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OPTIMAL CONTROL OF A QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH HISTORY-DEPENDENT OPERATORS

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Abstract. In this paper, we are concerned with an optimal control problem of a quasistatic frictional contact model with history-dependent operators. The contact boundary of the model is divided into two parts where different contact conditions are specified. For the contact problem, we first derive its weak formulation and prove the existence and uniqueness of the solution to the weak formulation. Then we give a priori estimate of the unique solution and prove a continuous dependence result for the solution map. Finally, an optimal control problem that contains boundary and initial condition controls is proposed, and the existence of optimal solutions to the control problem is established.

Key words. Variational inequality, contact problem, history-dependent operator, optimal control.

1. Introduction

Contact models play a significant role in mechanical engineering and have long been an important topic of research for scholars. The theory of variational inequalities [1, 2, 16, 17, 18] provides an effective way to study contact problems. As the research progresses, the concept of history-dependent operators was first introduced in [19]. These operators are used to model contact problems with long memory. The recent references related to history-denpendent operators can be found in [3, 11, 12, 13, 14, 20, 21, 29].

From the point of view of practical applications, it is great meaningful to study optimal control problems in contact mechanics. The subject of optimal control of variational inequalities was first studied in [23] and was developed by [24, 25, 26]. In [27], the existence of the solution to an optimal control problem is proved and the convergence for the regularized control problem is studied. The reference [6] and [7] prove the existence and approximation results of optimal solutions to a class of quasilinear elliptic variational inequalities and a nonlinear elliptic inclusion, respectively. In [28], the authors consider the numerical solutions for the optimal control of a class of variational-hemivariational inequalities and deduce the convergence result. As for evolutional case, the reference [14] studies an optimal control for a class of subdifferential evolution inclusions involving history-dependent operators and [4] focuses on the boundary optimal control of a dynamic frictional contact problem. The works of these two papers give us a great inspiration.

In this paper, we study an optimal control problem of a quasistatic friction contact problem involving history-dependent operators. The contact model was proposed in [8], and the special feature of the model lies on its contact boundary, which is divided into two parts with different contact conditions. The difference

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is that we consider boundary conditions related to the diaplacement field instead of the velocity field. In [8], the existence and uniqueness of the solution to the weak problem is proved and the error estimate for a discrete scheme is derived. However, the work of this paper is a useful exploration of the problem from another perspective. The main novelty is that we prove a continuous dependence result for the solution map of a quasistatic problem. Compared with dynamic problems, quasistatic problems [5, 10] are more difficult to derive a continuous dependence result and there is little relevant literature. Moreover, we consider control variables with regard to both boundary and initial conditions, and a cost functional that combines observations within the domain, on the boundary and at the terminal time. The techniques used in this work can also be applied to study some forms of variational-hemivariational inequalities, that is, weak formulations of some contact problems involving both convex and Clarke subdifferentials.

The rest of the paper is structured as follows. In Section 2 we recall some basic notation and present several preliminary results. In Section 3 we introduce a quasistatic contact problem with history-dependent operators. The existence and uniqueness of the solution is given and a priori estimate for the solution is proved. In Section 4, we deduce a continuous dependence result for the solution map based on the evolution inclusion. In Section 5, we prove that an optimal control problem has at least one solution, based on the continuous dependence result.

2. Notation and preliminaries

In this section, we recall some basic notation and known results that will be used later in the paper. Let X be a real Banach space. Throughout the paper, we denote by $\|\cdot\|_X$ and X^* the norm in X and its dual space, respectively. The notation X_w denotes X equipped with the weak topology. Furthermore, if X is a real Hilbert space, we denote by $(\cdot,\cdot)_X$ the inner product on X. We start with the definitions of the (convex) subdifferential and subgradient.

Definition 2.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Assume that $u \in X$ is such that $f(u) \neq \infty$. Then, the subdifferential of f at u is the set*

$$\partial f(x) = \{\xi \in X^* \mid f(v) - f(u) \geq \langle \xi, v-u\rangle_{X^*\times X}\,,\ \forall\ v \in X\}.$$

Each element $\xi \in \partial f(u)$ is called a subgradient of f at u.

For a function $\psi : X \to \mathbb{R} \cup \{+\infty\}$, we use the notation $D(\psi)$ for the effective domain of ψ, i.e.

$$D(\psi) = \{u \in X \mid \psi(u) \neq \infty\}.$$

The following lemma will be used in Section 3 to prove the unique weak solvability of a contact problem, and its proof can be found in [24], page 35.

Lemma 2.2. *Let X be a real Hilbert space and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex proper lower semicontinuous function. Then, for every $f \in L^2(0,T;X)$ and $u_0 \in D(\psi)$, there exists a unique function $u \in H^1(0,T;X)$ which satisfies*

$$\begin{aligned} &u'(t) + \partial\psi(u(t)) \ni f(t) \quad a.e.\ t \in (0,T),\\ &u(0) = u_0. \end{aligned}$$

Then we recall the following consequence of the Banach fixed point theorem ([3], Lemma 3).

Theorem 2.3. *Let X be a Banach space and $0 < T < +\infty$. Let $\Lambda : L^2(0,T;X) \to L^2(0,T;X)$ be an operator such that*

$$\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_X^2 \le c\int_0^t \|\eta_1(s) - \eta_2(s)\|_X^2 ds,$$

for all $\eta_1, \eta_2 \in L^2(0,T;X)$, a.e. $t \in (0,T)$ with a constant $c > 0$. Then Λ has a unique fixed point in $L^2(0,T;X)$, i.e., there exists a unique $\eta^* \in L^2(0,T;X)$ such that $\Lambda\eta^* = \eta^*$.

Next, we introduce a result related to upper semicontinuous multivalued functions, which can be found in the appendix of [4].

Lemma 2.4. *Let X be a topological space and Y be a Banach space. Assume that $G : X \to 2^Y$ is a multivalued mapping such that*

(i) *G has nonempty, closed and convex values in Y.*
(ii) *G is upper semicontinuous from X to Y_w.*

If $x_n : (0,T) \to X$ and $y_n : (0,T) \to Y$ are measurable functions such that $x_n(t) \to x(t)$ in X for a.e. $t \in (0,T)$, $y_n \to y$ weakly in $L^1(0,T;Y)$ and $y_n(t) \in G(x_n(t))$ for a.e. $t \in (0,T)$. Then we have $y(t) \in G(x(t))$ for a.e. $t \in (0,T)$.

