \tag{3.3}
$$

The symmetry of a_h, Eq. (3.3) and the continuity of a^E yield the continuity of a_h, i.e. for all $\boldsymbol{w}_h,\boldsymbol{v}_h\in\boldsymbol{V}_h^E$ we have

$$
\begin{aligned}
a_h^E(\boldsymbol{w}_h,\boldsymbol{v}_h) &\le \left(a_h^E(\boldsymbol{w}_h,\boldsymbol{w}_h)\right)^{1/2}\left(a_h^E(\boldsymbol{v}_h,\boldsymbol{v}_h)\right)^{1/2}\\
&\le \alpha^*\left(a^E(\boldsymbol{w}_h,\boldsymbol{w}_h)\right)^{1/2}\left(a^E(\boldsymbol{v}_h,\boldsymbol{v}_h)\right)^{1/2}\\
&\le \alpha^*\|a\|\|\boldsymbol{w}_h\|_{0,E}\|\boldsymbol{v}_h\|_{0,E}.
\end{aligned}
$$

On the other hand, for all $\boldsymbol{w}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h^E$, the following estimates hold:

$$\alpha_* a(\boldsymbol{v}_h, \boldsymbol{v}_h) \le a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \le \alpha^* a(\boldsymbol{v}_h, \boldsymbol{v}_h), \tag{3.4}$$

$$a_h(\boldsymbol{w}_h, \boldsymbol{v}_h) \le \alpha^* \|a\| \|\boldsymbol{w}_h\|_{\boldsymbol{H}} \|\boldsymbol{v}_h\|_{\boldsymbol{H}}. \tag{3.5}$$

Proposition 3.2 (cf. Meng *et al.* [17]). *The discrete inf-sup condition holds — i.e. there is* $\zeta > 0$ *such that*

$$\inf_{w_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{| b(\boldsymbol{v}_h, w_h) |}{\|w_h\|_Q \|\boldsymbol{v}_h\|_V} \ge \zeta.$$

3.2. Virtual element approximation for optimal control problem

The virtual element approximation for optimal control problem (1.1)-(1.2) consists in finding a triple $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times Q_h \times U_{ad}$ such that

$$\min_{u_h \in U_{ad}} J(\boldsymbol{p}_h, y_h, u_h) := \frac{1}{2} \sum_{E \in \mathscr{T}_h} \int_E \left(\boldsymbol{\Pi}_1^0 \boldsymbol{p}_h - \boldsymbol{p}_d \right)^2 + \frac{1}{2} \int_\Omega (y_h - y_d)^2 + \frac{\gamma}{2} \int_\Omega u_h^2 \tag{3.6}$$

subject to

$$\begin{aligned} a_h(\boldsymbol{p}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, y_h) &= 0 \quad \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{p}_h, w_h) &= (f + u_h, w_h) \quad \text{for all} \quad w_h \in Q_h. \end{aligned} \tag{3.7}$$

Here we employ the implicit discrete approach for the control variable — cf. [7]. The control variable u_h belongs to the infinite dimensional set U_{ad} instead of the virtual element space $\boldsymbol{V}_h$. In order to guarantee the computability of the discrete adjoint problem, $\boldsymbol{\Pi}_1^0 \boldsymbol{p}_h$ in (3.6) is replaced by $\boldsymbol{p}_h$.

To derive the discrete first-order optimality system, we consider the following Lagrangian functional:

$$\begin{aligned} &\mathscr{L}(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \\ &= J(\boldsymbol{p}_h, y_h, u_h) + (f + u_h, z_h) - a_h(\boldsymbol{p}_h, \boldsymbol{q}_h) \\ &\quad + b(\boldsymbol{q}_h, y_h) - b(\boldsymbol{p}_h, z_h). \end{aligned}$$

Differentiating $\mathscr{L}$ in $\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h$ leads to a discrete first-order optimality condition for the problem (3.6)-(3.7), viz.

$$\begin{aligned} &a_h(\boldsymbol{p}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, y_h) = 0 && \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ &b(\boldsymbol{p}_h, w_h) = (f + u_h, w_h) && \text{for all} \quad w_h \in Q_h, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &a_h(\boldsymbol{q}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, z_h) = \sum_{E \in \mathscr{T}_h} \left(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{\Pi}_1^0 \boldsymbol{v}_h \right)_{0,E} && \text{for all} \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ &b(\boldsymbol{q}_h, w_h) = -(y_h - y_d, w_h) && \text{for all} \quad w_h \in Q_h, \end{aligned} \tag{3.9}$$

$$(\gamma u_h + z_h, \tilde{u}_h - u_h) \ge 0 \quad \text{for all} \quad \tilde{u}_h \in U_{ad}. \tag{3.10}$$

In the next section, we derive a priori error estimates for the discrete virtual element scheme.

4. A Priori Error Estimates

In order to derive a priori error estimates, we consider the following auxiliary problems for the pairs $(\boldsymbol{v}_h, w_h) \in \boldsymbol{V}_h \times Q_h$:

$$a_h\big(\boldsymbol{q}_h(y), \boldsymbol{v}_h\big) + b\big(\boldsymbol{v}_h, z_h(y)\big) = \sum_{E \in \mathscr{T}_h} \big(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{\Pi}_1^0 \boldsymbol{v}_h\big)_{0,E}, \tag{4.1a}$$

$$b\big(\boldsymbol{q}_h(y), w_h\big) = -(y - y_d, w_h), \tag{4.1b}$$

and

$$\begin{aligned} &a_h\big(\boldsymbol{q}_h(u), \boldsymbol{v}_h\big) + b\big(\boldsymbol{v}_h, z_h(u)\big) = \sum_{E \in \mathscr{T}_h} \big(\boldsymbol{p}_h(u) - \boldsymbol{p}_d, \boldsymbol{\Pi}_1^0 \boldsymbol{v}_h\big)_{0,E}, \\ &b\big(\boldsymbol{q}_h(u), w_h\big) = -\big(y_h(u) - y_d, w_h\big). \end{aligned} \tag{4.2}$$

Consequently, the errors $\boldsymbol{p} - \boldsymbol{p}_h$, $y - y_h$, $\boldsymbol{q} - \boldsymbol{q}_h$ and $z - z_h$ can be represented in the form

$$\begin{aligned} \boldsymbol{p} - \boldsymbol{p}_h &= \boldsymbol{p} - \boldsymbol{p}_h(u) + \boldsymbol{p}_h(u) - \boldsymbol{p}_h, \\ y - y_h &= y - y_h(u) + y_h(u) - y_h, \\ \boldsymbol{q} - \boldsymbol{q}_h &= \boldsymbol{q} - \boldsymbol{q}_h(y) + \boldsymbol{q}_h(y) - \boldsymbol{q}_h, \\ z - z_h &= z - z_h(y) + z_h(y) - z_h. \end{aligned}$$

It is easily seen that $(\boldsymbol{p}_h(u), y_h(u))$ and $(\boldsymbol{q}_h(y), z_h(y))$ are the virtual element approximations of $(\boldsymbol{p}, y)$ and $(\boldsymbol{q}, z)$, respectively. Therefore, using results from [5], we obtain the following error estimates.