We conclude this section with a well-known Young's inequality

$$ab \le \varepsilon a^2 + c(\varepsilon)b^2,$$

for all $a, b \in \mathbb{R}, \varepsilon > 0$, where $c(\varepsilon) = 1/4\varepsilon$. Hereafter, we denote by $c(\varepsilon)$ a positive constant dependent on ε and its value can differ from line to line.

3. A quasistatic frictional contact problem

In this section, we present a quasistatic frictional contact problem with a history-dependent operator. The unique weak solvability for the problem will be proved.

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $d = 2, 3$, occupied in its reference configuration by a viscoelastic body. The boundary $\partial\Omega$ is Lipschitz continuous. The symbol $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$. We use $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ to denote the displacement vector, the stress tensor, and the strain tensor, respectively. Here $\varepsilon_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2$, where $u_{i,j} = \partial u_i/\partial x_j$. We denote by $\boldsymbol{\nu}$ the unit outward normal vector. For a vector field $\boldsymbol{v}$ defined on $\partial\Omega$, its normal and tangential components are $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. The normal and tangential components of the stress field $\boldsymbol{\sigma}$ are $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$.

We adopt the summation convention on a repeated index. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are respectively given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}, \qquad \|\boldsymbol{\varepsilon}\| = (\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon})^{1/2} \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d.$$

The boundary $\partial\Omega$ is split into three disjoint measurable parts Γ_1, Γ_2 and Γ_3. And Γ_3 is further divided into two parts $\Gamma_{3,1}$ and $\Gamma_{3,2}$ where different contact conditions will be specified. Assume that the measure of Γ_1 is positive and the body is clamped on it. Moreover, the measure of $\Gamma_{3,1}$ and $\Gamma_{3,2}$ cannot be zero at the same time. When one of them is zero, the corresponding contact condition below is suppressed from the problem. The volume forces of density $\boldsymbol{f}_0$ act in the body Ω, and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. We are interested in the evolutionary process of the mechanical state of the body on the time interval $[0,T]$ with $T>0$. The classical formulation of the contact problem is following.

Problem 3.1. *Find a displacement field* $\boldsymbol{u}:\Omega\times[0,T]\to\mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}:\Omega\times[0,T]\to\mathbb{S}^d$ *such that for all* $t\in[0,T]$,

$$\boldsymbol{\sigma}(t)=\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \qquad \text{in } \Omega, \tag{1}$$

$$\mathrm{Div}\boldsymbol{\sigma}(t)+\boldsymbol{f}_0(t)=\boldsymbol{0} \qquad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u}(t)=\boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu}=\boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2, \tag{4}$$

$$u_\nu(t)\le 0,\ \sigma_\nu(t)\le 0,\ \sigma_\nu(t)u_\nu(t)=0,\ \boldsymbol{\sigma}_\tau(t)=\boldsymbol{0} \qquad \text{on } \Gamma_{3,1}, \tag{5}$$

$$-\sigma_\nu(t)=F \qquad \text{on } \Gamma_{3,2}, \tag{6}$$

$$\|\boldsymbol{\sigma}_\tau(t)\|\le\mu|\sigma_\nu(t)|,\ -\boldsymbol{\sigma}_\tau(t)=\mu|\sigma_\nu(t)|\frac{\boldsymbol{u}_\tau(t)}{\|\boldsymbol{u}_\tau(t)\|} \text{ if } \boldsymbol{u}_\tau(t)\ne\boldsymbol{0} \qquad \text{on } \Gamma_{3,2}, \tag{7}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{8}$$

For a brief description on the mechanical interpretations of the problem above, the reader is referred to [8]. The difference is that we describe the unilateral constraints (5) and Coulomb's law of dry friction (7) on the displacement field.

Subsequently, we introduce the following functional spaces.

$$V=\{\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^d)\mid \boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\},$$
$$H=L^2(\Omega;\mathbb{R}^d),$$
$$Q=L^2(\Omega;\mathbb{S}^d).$$

The inner product in V is defined by

$$(\boldsymbol{u},\boldsymbol{v})_V=(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \text{for } \boldsymbol{u},\boldsymbol{v}\in V.$$

Since the measure of Γ_1 is positive, from the Korn inequality, we deduce from the assumption $H(\mathcal{A})$ introduced later that the space V is a real Hilbert space and the norm $\|\cdot\|_V$ is equivalent with the usual Sobolev norm on $H^1(\Omega;\mathbb{R}^d)$. Specifically,

$$\frac{1}{\sqrt{\|\mathcal{A}\|}}\|\boldsymbol{u}\|_V\le\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_Q\le\frac{1}{\sqrt{m_\mathcal{A}}}\|\boldsymbol{u}\|_V$$

In addition, we note that the embedding $V\subset H$ is compact. Given $0<T<+\infty$, we introduce spaces $\mathcal{V}=L^2(0,T;V)$, $\mathcal{H}=L^2(0,T;H)$. The set of admissible displacement fields is

$$K=\{\boldsymbol{v}\in V\mid v_\nu\le 0 \text{ a.e. on } \Gamma_{3,1}\}.$$

Moreover, we define a space of fourth-order tensor fields,

$$\mathcal{Q}_\infty=\{\mathcal{E}=(\mathcal{E}_{ijkl})\mid \mathcal{E}_{ijkl}=\mathcal{E}_{jikl}=\mathcal{E}_{klij}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d\},$$

and a convex and closed subset of $L^2(\Gamma_{3,2})$,

$$M=\{\mu\in L^2(\Gamma_{3,2})\mid 0<a\le\mu(\boldsymbol{x})\le b,\ \text{a.e. } \boldsymbol{x}\in\Gamma_{3,2}\}.$$

Now we introduce assumptions on the data of Problem 3.1.

$\underline{H(\mathcal{A})}$. The viscosity tensor $\mathcal{A}=(a_{ijkl}):\Omega\times\mathbb{S}^d\to\mathbb{S}^d$ is such that

(i) $\mathcal{A}\in\mathcal{Q}_\infty$;

(ii) $\mathcal{A}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} \geq m_{\mathcal{A}}\|\boldsymbol{\varepsilon}\|^2$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ a.e. in Ω with $m_{\mathcal{A}} > 0$.

$\underline{H(\mathcal{B})}$. The elasticity tensor $\mathcal{B} = (b_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) $b_{ijkl} \in L^\infty(\Omega)$, $1 \leq i,j,k,l \leq d$;
(ii) $\mathcal{B}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} \geq m_{\mathcal{B}}\|\boldsymbol{\varepsilon}\|^2$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ a.e. in Ω with $m_{\mathcal{B}} > 0$.

$\underline{H(\mathcal{R})}$. The relaxation tensor $\mathcal{R} : [0,T] \to \mathcal{Q}_\infty$ is Lipschitz continuous with constant $L_{\mathcal{R}} > 0$.

$\underline{H(\mathcal{C})}$. $F : \Gamma_{3,2} \to \mathbb{R}$ and $\mu : \Gamma_{3,2} \to \mathbb{R}$ satisfy

(i) $F \in L^2(\Gamma_{3,2})$, $F(\boldsymbol{x}) \geq 0$ for a.e. $\boldsymbol{x} \in \Gamma_{3,2}$;
(ii) $\mu \in M$.

$\underline{H(\boldsymbol{f})}$. The densities of forces, surface tractions and initial displacement satisfy

$$\boldsymbol{f}_0 \in \mathcal{H},\ \boldsymbol{f}_2 \in L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d)),$$
$$\boldsymbol{u}_0 \in V,\ u_{0,\nu}(x) < 0 \text{ for a.e. } x \in \Gamma_{3,1}.$$

By a standard procedure, one can obtain the following weak problem of Problem 3.1.

Problem 3.2. *Find $\boldsymbol{u} : [0,T] \to V$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and for a.e. $t \in (0,T)$, $\boldsymbol{u}(t) \in K$,*

$$\begin{aligned}&(\boldsymbol{u}'(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + \left(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}(t))\right)_Q \\ &+ \int_{\Gamma_{3,2}} [F(v_\nu - u_\nu(t)) + \mu F(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau(t)\|)]d\Gamma \\ &\geq (\boldsymbol{f}_0(t), \boldsymbol{v} - \boldsymbol{u}(t))_H + (\boldsymbol{f}_2(t), \boldsymbol{v} - \boldsymbol{u}(t))_{L^2(\Gamma_2;\mathbb{R}^d)}, \quad \text{for all } \boldsymbol{v} \in K.\end{aligned}$$

For the sake of simplicity, we introduce the following notations. Under the assumptions of $H(\mathcal{B})$, $H(\mathcal{R})$ and $H(\boldsymbol{f})$, according to Riesz representation theorem, we can define the operator $\mathcal{E} : \mathcal{V} \to \mathcal{V}$ by

$$((\mathcal{E}\boldsymbol{u})(t), \boldsymbol{v})_V = \left(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_Q,$$

for all $\boldsymbol{u} \in \mathcal{V}$, $\boldsymbol{v} \in V$, a.e. $t \in [0,T]$, and the operator $\boldsymbol{f} : [0,T] \to V$ by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)},$$

for all $\boldsymbol{v} \in V$, a.e. $t \in [0,T]$. Moreover, we introduce functionals $\varphi_1 : V \to \mathbb{R}$, $\varphi_2 : V \to \mathbb{R}$ and $\varphi_3 : V \to \mathbb{R} \cup \{+\infty\}$ defined as follows

$$\varphi_1(\boldsymbol{v}) = \int_{\Gamma_{3,2}} F v_\nu \, d\Gamma, \qquad \boldsymbol{v} \in V,$$
$$\varphi_2(\boldsymbol{v}) = \int_{\Gamma_{3,2}} \mu F \|\boldsymbol{v}_\tau\| \, d\Gamma, \qquad \boldsymbol{v} \in V,$$
$$\varphi_3(\boldsymbol{v}) = \int_{\Gamma_{3,1}} I_{(-\infty,0]}(v_\nu) \, d\Gamma, \qquad \boldsymbol{v} \in V,$$

where $I_{(-\infty,0]} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of interval $(-\infty, 0]$ defined by

$$I_{(-\infty,0]}(s) = \begin{cases} 0 & \text{if } s \in (-\infty, 0], \\ +\infty, & \text{otherwise.} \end{cases}$$

Using the above notations, we can write the following equivalent form of Problem 3.2.

Problem 3.3. *Find $\boldsymbol{u} : [0,T] \to V$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and for a.e. $t \in (0,T)$,*

$$\begin{aligned} &(\boldsymbol{u}'(t) + (\mathcal{E}\boldsymbol{u})(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + \varphi_1(\boldsymbol{v}) - \varphi_1(\boldsymbol{u}(t)) + \varphi_2(\boldsymbol{v}) - \varphi_2(\boldsymbol{u}(t)) \\ &+ \varphi_3(\boldsymbol{v}) - \varphi_3(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \text{for all } \boldsymbol{v} \in V. \end{aligned} \tag{9}$$

We have the following existence and uniqueness result.

Theorem 3.4. *Under the assumptions $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mathcal{R})$, $H(\mathcal{C})$ and $H(\boldsymbol{f})$, Problem* 3.3 *has a unique solution $\boldsymbol{u} \in H^1(0,T;V)$.*

Proof. The proof is carried out in three steps based on Lemma 2.2 and Theorem 2.3.

Step 1. Let $\boldsymbol{\eta} \in \mathcal{V}$ and consider the following auxiliary problem. Find $\boldsymbol{u}_{\boldsymbol{\eta}} : [0,T] \to V$ such that $\boldsymbol{u}_{\boldsymbol{\eta}}(0) = \boldsymbol{u}_0$ and for a.e. $t \in (0,T)$,

$$\begin{aligned} &\big(\boldsymbol{u}_{\boldsymbol{\eta}}'(t) + \boldsymbol{\eta}(t), \boldsymbol{v} - \boldsymbol{u}_{\boldsymbol{\eta}}(t)\big)_V + \varphi_1(\boldsymbol{v}) - \varphi_1(\boldsymbol{u}_{\boldsymbol{\eta}}(t)) + \varphi_2(\boldsymbol{v}) - \varphi_2(\boldsymbol{u}_{\boldsymbol{\eta}}(t)) \\ &+ \varphi_3(\boldsymbol{v}) - \varphi_3(\boldsymbol{u}_{\boldsymbol{\eta}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_{\boldsymbol{\eta}}(t))_V \quad \text{for all } \boldsymbol{v} \in V. \end{aligned} \tag{10}$$