Lemma 4.1. *If $(\boldsymbol{p}, y)$ and $(\boldsymbol{p}_h(u), y_h(u))$ are respective unique solutions of the continuous and discrete schemes Eqs.* (2.3) *and* (3.1)*, then*

$$\begin{aligned} \|\boldsymbol{p} - \boldsymbol{p}_h(u)\|_V &\le Ch\big(\|\boldsymbol{p}\|_{1,\Omega} + \|f + u\|_{1,\Omega}\big), \\ \|y - y_h(u)\|_Q &\le Ch\big(\|\boldsymbol{p}\|_{1,\Omega} + \|y\|_{1,\Omega}\big). \end{aligned}$$

Lemma 4.2. *If $(\boldsymbol{q}, z)$ and $(\boldsymbol{q}_h(y), z_h(y))$ are respective solutions of the Eqs.* (2.4) *and* (4.1)*, then there exists a positive constant C independent of the mesh size h such that*

$$\begin{aligned} \|\boldsymbol{q} - \boldsymbol{q}_h(y)\|_V &\le Ch\big(\|\boldsymbol{q}\|_{1,\Omega} + \|\boldsymbol{p} - \boldsymbol{p}_d\|_{1,\Omega} + \|y - y_d\|_{1,\Omega}\big), \\ \|z - z_h(y)\|_Q &\le Ch\big(\|z\|_{1,\Omega} + \|\boldsymbol{q}\|_{1,\Omega} + \|\boldsymbol{p} - \boldsymbol{p}_d\|_{1,\Omega}\big). \end{aligned}$$

Proof. Before proving Lemma 4.2, we establish the inequalities

$$\|\boldsymbol{q} - \boldsymbol{q}_h(y)\|_H \le C_1\big(\|\boldsymbol{q} - \boldsymbol{q}_I\|_H + \|\boldsymbol{q} - \boldsymbol{q}_\pi\|_H + \aleph\big), \tag{4.3}$$

$$\|z_I - z_h(y)\|_Q \le C_2\big(\|\boldsymbol{q} - \boldsymbol{q}_h(y)\|_H + \|\boldsymbol{q} - \boldsymbol{q}_\pi\|_H + \aleph\big), \tag{4.4}$$

where $\boldsymbol{q}_I$ is the interpolant of $\boldsymbol{q}$, z_I the interpolant z, and $\boldsymbol{q}_\pi$ the piecewise approximation of $\boldsymbol{q}$ in $\widehat{\boldsymbol{V}}^E$. Besides, C_1 is a constant depending only on $\mathbb{K}$, α^*, α_*, whereas C_2 is a constant depending only on $\mathbb{K}$, α^*, ζ, and

$$\aleph = \big\|\boldsymbol{\Pi}_1^0(\boldsymbol{p} - \boldsymbol{p}_d) - (\boldsymbol{p} - \boldsymbol{p}_d)\big\|_H.$$

We define

$$\boldsymbol{\delta}_h := \boldsymbol{q}_h(y) - \boldsymbol{q}_I. \tag{4.5}$$

It follows from the Eq. (4.1) that $\mathrm{div}\boldsymbol{q}_h(y) = -\Pi_{Q_h}(y - y_d)$. Noting that

$$\mathrm{div}\boldsymbol{q}_I = \Pi_{Q_h}\mathrm{div}\boldsymbol{q} = -\Pi_{Q_h}(y - y_d),$$

we obtain that

$$\mathrm{div}\boldsymbol{\delta}_h = 0 \tag{4.6}$$

and $\|\boldsymbol{\delta}_h\|_V = \|\boldsymbol{\delta}_h\|_{\boldsymbol{H}}$. Consequently,

$$\begin{aligned}
\alpha_* \alpha \|\boldsymbol{\delta}_h\|_{\boldsymbol{H}}^2 \le a_h(\boldsymbol{\delta}_h, \boldsymbol{\delta}_h) &= a_h\big(\boldsymbol{q}_h(y), \boldsymbol{\delta}_h\big) - a_h\big(\boldsymbol{q}_I, \boldsymbol{\delta}_h\big) \quad \text{(use (4.1) and (4.6))}\\
&= \sum_{E\in\mathscr{T}_h} \big(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{\Pi}_1^0 \boldsymbol{\delta}_h\big) - \sum_{E\in\mathscr{T}_h} a_h^E(\boldsymbol{q}_I, \boldsymbol{\delta}_h) \qquad \text{(use Definition 2.1 and } \pm\boldsymbol{q}_\pi)\\
&= \sum_{E\in\mathscr{T}_h} \big(\boldsymbol{\Pi}_1^0(\boldsymbol{p} - \boldsymbol{p}_d), \boldsymbol{\delta}_h\big) - \sum_{E\in\mathscr{T}_h} \big(a_h^E(\boldsymbol{q}_I - \boldsymbol{q}_\pi, \boldsymbol{\delta}_h) + a_h^E(\boldsymbol{q}_\pi, \boldsymbol{\delta}_h)\big) \quad \text{(use (3.2))}\\
&= \sum_{E\in\mathscr{T}_h} \big(\boldsymbol{\Pi}_1^0(\boldsymbol{p} - \boldsymbol{p}_d), \boldsymbol{\delta}_h\big) - \sum_{E\in\mathscr{T}_h} \big(a_h^E(\boldsymbol{q}_I - \boldsymbol{q}_\pi, \boldsymbol{\delta}_h) + a^E(\boldsymbol{q}_\pi, \boldsymbol{\delta}_h)\big) \quad \text{(use } \pm\boldsymbol{q})\\
&= \sum_{E\in\mathscr{T}_h} \big(\boldsymbol{\Pi}_1^0(\boldsymbol{p} - \boldsymbol{p}_d), \boldsymbol{\delta}_h\big) - \sum_{E\in\mathscr{T}_h} \big(a_h^E(\boldsymbol{q}_I - \boldsymbol{q}_\pi, \boldsymbol{\delta}_h) + a^E(\boldsymbol{q}_\pi - \boldsymbol{q}, \boldsymbol{\delta}_h)\big)\\
&\quad - a(\boldsymbol{q}, \boldsymbol{\delta}_h) \quad \text{(use (2.4) with (4.6))}\\
&= \sum_{E\in\mathscr{T}_h} \big((\boldsymbol{\Pi}_1^0(\boldsymbol{p} - \boldsymbol{p}_d) - (\boldsymbol{p} - \boldsymbol{p}_d), \boldsymbol{\delta}_h)_{0,E} - a_h^E(\boldsymbol{q}_I - \boldsymbol{q}_\pi, \boldsymbol{\delta}_h) - a^E(\boldsymbol{q}_\pi - \boldsymbol{q}, \boldsymbol{\delta}_h)\big).
\end{aligned}$$

The continuity of a^E and a_h^E yields

$$\|\boldsymbol{\delta}_h\|_{\boldsymbol{H}}^2 \le C\big(\|\boldsymbol{q}_I - \boldsymbol{q}_\pi\|_{\boldsymbol{H}} + \|\boldsymbol{q} - \boldsymbol{q}_\pi\|_{\boldsymbol{H}} + \aleph\big)\|\boldsymbol{\delta}_h\|_{\boldsymbol{H}},$$

and (4.3) follows from the triangle inequality.