To study the inequality (10), we define a functional $\psi : V \to \mathbb{R}$ by

$$\psi(\boldsymbol{v}) = \varphi_1(\boldsymbol{v}) + \varphi_2(\boldsymbol{v}) + \varphi_3(\boldsymbol{v}), \tag{11}$$

for all $\boldsymbol{v} \in V$, and consider the following evolutionary inclusion. Find $\boldsymbol{u}_{\boldsymbol{\eta}} : [0,T] \to V$ such that $\boldsymbol{u}_{\boldsymbol{\eta}}(0) = \boldsymbol{u}_0$ and for a.e. $t \in (0,T)$,

$$\boldsymbol{u}_{\boldsymbol{\eta}}'(t) + \partial\psi(\boldsymbol{u}_{\boldsymbol{\eta}}(t)) + \boldsymbol{\eta}(t) \ni \boldsymbol{f}(t). \tag{12}$$

From (11) and the definitions of φ_1, φ_2 and φ_3, we find that ψ is a convex proper lower semicontinuous function and $\boldsymbol{u}_0 \in D(\psi)$. Moreover, it is clear that $\boldsymbol{f} - \boldsymbol{\eta} \in \mathcal{V}$. Then it follows from Lemma 2.2 that problem (12) has a unique solution $\boldsymbol{u}_{\boldsymbol{\eta}} \in H^1(0,T;V)$.

Note that, from the definition of the convex subdifferential and (11), every solution to problem (12) is also a solution to problem (10). Now we prove the uniqueness of the solution to problem (10).

Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in H^1(0,T;V)$ be solutions to problem (10). Here we skip the subscripts $\boldsymbol{\eta}$. Take $\boldsymbol{u}_2(t)$ as the test function in the inequality for $\boldsymbol{u}_1$, take $\boldsymbol{u}_1(t)$ as the test function in the inequality for $\boldsymbol{u}_2$, and add the two resulting inequalities to get

$$(\boldsymbol{u}_1'(t) - \boldsymbol{u}_2'(t), \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t))_V \leq 0,$$

for a.e. $t \in (0,T)$. Integrating the above inequality over the time interval $[0,t]$, noting $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) = \boldsymbol{u}_0$, we have

$$\frac{1}{2}\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V^2 \leq 0,$$

for all $t \in [0,T]$. Thus, $\boldsymbol{u}_1 = \boldsymbol{u}_2$. So far, we have proved that problem (10) has a unique solution.

Step 2. According to the Riesz representation theorem, we can define $\Lambda\boldsymbol{\eta}(t) \in V$ by

$$(\Lambda\boldsymbol{\eta}(t), \boldsymbol{v})_V = ((\mathcal{E}\boldsymbol{u}_{\boldsymbol{\eta}})(t), \boldsymbol{v})_V\,, \tag{13}$$

for all $\boldsymbol{v} \in V$ and $t \in [0, T]$, where $\boldsymbol{u}_{\boldsymbol{\eta}} \in H^1(0, T; V)$ is the unique solution to problem (10).

We first show that $\Lambda\boldsymbol{\eta} \in H^1(0, T; V)$. Let $t_1, t_2 \in [0, T]$, from (13), we have

$$\begin{aligned}
&(\Lambda\boldsymbol{\eta}(t_1) - \Lambda\boldsymbol{\eta}(t_2), \boldsymbol{v})_V = (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(t_1)) - \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(t_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \\
&+ \left(\int_0^{t_1} \mathcal{R}(t_1 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds - \int_0^{t_2} \mathcal{R}(t_2 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_Q \\
\le& \left\|\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(t_1)) - \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(t_2))\right\|_Q \left\|\boldsymbol{\varepsilon}(\boldsymbol{v})\right\|_Q \\
&+ \left\|\int_0^{t_1} \mathcal{R}(t_1 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds - \int_0^{t_2} \mathcal{R}(t_2 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds\right\|_Q \left\|\boldsymbol{\varepsilon}(\boldsymbol{v})\right\|_Q\,,
\end{aligned} \tag{14}$$

for all $\boldsymbol{v} \in V$. By assumptions $H(\mathcal{B})$ and $H(\mathcal{R})$, we have

$$\begin{aligned}
&\left\|\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(t_1)) - \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(t_2))\right\|_Q \le \|\mathcal{B}\| \left\|\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(t_1)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(t_2))\right\|_Q \\
\le& \frac{\|\mathcal{B}\|}{\sqrt{m_{\mathcal{A}}}} \left\|\boldsymbol{u}_{\boldsymbol{\eta}}(t_1) - \boldsymbol{u}_{\boldsymbol{\eta}}(t_2)\right\|_V\,,
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\left\|\int_0^{t_1} \mathcal{R}(t_1 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds - \int_0^{t_2} \mathcal{R}(t_2 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds\right\|_Q \\
\le& \left\|\int_0^{t_1} \mathcal{R}(t_1 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds - \int_0^{t_1} \mathcal{R}(t_2 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds\right\|_Q \\
&+ \left\|\int_0^{t_1} \mathcal{R}(t_2 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds - \int_0^{t_2} \mathcal{R}(t_2 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))ds\right\|_Q \\
\le& \int_0^{t_1} L_{\mathcal{R}}|t_1 - t_2| \|\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))\|_Q ds + \left|\int_{t_1}^{t_2} \left\|\mathcal{R}(t_2 - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))\right\|_Q ds\right| \\
\le& L_{\mathcal{R}}|t_1 - t_2|\sqrt{T}\left(\int_0^T \|\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))\|_Q^2 ds\right)^{\frac{1}{2}} + c_{\mathcal{R}}\left|\int_{t_1}^{t_2} \|\boldsymbol{\varepsilon}(\boldsymbol{u}_{\boldsymbol{\eta}}(s))\|_Q ds\right| \\
\le& \left(L_{\mathcal{R}}\sqrt{\frac{T}{m_{\mathcal{A}}}}\|\boldsymbol{u}_{\boldsymbol{\eta}}\|_{\mathcal{V}} + \frac{c_{\mathcal{R}}}{\sqrt{m_{\mathcal{A}}}}\|\boldsymbol{u}_{\boldsymbol{\eta}}\|_{C([0,T];V)}\right)|t_1 - t_2|,
\end{aligned} \tag{16}$$

where $c_R = \|\mathcal{R}\|_{C([0,T];\mathcal{Q}_\infty)}$. Then we apply (15) and (16) to (14) to get

$$\begin{aligned}
(\Lambda\boldsymbol{\eta}(t_1) - \Lambda\boldsymbol{\eta}(t_2), \boldsymbol{v})_V \le& \frac{\|\mathcal{B}\|}{m_{\mathcal{A}}} \left\|\boldsymbol{u}_{\boldsymbol{\eta}}(t_1) - \boldsymbol{u}_{\boldsymbol{\eta}}(t_2)\right\|_V \|\boldsymbol{v}\|_V \\
+ \left(\frac{L_{\mathcal{R}}\sqrt{T}}{m_{\mathcal{A}}}\|\boldsymbol{u}_{\boldsymbol{\eta}}\|_{\mathcal{V}} + \frac{c_{\mathcal{R}}}{m_{\mathcal{A}}}\|\boldsymbol{u}_{\boldsymbol{\eta}}\|_{C([0,T];V)}\right)&|t_1 - t_2|\|\boldsymbol{v}\|_V
\end{aligned}$$