We turn next to the proof of (4.4). According to [5], for $z^* := z_h(y) - z_I$ there exists $\boldsymbol{w}_h^* \in \boldsymbol{V}_h$ such that $\mathrm{div}\boldsymbol{w}_h^* = z^*$ and

$$\|\boldsymbol{w}_h^*\|_{\boldsymbol{H}} \le C\|z^*\|_Q = C\|z_h(y) - z_I\|_Q.$$

Since $\mathrm{div}\boldsymbol{w}_h^* \in Q_h$, we can use the definition of z_I, (2.4a) and (4.1a) to obtain

$$\begin{aligned}
\|z_h(y) - z_I\|_Q^2 &= \big(z_h(y) - z_I, \mathrm{div}\boldsymbol{w}_h^*\big) = \big(z_h(y) - z, \mathrm{div}\boldsymbol{w}_h^*\big)\\
&= b\big(\boldsymbol{w}_h^*, z_h(y)\big) - b\big(\boldsymbol{w}_h^*, z\big)\\
&= -a_h\big(\boldsymbol{q}_h(y), \boldsymbol{w}_h^*\big) + a\big(\boldsymbol{q}, \boldsymbol{w}_h^*\big)\\
&\quad + \sum_{E\in\mathscr{T}_h} \big(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{\Pi}_1^0 \boldsymbol{w}_h^*\big) - \big(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{w}_h^*\big).
\end{aligned}$$

Taking into account Definition 2.1, and using the Eq. (3.2), we can further write

$$\begin{aligned}\|z_h(y)-z_I\|_Q^2 &= \sum_{E\in\mathcal{T}_h}\Big(-a_h^E\big(\boldsymbol{q}_h(y),\boldsymbol{w}_h^*\big)+a^E\big(\boldsymbol{q},\boldsymbol{w}_h^*\big)\Big)\\&\quad+\sum_{E\in\mathcal{T}_h}\big(\boldsymbol{\Pi}_1^0(\boldsymbol{p}-\boldsymbol{p}_d)-(\boldsymbol{p}-\boldsymbol{p}_d),\boldsymbol{w}_h^*\big)\\&=\sum_{E\in\mathcal{T}_h}\Big(-a_h^E\big(\boldsymbol{q}_h(y)-\boldsymbol{q}_\pi,\boldsymbol{w}_h^*\big)+a^E\big(\boldsymbol{q}-\boldsymbol{q}_\pi,\boldsymbol{w}_h^*\big)\Big)\\&\quad+\sum_{E\in\mathcal{T}_h}\big(\boldsymbol{\Pi}_1^0(\boldsymbol{p}-\boldsymbol{p}_d)-(\boldsymbol{p}-\boldsymbol{p}_d),\boldsymbol{w}_h^*\big)\\&\le C\big(\|\boldsymbol{q}_h(y)-\boldsymbol{q}_\pi\|_{\boldsymbol{H}}+\|\boldsymbol{q}-\boldsymbol{q}_\pi\|_{\boldsymbol{H}}+\aleph\big)\|\boldsymbol{w}_h^*\|_{\boldsymbol{H}}\\&\le C\big(\|\boldsymbol{q}_h(y)-\boldsymbol{q}_\pi\|_{\boldsymbol{H}}+\|\boldsymbol{q}-\boldsymbol{q}_\pi\|_{\boldsymbol{H}}+\aleph\big)\|z_h(y)-z_I\|_Q.\end{aligned}$$

Thus (4.4) is obtained by the triangle inequality. Applying Lemmas 2.1 and 2.2 to (4.3) and (4.4) finishes the proof. □

Theorem 4.1 (A Priori Error Estimates). *Assume that* $(\boldsymbol{p},y,\boldsymbol{q},z,u)$ *and* $(\boldsymbol{p}_h,y_h,\boldsymbol{q}_h,z_h,u_h)$ *are the solutions of the Eqs.* (2.3)-(2.5) *and* (3.8)-(3.10)*, respectively. If* $y,z,u\in H^1(\Omega)$ *and* $\boldsymbol{p},\boldsymbol{q}\in(H^1(\Omega))^2$*, then*

$$\begin{aligned}&\|u-u_h\|_Q\le Ch,\\&\|\boldsymbol{p}-\boldsymbol{p}_h\|_V+\|\boldsymbol{q}-\boldsymbol{q}_h\|_V\le Ch,\\&\|y-y_h\|_Q+\|z-z_h\|_Q\le Ch.\end{aligned}$$

Proof. It follows from Lemma 4.1 that

$$\begin{aligned}&\|\boldsymbol{p}-\boldsymbol{p}_h(u)\|_V\le Ch,\\&\|y-y_h(u)\|_Q\le Ch.\end{aligned}$$

Using the governing equation of $(\boldsymbol{p}_h(u),y_h(u))$ and $(\boldsymbol{p}_h,y_h)$ gives

$$a_h\big(\boldsymbol{p}_h(u)-\boldsymbol{p}_h,\boldsymbol{v}_h\big)-b\big(\boldsymbol{v}_h,y_h(u)-y_h\big)=0, \tag{4.7a}$$
$$b\big(\boldsymbol{p}_h(u)-\boldsymbol{p}_h,w_h\big)=(u-u_h,w_h). \tag{4.7b}$$

Taking into account Proposition 3.2, we write

$$\zeta\|y_h(u)-y_h\|_Q\le\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{b(\boldsymbol{v}_h,y_h(u)-y_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}}=\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{a_h(\boldsymbol{p}_h(u)-\boldsymbol{p}_h,\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}}.$$

It follows from (3.5) that

$$a_h\big(\boldsymbol{p}_h(u)-\boldsymbol{p}_h,\boldsymbol{v}_h\big)\le\alpha^*\|a\|\|\boldsymbol{p}_h(u)-\boldsymbol{p}_h\|_{\boldsymbol{H}}\|\boldsymbol{v}_h\|_{\boldsymbol{H}}.$$

Consequently, we have

$$\|y_h(u)-y_h\|_Q \le C\|\boldsymbol{p}_h(u)-\boldsymbol{p}_h\|_{\boldsymbol{H}}.$$

Choosing $\boldsymbol{v}_h=\boldsymbol{p}_h(u)-\boldsymbol{p}_h$, $w_h=y_h(u)-y_h$ and using (2.2) and (3.4) yields

$$\begin{aligned}\alpha_*\alpha\|\boldsymbol{p}_h(u)-\boldsymbol{p}_h\|_{\boldsymbol{H}}^2 &\le \alpha_* a\big(\boldsymbol{p}_h(u)-\boldsymbol{p}_h,\boldsymbol{p}_h(u)-\boldsymbol{p}_h\big)\\ &\le a_h\big(\boldsymbol{p}_h(u)-\boldsymbol{p}_h,\boldsymbol{p}_h(u)-\boldsymbol{p}_h\big)\\ &=\big(u-u_h,y_h(u)-y_h\big)\\ &\le C\|u-u_h\|_Q\|y_h(u)-y_h\|_Q\\ &\le C\|u-u_h\|_Q\|\boldsymbol{p}_h(u)-\boldsymbol{p}_h\|_{\boldsymbol{H}}.\end{aligned}$$

This implies

$$\begin{aligned}\|\boldsymbol{p}_h(u)-\boldsymbol{p}_h\|_{\boldsymbol{H}} &\le C\|u-u_h\|_Q,\\ \|y_h(u)-y_h\|_Q &\le C\|u-u_h\|_Q.\end{aligned}$$

Substituting $w_h=\mathrm{div}(\boldsymbol{p}_h(u)-\boldsymbol{p}_h)$ into (4.7b), we obtain

$$\begin{aligned}\|\mathrm{div}(\boldsymbol{p}_h(u)-\boldsymbol{p}_h)\|_0^2 &= b\big(\boldsymbol{p}_h(u)-\boldsymbol{p}_h,\mathrm{div}(\boldsymbol{p}_h(u)-\boldsymbol{p}_h)\big)\\ &=\big(u-u_h,\mathrm{div}(\boldsymbol{p}_h(u)-\boldsymbol{p}_h)\big)\\ &\le C\|u-u_h\|_Q\|\mathrm{div}(\boldsymbol{p}_h(u)-\boldsymbol{p}_h)\|_0.\end{aligned}$$

Therefore,

$$\|\mathrm{div}(\boldsymbol{p}_h(u)-\boldsymbol{p}_h)\|_0 \le C\|u-u_h\|_Q.$$

Taking into account the above inequalities, we arrive at the estimates

$$\begin{aligned}\|\boldsymbol{p}-\boldsymbol{p}_h\|_V &\le C\big(h+\|u-u_h\|_Q\big),\\ \|y-y_h\|_Q &\le C\big(h+\|u-u_h\|_Q\big).\end{aligned}$$

Now we consider error estimates for adjoint variables. It follows from Lemma 4.2 that

$$\begin{aligned}\|\boldsymbol{q}-\boldsymbol{q}_h(y)\|_V &\le Ch,\\ \|z-z_h(y)\|_Q &\le Ch.\end{aligned}$$

In order to evaluate the residues $\boldsymbol{q}_h(y)-\boldsymbol{q}_h$ and $z_h(y)-z_h$, we employ the following equations:

$$\begin{aligned}&a_h\big(\boldsymbol{q}_h(y)-\boldsymbol{q}_h,\boldsymbol{v}_h\big)+b\big(\boldsymbol{v}_h,z_h(y)-z_h\big)=\sum_{E\in\mathscr{T}_h}\big(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{\Pi}_1^0\boldsymbol{v}_h\big)_{0,E},\\ &b\big(\boldsymbol{q}_h(y)-\boldsymbol{q}_h,w_h\big)=-(y-y_h,w_h),\end{aligned} \tag{4.8}$$

which can be obtained from (4.1) and (3.9). Using (3.5) and Definition 2.1 shows that

$$\begin{aligned}b\big(\boldsymbol{v}_h,z_h(y)-z_h\big) &= a_h\big(\boldsymbol{q}_h-\boldsymbol{q}_h(y),\boldsymbol{v}_h\big)+\sum_{E\in\mathscr{T}_h}\big(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{\Pi}_1^0\boldsymbol{v}_h\big)_{0,E}\\ &\le \alpha^*\|a\|\|\boldsymbol{q}_h(y)-\boldsymbol{q}_h\|_{\boldsymbol{H}}\|\boldsymbol{v}_h\|_{\boldsymbol{H}}+C\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\boldsymbol{H}}\|\boldsymbol{v}_h\|_{\boldsymbol{H}},\end{aligned}$$

and recalling Proposition 3.2 gives

$$\begin{aligned}\zeta\|z_h(y)-z_h\|_Q &\le \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{b(\boldsymbol{v}_h,z_h(y)-z_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}}\\ &\le C\big(\|\boldsymbol{q}_h(y)-\boldsymbol{q}_h\|_{\boldsymbol{H}}+\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\boldsymbol{H}}\big),\end{aligned}$$

so that

$$\|z_h(y)-z_h\|_Q \le C\big(\|\boldsymbol{q}_h(y)-\boldsymbol{q}_h\|_{\boldsymbol{H}}+\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\boldsymbol{H}}\big).$$

Further, choosing $\boldsymbol{v}_h=\boldsymbol{q}_h(y)-\boldsymbol{q}_h$, $w_h=z_h(y)-z_h$ in (4.8), and using (2.2) and (3.4) yields

$$\begin{aligned}\alpha_*\alpha\|\boldsymbol{q}_h(y)-\boldsymbol{q}_h\|_{\boldsymbol{H}}^2 &\le \alpha_* a\big(\boldsymbol{q}_h(y)-\boldsymbol{q}_h,\boldsymbol{q}_h(y)-\boldsymbol{q}_h\big)\\ &\le a_h\big(\boldsymbol{q}_h(y)-\boldsymbol{q}_h,\boldsymbol{q}_h(y)-\boldsymbol{q}_h\big)\\ &=\sum_{E\in\mathscr{T}_h}\big(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{\Pi}_1^0(\boldsymbol{q}_h(y)-\boldsymbol{q}_h)\big)_{0,E}+\big(y-y_h,z_h(y)-z_h\big)\\ &\le C\big(\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\boldsymbol{H}}\|\boldsymbol{q}_h(y)-\boldsymbol{q}_h\|_{\boldsymbol{H}}+\|y-y_h\|_Q\|z_h(y)-z_h\|_Q\big).\end{aligned}$$

Consequently, using the above estimate for $\|z_h(y)-z_h\|_Q$ and the Young's inequality we first obtain

$$\|\boldsymbol{q}_h(y)-\boldsymbol{q}_h\|_{\boldsymbol{H}}^2 \le C\big(\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\boldsymbol{H}}^2+\|y-y_h\|_Q^2\big),$$

and then

$$\begin{aligned}\|\boldsymbol{q}_h(y)-\boldsymbol{q}_h\|_{\boldsymbol{H}} &\le C\big(h+\|u-u_h\|_Q\big),\\ \|z_h(y)-z_h\|_Q &\le C\big(h+\|u-u_h\|_Q\big).\end{aligned}$$