Take $\boldsymbol{v} = \Lambda\boldsymbol{\eta}(t_1) - \Lambda\boldsymbol{\eta}(t_2)$ in the above inequality and obtain

$$\left\|\Lambda\boldsymbol{\eta}(t_1) - \Lambda\boldsymbol{\eta}(t_2)\right\|_V \le \frac{\|\mathcal{B}\|}{m_{\mathcal{A}}} \left\|\boldsymbol{u}_{\boldsymbol{\eta}}(t_1) - \boldsymbol{u}_{\boldsymbol{\eta}}(t_2)\right\|_V$$

$$+\left(\frac{L_{\mathcal{R}}\sqrt{T}}{m_{\mathcal{A}}}\|\boldsymbol{u}_{\boldsymbol{\eta}}\|_{\mathcal{V}}+\frac{c_{\mathcal{R}}}{m_{\mathcal{A}}}\|\boldsymbol{u}_{\boldsymbol{\eta}}\|_{C([0,T];V)}\right)|t_1-t_2|. \tag{17}$$

Since $\boldsymbol{u}_{\boldsymbol{\eta}} \in H^1(0,T;V)$, we deduce from (17) that $\Lambda\boldsymbol{\eta} \in H^1(0,T;V)$.

Now, let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in \mathcal{V}$ and let $t \in [0,T]$, we use the notation $\boldsymbol{u}_1 = \boldsymbol{u}_{\boldsymbol{\eta}_1}$ and $\boldsymbol{u}_2 = \boldsymbol{u}_{\boldsymbol{\eta}_2}$. From (13), we have

$$(\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t),\boldsymbol{v})_V = (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t))-\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_2(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$$
$$+\left(\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s))ds-\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_2(s))ds,\boldsymbol{\varepsilon}(\boldsymbol{v})\right)_Q,$$

for all $\boldsymbol{v} \in V$. Arguments similar to (14)–(17) lead to

$$\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_V \le \frac{L_{\mathcal{B}}}{m_{\mathcal{A}}}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V$$
$$+\frac{c_{\mathcal{R}}}{m_{\mathcal{A}}}\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V ds. \tag{18}$$

We get from (10) that for a.e. $t \in (0,T)$,

$$(\boldsymbol{u}_1'(t)-\boldsymbol{u}_2'(t),\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))_V \le (\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\boldsymbol{u}_2(t)-\boldsymbol{u}_1(t))_V.$$

Integrating the above inequality over the time interval $[0,t]$, we have

$$\frac{1}{2}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \le \int_0^t (\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s),\boldsymbol{u}_2(s)-\boldsymbol{u}_1(s))_V ds$$
$$\le \int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V ds$$
$$\le \frac{1}{2}\int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2 ds+\frac{1}{2}\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2 ds. \tag{19}$$

We use Gronwall's inequality in (19) to get

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \le c\int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2 ds. \tag{20}$$

Applying (20) to (18), we obtain

$$\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_V^2 \le c\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+c\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2 ds$$
$$\le c\int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2 ds+c\int_0^t\int_0^s \|\boldsymbol{\eta}_1(r)-\boldsymbol{\eta}_2(r)\|_V^2 dr ds$$
$$\le c\int_0^t \|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2 ds,$$

for a.e. $t \in (0,T)$. By Theorem 2.3, we deduce that there exists a unique fixed point $\boldsymbol{\eta}^*$ of Λ.

Step 3. Let $\boldsymbol{\eta}^* \in \mathcal{V}$ be the unique fixed point of the operator Λ. Let $\boldsymbol{u}_{\boldsymbol{\eta}^*} \in H^1(0,T;V)$ be the unique solution to problem (10) corresponding to $\boldsymbol{\eta}^*$. From the definition of the operator Λ, we have

$$(\boldsymbol{\eta}^*(t),\boldsymbol{v})_V = (\Lambda\boldsymbol{\eta}^*(t),\boldsymbol{v})_V = ((\mathcal{E}\boldsymbol{u}_{\boldsymbol{\eta}^*})(t),\boldsymbol{v})_V\,,$$

for all $\boldsymbol{v} \in V$ and $t \in [0,T]$. Thus, $\boldsymbol{u}_{\boldsymbol{\eta}^*} \in H^1(0,T;V)$ is the unique solution to Problem 3.3. This completes the proof of the theorem. □

4. Continuous dependence result

In this section, we first introduce an evolutionary inclusion of the subdifferential type, which share the same unique solution with Problem 3.3. Then, some priori estimates for the solution to Problem 3.3 will be established. Finally, we prove a continuous dependence result for the solution map.

We introduce functionals $\tilde{\varphi}_1 : L^2(\Gamma_{3,2}) \to \mathbb{R}$, $\tilde{\varphi}_2 : M \times L^2(\Gamma_{3,2}; \mathbb{R}^d) \to \mathbb{R}$ and $\tilde{\varphi}_3 : L^2(\Gamma_{3,1}) \to \mathbb{R} \cup \{+\infty\}$ defined as follows

$$\tilde{\varphi}_1(v) = \int_{\Gamma_{3,2}} Fv \, d\Gamma, \qquad v \in L^2(\Gamma_{3,2}),$$

$$\tilde{\varphi}_2(\mu, \boldsymbol{v}) = \int_{\Gamma_{3,2}} \mu F \|\boldsymbol{v}\| \, d\Gamma, \qquad \mu \in M, \ \boldsymbol{v} \in L^2(\Gamma_{3,2}; \mathbb{R}^d),$$

$$\tilde{\varphi}_3(v) = \int_{\Gamma_{3,1}} I_{(-\infty,0]}(v) \, d\Gamma, \qquad v \in L^2(\Gamma_{3,1}).$$

Let $\gamma_1 : V \to L^2(\Gamma_{3,2})$ and $\gamma_2 : V \to L^2(\Gamma_{3,2}; \mathbb{R}^d)$ be operators of the normal and tangential traces on $\Gamma_{3,2}$, respectively, defined by $\gamma_1(\boldsymbol{v}) = v_\nu$ and $\gamma_2(\boldsymbol{v}) = \boldsymbol{v}_\tau$ for $\boldsymbol{v} \in V$. Let $\gamma_3 : V \to L^2(\Gamma_{3,1})$ be the operator of the normal trace on $\Gamma_{3,1}$. We can easily find that the following problem is equivalent to Problem 3.3.