Substituting $w_h=\operatorname{div}(\boldsymbol{q}_h(y)-\boldsymbol{q}_h)$ into (4.8) yields

$$\begin{aligned}\|\operatorname{div}(\boldsymbol{q}_h(y)-\boldsymbol{q}_h)\|_0^2 &= b\big(\boldsymbol{q}_h(y)-\boldsymbol{q}_h,\boldsymbol{q}_h(y)-\boldsymbol{q}_h\big)\\ &=\big(y_h-y,\operatorname{div}(\boldsymbol{q}_h(y)-\boldsymbol{q}_h)\big)\\ &\le C\|y-y_h\|_Q\|\operatorname{div}(\boldsymbol{q}_h(y)-\boldsymbol{q}_h)\|_0,\end{aligned}$$

so that

$$\|\operatorname{div}(\boldsymbol{q}_h(y)-\boldsymbol{q}_h)\|_0 \le C\|y-y_h\|_Q \le C\big(h+\|u-u_h\|_Q\big).$$

Consequently, we obtain

$$\|\boldsymbol{q}(y)-\boldsymbol{q}_h\|_{\boldsymbol{V}} \le C\big(h+\|u-u_h\|_Q\big).$$

Combining the above estimates gives

$$\begin{aligned}\|\boldsymbol{q}-\boldsymbol{q}_h\|_{\boldsymbol{V}} &\le C\big(h+\|u-u_h\|_Q\big),\\ \|z-z_h\|_Q &\le C\big(h+\|u-u_h\|_Q\big).\end{aligned}$$

Note that the state and adjoint state estimates depend on the control variable. Therefore, we have to evaluate the norm $\|u-u_h\|_0$. Setting

$$\hat{J}_h(u)(\widetilde{u}-u) := \int_\Omega \big(\gamma u + z_h(u)\big)(\widetilde{u}-u)\,dx,$$

we can show that

$$\hat{J}_h(\widetilde{u})(\widetilde{u}-u) - \hat{J}_h(u)(\widetilde{u}-u) \geq \gamma \|\widetilde{u}-u\|_Q^2. \tag{4.9}$$

In fact

$$\begin{aligned}
&\hat{J}_h(\widetilde{u})(\widetilde{u}-u) - \hat{J}_h(u)(\widetilde{u}-u)\\
=&\int_\Omega \big(\gamma\widetilde{u} + z_h(\widetilde{u}) - \gamma u - z_h(u)\big)(\widetilde{u}-u)\,dx\\
=&\,\gamma\int_\Omega (\widetilde{u}-u)^2\,dx + \int_\Omega \big(z_h(\widetilde{u}) - z_h(u)\big)(\widetilde{u}-u)\,dx.
\end{aligned}$$

Taking into account the Eq. (3.1), we write

$$\begin{aligned}
&\int_\Omega \big(z_h(\widetilde{u}) - z_h(u)\big)(\widetilde{u}-u)\,dx\\
=&\, b\big(\boldsymbol{p}_h(\widetilde{u}) - \boldsymbol{p}_h(u), z_h(\widetilde{u}) - z_h(u)\big)\\
=&\sum_{E\in\mathscr{T}_h} \big(\boldsymbol{p}_h(\widetilde{u}) - \boldsymbol{p}_h(u), \boldsymbol{\Pi}_1^0\big(\boldsymbol{p}_h(\widetilde{u}) - \boldsymbol{p}_h(\widetilde{u})\big)_{0,E}\\
&\quad - a_h\big(\boldsymbol{q}_h(\widetilde{u}) - \boldsymbol{q}_h(u), \boldsymbol{p}_h(\widetilde{u}) - \boldsymbol{p}_h(u)\big)\\
=&\sum_{E\in\mathscr{T}_h} \big(\boldsymbol{\Pi}_1^0\boldsymbol{p}_h(\widetilde{u}) - \boldsymbol{p}_h(u), \boldsymbol{\Pi}_1^0\big(\boldsymbol{p}_h(\widetilde{u}) - \boldsymbol{p}_h(\widetilde{u})\big)_{0,E}\\
&\quad - a_h\big(\boldsymbol{q}_h(\widetilde{u}) - \boldsymbol{q}_h(u), \boldsymbol{p}_h(\widetilde{u}) - \boldsymbol{p}_h(u)\big).
\end{aligned}$$

Besides, the Eqs. (3.1) and (4.2) give

$$\begin{aligned}
&a_h\big(\boldsymbol{p}_h(\widetilde{u}) - \boldsymbol{p}_h(u), \boldsymbol{q}_h(\widetilde{u}) - \boldsymbol{q}_h(u)\big)\\
=&\, b\big(\boldsymbol{q}_h(\widetilde{u}) - \boldsymbol{q}_h(u), y_h(\widetilde{u}) - y_h(u)\big)\\
=&-\big(y_h(\widetilde{u}) - y_h(u), y_h(\widetilde{u}) - y_h(u)\big) \leq 0.
\end{aligned}$$

Recalling Definition 2.1, we deduce

$$\int_\Omega \big(z_h(\widetilde{u}) - z_h(u)\big)(\widetilde{u}-u)\,dx \geq 0,$$

and the inequality (4.9) implies

$$\gamma\|u-u_h\|^2 \leq \hat{J}_h(u)(u-u_h) - \hat{J}_h(u_h)(u-u_h)$$

$$
\begin{aligned}
&= \int_\Omega \big(\gamma u + z_h(u) - \gamma u_h - z_h(u_h)\big)(u-u_h)\,dx \\
&= (\gamma u + z, u - u_h) + \int_\Omega \big(\gamma u_h + z_h(u_h)\big)(u_h - u)\,dx \\
&\quad + \int_\Omega \big(z_h(u) - z\big)(u-u_h)\,dx \\
&\le 0 + 0 + \int_\Omega \big(z_h(u) - z\big)(u-u_h)\,dx \\
&\le C\|z_h(u) - z\|_Q \|u - u_h\|_Q.
\end{aligned}
$$

Now we have to estimate the norm $\|z_h(u) - z\|_Q$. Writing

$$z_h(u) - z = z_h(u) - z_h(y) + z_h(y) - z$$

and recalling Lemma 4.2 gives

$$\|z_h(y) - z\|_Q \le Ch.$$

Subtracting (4.1) from (4.2), we write

$$
\begin{aligned}
&a_h\big(\boldsymbol{q}_h(u) - \boldsymbol{q}_h(y), \boldsymbol{v}_h\big) + b\big(\boldsymbol{v}_h, z_h(u) - z_h(y)\big) = \sum_{E\in\mathscr{T}_h} \big(\boldsymbol{p}_h(u) - \boldsymbol{p}, \boldsymbol{\Pi}_1^0 \boldsymbol{v}_h\big)_{0,E}, \\
&b\big(\boldsymbol{q}_h(u) - \boldsymbol{q}_h(y), w_h\big) = -\big(y_h(u) - y, w_h\big).
\end{aligned}
$$

The terms $\boldsymbol{q}_h(y) - \boldsymbol{q}_h$ and $z_h(y) - z_h$ can be estimated analogously. In particular,

$$\|z_h(u) - z_h(y)\|_Q \le Ch,$$

so that

$$\|u - u_h\|_0 \le C\|z_h(u) - z\|_Q \le Ch.$$

Combining the above estimates finishes the proof. □

5. Numerical Experiments

We consider two examples with the domain $\Omega = [0,1]\times[0,1]$ aimed to verify the theoretical results.

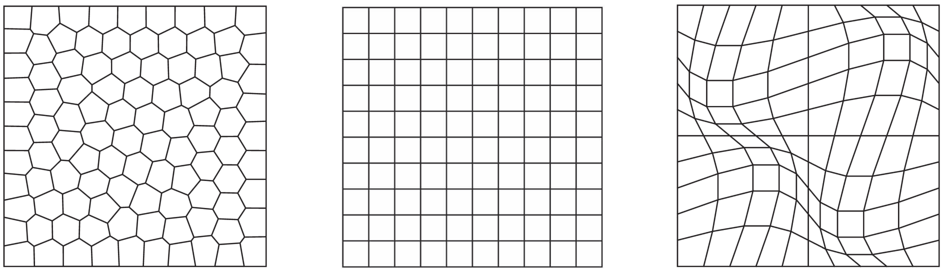
We display the errors between of the numerical solutions $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ to confirm the convergence results. Note that $\text{err}(y, L^2)$, $\text{err}(z, L^2)$, and $\text{err}(u, L^2)$ denotes the corresponding errors in the L^2 norm. Since inside of the related elements the VEM solutions $(\boldsymbol{p}_h, \boldsymbol{q}_h)$ are not known, we replace the projection errors by numerical solution errors. Analogously, the differences between the $\boldsymbol{\Pi}_1^0 \boldsymbol{p}_h, \boldsymbol{\Pi}_1^0 \boldsymbol{q}_h$ and the exact solutions $\boldsymbol{p}, \boldsymbol{q}$ evaluated in the $\|\cdot\|_V$ norm are denoted by $\text{err}(\boldsymbol{p}, \boldsymbol{V})$ and $\text{err}(\boldsymbol{q}, \boldsymbol{V})$.

In numerical examples, we consider the following optimal control problem:

$$
\begin{aligned}
&\min_{u\in U_{ad}} J(\boldsymbol{p},y,u)=\frac{1}{2}\int_{\Omega}(\boldsymbol{p}-\boldsymbol{p}_d)^2\,d\Omega+\frac{1}{2}\int_{\Omega}(y-y_d)^2\,d\Omega+\frac{1}{2}\int_{\Omega}u^2\,d\Omega,\\
&\mathrm{div}\boldsymbol{p}=f+u, \qquad \boldsymbol{p}=-\nabla y,\\
&\mathrm{div}\boldsymbol{q}=-(y-y_d), \quad \boldsymbol{q}=\nabla z+(\boldsymbol{p}-\boldsymbol{p}_d),
\end{aligned}
$$

and use three different mesh sequences — viz.

1. Voronoi meshes (Lloyd).
2. Square meshes (Square).
3. Distorted quadrilateral meshes (Distorted Square).



(a) Lloyd (b) Square (c) Distorted Square

Figure 1: Basic sample meshes on the square domain Ω.

Example 5.1. Set $\Gamma_1=\{(x_1,x_2):x_1=0,1;0\le x_2\le 1\}$ and choose the following functions:

$$
\begin{aligned}
&y=\sin(\pi x_1)\cos(\pi x_2),\\
&z=\sin(2\pi x_1)\cos(2\pi x_2),\\
&u=\max(-z,0),\\
&f=2\pi^2 y-u,\\
&y_d=-8\pi^2 z+y,\\
&\boldsymbol{p}_d=\begin{pmatrix}-\pi\cos(\pi x_1)\cos(\pi x_2)\\ \pi\sin(\pi x_1)\sin(\pi x_2)\end{pmatrix}.
\end{aligned}
$$

Let *NE* and h denote the number of the mesh elements and the mesh size, respectively. The corresponding numerical results for variables $\boldsymbol{p},\boldsymbol{q},y,z$, and u are shown in Tables 1-6. Figs. 2-5 demonstrate the numerical and exact solutions $\boldsymbol{p}_h, y_h$, and u_h. Since the numerical solution y_h and u_h are obtained by using piecewise constant approximations, the corresponding graphs look discontinuous.

Table 1: Example 5.1. Lloyd mesh: Errors and convergence rates.

NE	h	err($\boldsymbol{p}, \boldsymbol{V}$)	rate	err($\boldsymbol{q}, \boldsymbol{V}$)	rate
100	1.000e-01	1.43184e-01	-	2.58594e-01	-
400	5.000e-02	7.04590e-02	1.02	1.29739e-01	1.00
900	3.333e-02	4.66296e-02	1.02	8.70785e-02	0.98
1600	2.500e-02	3.52747e-02	0.97	6.54631e-02	0.99

Table 2: Example 5.1. Lloyd mesh: Errors and convergence rates.

NE	h	err(y, L^2)	rate	err(z, L^2)	rate	err(u, L^2)	rate
100	1.000e-01	1.28923e-01	-	2.58216e-01	-	8.89665e-02	-
400	5.000e-02	6.35910e-02	1.02	1.27136e-01	1.02	4.49469e-02	0.99
900	3.333e-02	4.23962e-02	1.00	8.50062e-02	0.99	2.98367e-02	1.01
1600	2.500e-02	3.17756e-02	1.00	6.37588e-02	1.00	2.23150e-02	1.01

Table 3: Example 5.1. Square mesh: Errors and convergence rates.

NE	h	err($\boldsymbol{p}, \boldsymbol{V}$)	rate	err($\boldsymbol{q}, \boldsymbol{V}$)	rate
100	1.000e-01	1.43630e-01	-	2.61038e-01	-
400	5.000e-02	7.20185e-02	1.00	1.30791e-01	1.00
900	3.333e-02	4.81139e-02	1.00	8.71645e-02	1.00
1600	2.500e-02	3.61120e-02	1.00	6.62046e-02	1.00

Table 4: Example 5.1. Square mesh: Errors and convergence rates.

NE	h	err(y, L^2)	rate	err(z, L^2)	rate	err(u, L^2)	rate
100	1.000e-01	1.28804e-01	-	2.57846e-01	-	9.13153e-02	-
400	5.000e-02	6.41997e-02	1.00	1.28800e-01	1.00	4.55379e-02	1.00
900	3.333e-02	4.27733e-02	1.00	8.56728e-02	1.01	3.02882e-02	1.01
1600	2.500e-02	3.20729e-02	1.00	6.42002e-02	1.00	2.26982e-02	1.00

Table 5: Example 5.1. Distorted Square mesh: Errors and convergence rates.

NE	h	err($\boldsymbol{p}, \boldsymbol{V}$)	rate	err($\boldsymbol{q}, \boldsymbol{V}$)	rate
100	1.000e-01	2.02417e-01	-	3.08795e-01	-
400	5.000e-02	9.93049e-02	1.03	1.56118e-01	0.98
900	3.333e-02	6.74900e-02	0.95	1.05325e-01	0.97
1600	2.500e-02	5.03964e-02	1.02	7.89470e-02	1.00

Table 6: Example 5.1. Distorted Square mesh: Errors and convergence rates.