Problem 4.1. *Find $\boldsymbol{u} : [0,T] \to V$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and for a.e. $t \in (0,T)$,*

$$\begin{aligned}&(\boldsymbol{u}'(t) + (\mathcal{E}\boldsymbol{u})(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + \tilde{\varphi}_1(\gamma_1 \boldsymbol{v}) - \tilde{\varphi}_1(\gamma_1 \boldsymbol{u}(t))\\ &+ \tilde{\varphi}_2(\mu, \gamma_2 \boldsymbol{v}) - \tilde{\varphi}_2(\mu, \gamma_2 \boldsymbol{u}(t)) + \tilde{\varphi}_3(\gamma_3 \boldsymbol{v}) - \tilde{\varphi}_3(\gamma_3 \boldsymbol{u}(t))\\ &\geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \text{for all } \boldsymbol{v} \in V.\end{aligned} \tag{21}$$

We consider the following evolutionary inclusion problem.

Problem 4.2. *Find $\boldsymbol{u} : [0,T] \to V$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and for a.e. $t \in (0,T)$,*

$$\begin{aligned}&\boldsymbol{u}'(t) + (\mathcal{E}\boldsymbol{u})(t) + \gamma_1^* \partial \tilde{\varphi}_1(\gamma_1 \boldsymbol{u}(t))\\ &+ \gamma_2^* \partial \tilde{\varphi}_2(\mu, \gamma_2 \boldsymbol{u}(t)) + \gamma_3^* \partial \tilde{\varphi}_3(\gamma_3 \boldsymbol{u}(t)) \ni \boldsymbol{f}(t) \text{ in } V.\end{aligned} \tag{22}$$

Note that (22) can also be writed as

$$\boldsymbol{u}'(t) + (\mathcal{E}\boldsymbol{u})(t) + \gamma_1^* \xi_1(t) + \gamma_2^* \boldsymbol{\xi}_2(t) + \gamma_3^* \xi_3(t) = \boldsymbol{f}(t), \quad \text{in } V, \tag{23}$$

where

$$\xi_1(t) \in \partial \tilde{\varphi}_1(\gamma_1 \boldsymbol{u}(t)), \quad \boldsymbol{\xi}_2(t) \in \gamma_2^* \partial \tilde{\varphi}_2(\mu, \gamma_2 \boldsymbol{u}(t)), \quad \xi_3(t) \in \partial \tilde{\varphi}_3(\gamma_3 \boldsymbol{u}(t)). \tag{24}$$

Furthermore, using the corresponding Nemitsky operators, (23) can be equivalently written as

$$\boldsymbol{u}' + \mathcal{E}\boldsymbol{u} + \gamma_1^* \xi_1 + \gamma_2^* \boldsymbol{\xi}_2 + \gamma_3^* \xi_3 = \boldsymbol{f}, \quad \text{in } \mathcal{V}. \tag{25}$$

Here and after, we may not use new notations for Nemitsky operators and it will not cause confusion.

Lemma 4.3. *Problems* 4.1 *and* 4.2 *are equivalent.*

Proof. On the one hand, assume $\boldsymbol{u} \in H^1(0,T;V)$ is a solution of Problem 4.1. Since $\boldsymbol{u}$ is also the solution of Problem 3.3, we equivalently write (9) as

$$\begin{aligned}&(-\boldsymbol{u}'(t) - (\mathcal{E}\boldsymbol{u})(t) + \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \leq \varphi_1(\boldsymbol{v}) - \varphi_1(\boldsymbol{u}(t))\\ &+ \varphi_2(\boldsymbol{v}) - \varphi_2(\boldsymbol{u}(t)) + \varphi_3(\boldsymbol{v}) - \varphi_3(\boldsymbol{u}(t)) \quad \text{for all } \boldsymbol{v} \in V.\end{aligned}$$

From the definition of the convex subdifferential, we have

$$-\boldsymbol{u}'(t) - (\mathcal{E}\boldsymbol{u})(t) + \boldsymbol{f}(t) \in \ \partial\varphi_1(\boldsymbol{u}(t)) + \partial\varphi_2(\boldsymbol{u}(t)) + \partial\varphi_3(\boldsymbol{u}(t)) \quad \text{in } V.$$

In addition, we have the following properties

$$\begin{aligned}
\partial\varphi_1(\boldsymbol{v}) &= \gamma_1^*\partial\tilde{\varphi}_1(\gamma_1\boldsymbol{v}),\\
\partial\varphi_2(\boldsymbol{v}) &= \gamma_2^*\partial\tilde{\varphi}_2(\mu,\gamma_2\boldsymbol{v}),\\
\partial\varphi_3(\boldsymbol{v}) &= \gamma_3^*\partial\tilde{\varphi}_3(\gamma_3\boldsymbol{v}),
\end{aligned}$$

for all $\boldsymbol{v} \in V$. Thus, we obtain

$$\begin{aligned}
&-\boldsymbol{u}'(t) - (\mathcal{E}\boldsymbol{u})(t) + \boldsymbol{f}(t)\\
&\in \gamma_1^*\partial\tilde{\varphi}_1(\gamma_1\boldsymbol{u}(t)) + \gamma_2^*\partial\tilde{\varphi}_2(\mu,\gamma_2\boldsymbol{u}(t)) + \gamma_3^*\partial\tilde{\varphi}_3(\gamma_3\boldsymbol{u}(t)) \quad \text{in } V.
\end{aligned}$$