NE	h	err(y, L^2)	rate	err(z, L^2)	rate	err(u, L^2)	rate
100	1.000e-01	1.49095e-01	-	3.04757e-01	-	1.00776e-01	-
400	5.000e-02	7.32200e-02	1.03	1.48529e-01	1.04	5.04830e-02	1.00
900	3.333e-02	4.89895e-02	0.99	9.91443e-02	1.00	3.38778e-02	0.98
1600	2.500e-02	3.66600e-02	1.01	7.41133e-02	1.01	2.52909e-02	1.02

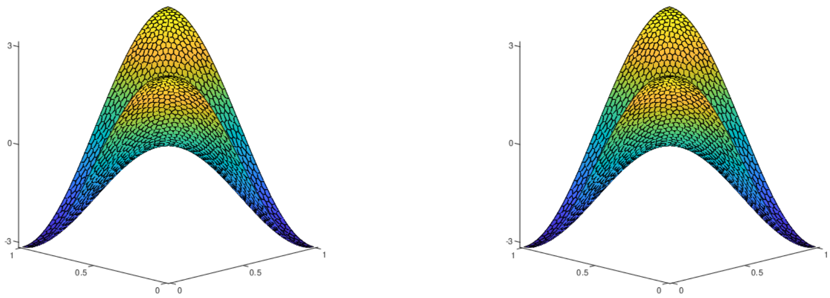


Figure 2: Example 5.1. Lloyd mesh. Left: Numerical solution $\boldsymbol{p}_{1,h}$. Right: Exact solution $\boldsymbol{p}_1$.

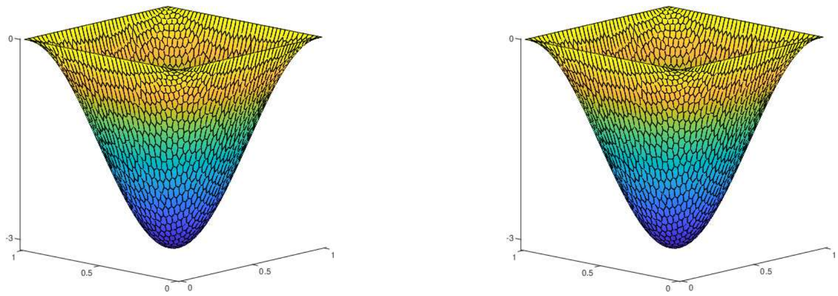


Figure 3: Example 5.1. Lloyd mesh. Left: Numerical solution $\boldsymbol{p}_{2,h}$. Right: Exact solution $\boldsymbol{p}_2$.

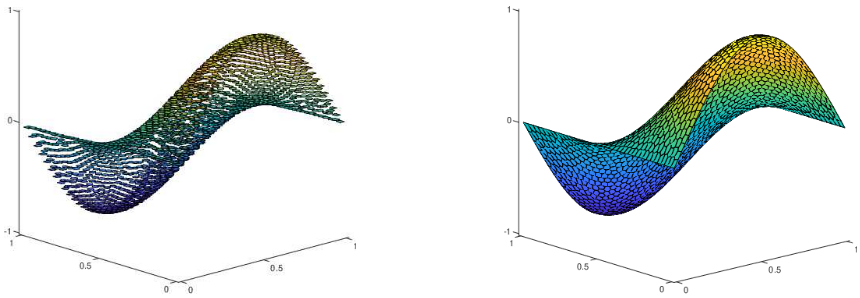


Figure 4: Example 5.1. Lloyd mesh. Left: Numerical solution y_h. Right: Exact solution y.

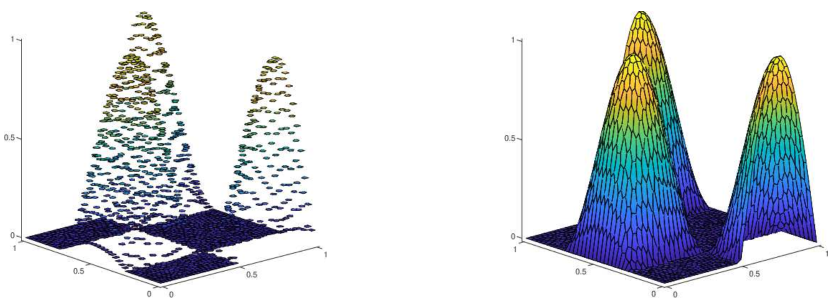


Figure 5: Example 5.1. Lloyd mesh. Left: Numerical solution u_h. Right: Exact solution u.

Example 5.2. Set $\Gamma_1 = \{(x_1, x_2) : x_1 = 0, 0 \le x_2 \le 1\} \cup \{(x_1, x_2) : 0 \le x_1 \le 1, x_2 = 0\}$ and choose the following functions:

$$\begin{aligned}
&y = (x_1 + x_2)\sin(2\pi x_1)\sin(2\pi x_2),\\
&z = 5(x_1 + x_2)\sin(2\pi x_1)\sin(2\pi x_2),\\
&u = \max(-z, 0),\\
&f = -4\pi\cos(2\pi x_1)\sin(2\pi x_2) - 4\pi\sin(2\pi x_1)\cos(2\pi x_2) + 8\pi^2 y - u,\\
&y_d = 20\pi\cos(2\pi x_1)\sin(2\pi x_2) + 20\pi\sin(2\pi x_1)\cos(2\pi x_2) + y(1 - 40\pi^2),\\
&\boldsymbol{p}_d = \begin{pmatrix} -2\pi(x_1 + x_2)\cos(2\pi x_1)\sin(2\pi x_2) - \sin(2\pi x_1)\sin(2\pi x_2) \\ -2\pi(x_1 + x_2)\sin(2\pi x_1)\cos(2\pi x_2) - \sin(2\pi x_1)\sin(2\pi x_2) \end{pmatrix}.
\end{aligned}$$

The corresponding numerical results for variables $\boldsymbol{p}, \boldsymbol{q}, y, z$, and u are shown in Tables 7-12, whereas numerical and exact solutions $\boldsymbol{p}_h$, y_h and u_h in Figs. 6-9.

Table 7: Example 5.2. Lloyd mesh: Errors and convergence rates.

NE	h	err($\boldsymbol{p}, \boldsymbol{V}$)	rate	err($\boldsymbol{q}, \boldsymbol{V}$)	rate
100	1.000e-01	2.58459e-01	-	2.57837e-01	-
400	5.000e-02	1.31584e-01	0.97	1.31110e-01	0.98
900	3.333e-02	8.76309e-02	1.00	8.74036e-02	1.00
1600	2.500e-02	6.52741e-02	1.02	6.52263e-02	1.02

Table 8: Example 5.2. Lloyd mesh: Errors and convergence rates.

NE	h	err(y, L^2)	rate	err(z, L^2)	rate	err(u, L^2)	rate
100	1.000e-01	2.57168e-01	-	2.58241e-01	-	4.57624e-01	-
400	5.000e-02	1.28500e-01	1.00	1.28652e-01	1.01	2.27478e-01	1.01
900	3.333e-02	8.53369e-02	1.01	8.53819e-02	1.01	1.51916e-01	1.00
1600	2.500e-02	6.35928e-02	1.02	6.36119e-02	1.02	1.14344e-01	0.99

Table 9: Example 5.2. Square mesh: Errors and convergence rates.