This is (22). On the other hand, assume $\boldsymbol{u} \in H^1(0,T;V)$ is a solution of Problem 4.2. According to (23), (24) and the definition of the convex subdifferential, we have

$$\begin{aligned}
&(-\boldsymbol{u}'(t) - (\mathcal{E}\boldsymbol{u})(t) + \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V\\
&= (\xi_1(t), \gamma_1\boldsymbol{v} - \gamma_1\boldsymbol{u}(t))_V + (\boldsymbol{\xi}_2(t), \gamma_2\boldsymbol{v} - \gamma_2\boldsymbol{u}(t))_V + (\xi_3(t), \gamma_3\boldsymbol{v} - \gamma_3\boldsymbol{u}(t))_V\\
&\le \tilde{\varphi}_1(\gamma_1\boldsymbol{v}) - \tilde{\varphi}_1(\gamma_1\boldsymbol{u}(t)) + \tilde{\varphi}_2(\mu,\gamma_2\boldsymbol{v}) - \tilde{\varphi}_2(\mu,\gamma_2\boldsymbol{u}(t)) + \tilde{\varphi}_3(\gamma_3\boldsymbol{v}) - \tilde{\varphi}_3(\gamma_3\boldsymbol{u}(t)),
\end{aligned}$$

which verifies (21) and completes the proof. □

Now we define operators $\mathcal{E}_0 : \mathcal{V} \to \mathcal{V}$, $\mathcal{N}_0 : \mathcal{V} \to L^2(0,T;L^2(\Gamma_{3,2}))$, $\bar{\mathcal{N}}_0 : \mathcal{V} \to L^2(0,T;L^2(\Gamma_{3,2};\mathbb{R}^d))$ and $\mathcal{M}_0 : \mathcal{V} \to L^2(0,T;L^2(\Gamma_{3,1}))$ by

$$\begin{aligned}
&(\mathcal{E}_0\boldsymbol{v})(t) = (\mathcal{E}(\boldsymbol{v} + \boldsymbol{u}_0))(t),\\
&(\mathcal{N}_0\boldsymbol{v})(t) = \{\xi \in L^2(0,T;L^2(\Gamma_{3,2})) \mid \xi(t) \in \partial\tilde{\varphi}_1(\gamma_1(\boldsymbol{v}(t) + \boldsymbol{u}_0))\},\\
&(\bar{\mathcal{N}}_0\boldsymbol{v})(t) = \{\boldsymbol{\xi} \in L^2(0,T;L^2(\Gamma_{3,2};\mathbb{R}^d)) \mid \boldsymbol{\xi}(t) \in \partial\tilde{\varphi}_2(\mu,\gamma_2(\boldsymbol{v}(t) + \boldsymbol{u}_0))\},\\
&(\mathcal{M}_0\boldsymbol{v})(t) = \{\xi \in L^2(0,T;L^2(\Gamma_{3,1})) \mid \xi(t) \in \partial\tilde{\varphi}_3(\gamma_3(\boldsymbol{v}(t) + \boldsymbol{u}_0))\},
\end{aligned}$$

for all $\boldsymbol{v} \in \mathcal{V}$ and a.e. $t \in (0,T)$. We propose the following evolutionary inclusion problem.

Problem 4.4. *Find $\boldsymbol{w} : [0,T] \to V$ such that $\boldsymbol{w}(0) = \boldsymbol{0}$ and*

$$\boldsymbol{w}' + \mathcal{E}_0\boldsymbol{w} + \gamma_1^*\zeta_1 + \gamma_2^*\boldsymbol{\zeta}_2 + \gamma_3^*\zeta_3 = \boldsymbol{f}, \quad \text{in } \mathcal{V}. \tag{26}$$

where

$$\zeta_1 \in \mathcal{N}_0\boldsymbol{w}, \quad \boldsymbol{\zeta}_2 \in \bar{\mathcal{N}}_0\boldsymbol{w}, \quad \zeta_3 \in \mathcal{M}_0\boldsymbol{w}. \tag{27}$$

It is obvious that $\boldsymbol{w} \in H^1(0,T;V)$ is a solution to Problem 4.4 if and only if $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_0$ is a solution to Problem 4.2.

Theorem 4.5. *Under the assumptions $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mathcal{R})$, $H(\mathcal{C})$ and $H(\boldsymbol{f})$, the unique solution to Problem 4.1 has the following estimate*

$$\|\boldsymbol{u}\|_{\mathcal{V}} + \|\boldsymbol{u}'\|_{\mathcal{V}} \le c(\boldsymbol{u}_0, \boldsymbol{f}), \tag{28}$$

where $c(\boldsymbol{u}_0,\boldsymbol{f})$ denote a constant dependent on $\|\boldsymbol{u}_0\|_V$ and $\|\boldsymbol{f}\|_{\mathcal{V}}$.

Proof. Take $\boldsymbol{v} = \boldsymbol{0}$ in (21), we have

$$(\boldsymbol{u}'(t), \boldsymbol{u}(t))_V + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + \left(\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\right)_Q$$

$$+\int_{\Gamma_{3,2}} F(\gamma_1 \boldsymbol{u}(t))d\Gamma + \int_{\Gamma_{3,2}} \mu F\|\gamma_2 \boldsymbol{u}(t)\|d\Gamma \le (\boldsymbol{f}(t), \boldsymbol{u}(t))_V. \tag{29}$$

We estimate each term of (29). First, from $H(\mathcal{B})$(ii), we have

$$(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \ge m_{\mathcal{B}}\|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|_Q^2 \ge \frac{m_{\mathcal{B}}}{\|\mathcal{A}\|}\|\boldsymbol{u}(t)\|_V^2. \tag{30}$$

Using $H(\mathcal{R})$, we get

$$\begin{aligned}
&\left(\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\right)_Q \\
&\ge -\varepsilon\|\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|_Q^2 - c(\varepsilon)\left\|\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds\right\|_Q^2 \\
&\ge -\frac{\varepsilon}{m_{\mathcal{A}}}\|\boldsymbol{u}(t)\|_V^2 - c(\varepsilon)\frac{c_{\mathcal{R}}^2 T}{m_{\mathcal{A}}}\int_0^t \|\boldsymbol{u}(s)\|_V^2 ds.
\end{aligned} \tag{31}$$