NE	h	err($\boldsymbol{p}, \boldsymbol{V}$)	rate	err($\boldsymbol{q}, \boldsymbol{V}$)	rate
100	1.000e-01	2.67876e-01	-	2.65872e-01	-
400	5.000e-02	1.33825e-01	1.00	1.33522e-01	1.00
900	3.333e-02	8.88719e-02	1.01	8.88702e-02	1.00
1600	2.500e-02	6.67404e-02	1.00	6.67394e-02	1.00

Table 10: Example 5.2. Square mesh: Errors and convergence rates.

NE	h	err(y, L^2)	rate	err(z, L^2)	rate	err(u, L^2)	rate
100	1.000e-01	2.61063e-01	-	2.61087e-01	-	4.65368e-01	-
400	5.000e-02	1.30122e-01	1.00	1.30259e-01	1.00	2.32445e-01	1.00
900	3.333e-02	8.65876e-02	1.00	8.66288e-02	1.01	1.54590e-01	1.01
1600	2.500e-02	6.48956e-02	1.00	6.49130e-02	1.00	1.15839e-01	1.00

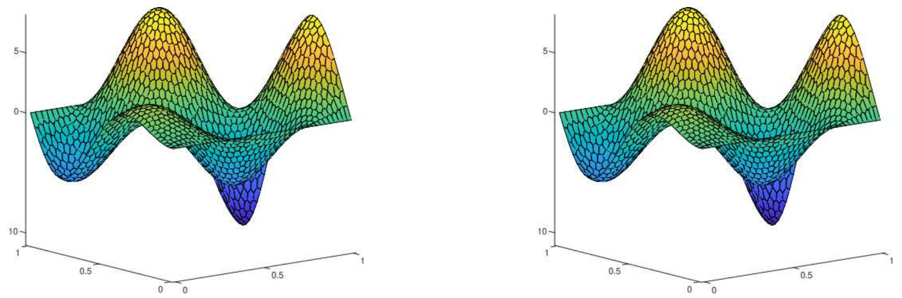


Figure 6: Example 5.2. Lloyd mesh. Left: Numerical solution $\boldsymbol{p}_{1,h}$. Right: Exact solution $\boldsymbol{p}_1$.

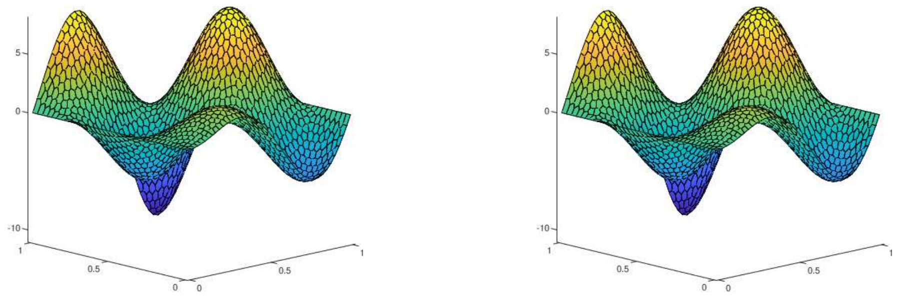


Figure 7: Example 5.2. Lloyd mesh. Left: Numerical solution $\boldsymbol{p}_{2,h}$. Right: Exact solution $\boldsymbol{p}_2$.

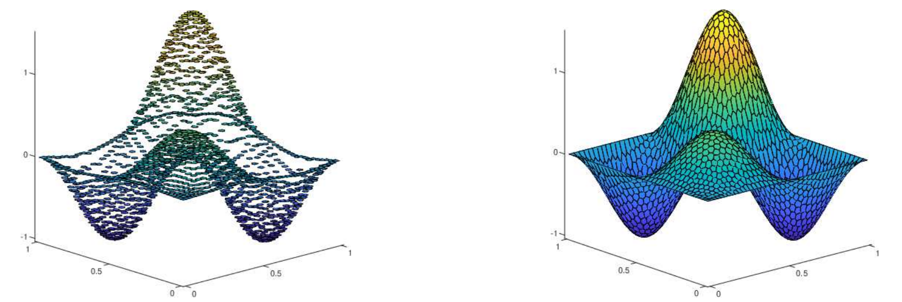


Figure 8: Example 5.2. Lloyd mesh. Left: Numerical solution y_h. Right: Exact solution y.

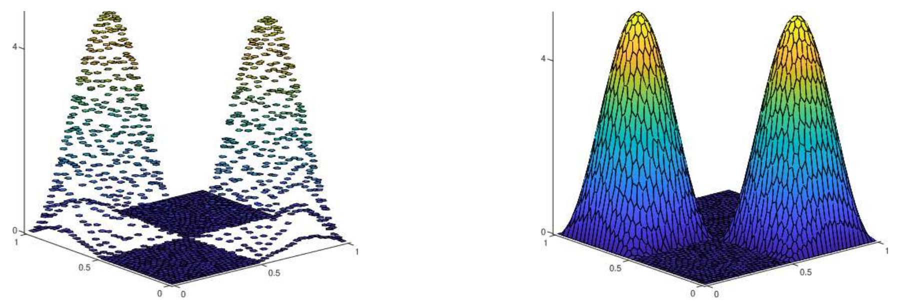


Figure 9: Example 5.2. Lloyd mesh. Left: Numerical solution u_h. Right: Exact solution u.

Table 11: Example 5.2. Distorted Square mesh: Errors and convergence rates.

NE	h	err$(\boldsymbol{p}, \boldsymbol{V})$	rate	err$(\boldsymbol{q}, \boldsymbol{V})$	rate
100	1.000e-01	3.07429e-01	-	2.99900e-01	-
400	5.000e-02	1.56848e-01	0.97	1.53022e-01	0.97
900	3.333e-02	1.06155e-01	0.96	1.03295e-01	0.97
1600	2.500e-02	7.96413e-02	1.00	7.75268e-02	1.00

Table 12: Example 5.2. Distorted Square mesh: Errors and convergence rates.

NE	h	err(y, L^2)	rate	err(z, L^2)	rate	err(u, L^2)	rate
100	1.000e-01	3.31522e-01	-	3.36504e-01	-	5.68256e-01	-
400	5.000e-02	1.51435e-01	1.13	1.52066e-01	1.15	2.77831e-01	1.03
900	3.333e-02	9.97310e-02	1.03	9.99264e-02	1.04	1.85581e-01	1.00
1600	2.500e-02	7.41774e-02	1.03	7.42598e-02	1.03	1.38719e-01	1.01

6. Conclusion

We solve the control constrained optimal control problem for the Darcy equation by a mixed virtual element method. A priori error estimates are derived for the state, adjoint state, and control variables. Compared with mixed finite element method, the mixed virtual element method is more flexible with respect to the mesh refinement. Adaptive mixed virtual element methods for optimal control problem can be also investigated.

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