From $H(\mathcal{C})$, we have

$$\begin{aligned}
&\int_{\Gamma_{3,2}} F(\gamma_1 \boldsymbol{u}(t))d\Gamma \\
&\ge -\varepsilon\int_{\Gamma_{3,2}} |\gamma_1 \boldsymbol{u}(t)|^2 d\Gamma - c(\varepsilon)\int_{\Gamma_{3,2}} F^2 d\Gamma \\
&\ge -\varepsilon\|\gamma_1\|^2\|\boldsymbol{u}(t)\|_V^2 - c(\varepsilon)\|F\|_{L^2(\Gamma_{3,2})}^2,
\end{aligned} \tag{32}$$

and

$$\int_{\Gamma_{3,2}} \mu F\|\gamma_2 \boldsymbol{u}(t)\|d\Gamma \ge 0. \tag{33}$$

Finally,

$$(\boldsymbol{f}(t), \boldsymbol{u}(t))_V \le \varepsilon\|\boldsymbol{u}(t)\|_V^2 + c(\varepsilon)\|\boldsymbol{f}(t)\|_V^2. \tag{34}$$

We apply (30)–(34) to (29) and obtain

$$\begin{aligned}
&(\boldsymbol{u}'(t), \boldsymbol{u}(t))_V + \left(\frac{m_{\mathcal{B}}}{\|\mathcal{A}\|} - \frac{\varepsilon}{m_{\mathcal{A}}} - \varepsilon\|\gamma_1\|^2 - \varepsilon\right)\|\boldsymbol{u}(t)\|_V^2 \\
\le& c(\varepsilon)\frac{c_{\mathcal{R}}^2 T}{m_{\mathcal{A}}}\int_0^t \|\boldsymbol{u}(s)\|_V^2 ds + c(\varepsilon)\|F\|_{L^2(\Gamma_{3,2})}^2 + c(\varepsilon)\|\boldsymbol{f}(t)\|_V^2.
\end{aligned} \tag{35}$$

Integrating (35) over the time interval $[0,t]$, we get

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{u}(t)\|_V^2 + \left(\frac{m_{\mathcal{B}}}{\|\mathcal{A}\|} - \frac{\varepsilon}{m_{\mathcal{A}}} - \varepsilon\|\gamma_1\|^2 - \varepsilon\right)\int_0^t \|\boldsymbol{u}(s)\|_V^2 ds \\
\le& c(\varepsilon)\frac{c_{\mathcal{R}}^2 T}{m_{\mathcal{A}}}\int_0^t\int_0^s \|\boldsymbol{u}(r)\|_V^2 dr ds + c(\varepsilon)T\|F\|_{L^2(\Gamma_{3,2})}^2 \\
&+ c(\varepsilon)\|\boldsymbol{f}\|_{\mathcal{V}}^2 + \frac{1}{2}\|\boldsymbol{u}_0\|_V^2.
\end{aligned} \tag{36}$$

Taking ε small enough and using Gronwall's inequality, we obtain

$$\|\boldsymbol{u}\|_{\mathcal{V}}^2 \le c(1 + \|\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{f}\|_{\mathcal{V}}^2).$$

That is,

$$\|\boldsymbol{u}\|_{\mathcal{V}} \le c(\boldsymbol{u}_0, \boldsymbol{f}). \tag{37}$$

Let $\boldsymbol{w} \in H^1(0,T;V)$ be the solution to Problem 4.4, from (26), (27) and Lemma 4.3, we have

$$\|\boldsymbol{u}'\|_{\mathcal{V}}^2 = \|\boldsymbol{w}'\|_{\mathcal{V}}^2 \le c\left(\|\mathcal{E}_0\boldsymbol{w}\|_{\mathcal{V}}^2 + \|\gamma_1^*\zeta_1\|_{\mathcal{V}}^2 + \|\gamma_2^*\boldsymbol{\zeta}_2\|_{\mathcal{V}}^2 + \|\gamma_3^*\zeta_3\|_{\mathcal{V}}^2 + \|\boldsymbol{f}\|_{\mathcal{V}}^2\right). \tag{38}$$

Since the relation $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_0$ holds, by (37), we get

$$\|\boldsymbol{w}\|_{\mathcal{V}} \le c(\boldsymbol{u}_0, \boldsymbol{f}). \tag{39}$$

According to the assumptions $H(\mathcal{B})$ and $H(\mathcal{R})$ and the definition of the operator $\mathcal{E}_0$, we find that

$$\|\mathcal{E}_0\boldsymbol{w}\|_{\mathcal{V}} \le c\,(1 + \|\boldsymbol{w}\|_{\mathcal{V}} + \|\boldsymbol{u}_0\|_V) \le c(\boldsymbol{u}_0, \boldsymbol{f}). \tag{40}$$

By direct calculation of convex subdifferentials, we have

$$\zeta_1(\boldsymbol{x},t) = F(\boldsymbol{x}), \quad \boldsymbol{\zeta}_2(\boldsymbol{x},t) \in \mu(\boldsymbol{x})F(\boldsymbol{x})\partial\|\boldsymbol{w}_\tau(\boldsymbol{x},t) + \boldsymbol{u}_{0,\tau}(\boldsymbol{x})\|, \tag{41}$$

for a.e. $(\boldsymbol{x},t) \in \Gamma_{3,2} \times (0,T)$. From (26), (39)–(41), we have

$$\begin{aligned}(\gamma_3^*\zeta_3, \boldsymbol{w})_{\mathcal{V}} &= (\boldsymbol{f} - \boldsymbol{w}' - \mathcal{E}_0\boldsymbol{w} - \gamma_1^*\zeta_1 - \gamma_2^*\boldsymbol{\zeta}_2, \boldsymbol{w})_{\mathcal{V}} \\ &\le (\|\boldsymbol{f}\|_{\mathcal{V}} + \|\mathcal{E}_0\boldsymbol{w}\|_{\mathcal{V}} + \|\gamma_1^*\zeta_1\|_{\mathcal{V}} + \|\gamma_2^*\boldsymbol{\zeta}_2\|_{\mathcal{V}})\,\|\boldsymbol{w}\|_{\mathcal{V}} \le c(\boldsymbol{u}_0, \boldsymbol{f}).\end{aligned} \tag{42}$$

According to Lemma 3.10 in [22], the operator $\gamma_3^*\mathcal{M}_0$ is strongly quasi-bounded. Then, by (39) and (42), we find that

$$\|\gamma_3^*\zeta_3\|_{\mathcal{V}} \le c(\boldsymbol{u}_0, \boldsymbol{f}). \tag{43}$$

Now we apply (40), (41) and (43) to (38) and obtain

$$\|\boldsymbol{u}'\|_{\mathcal{V}} \le c(\boldsymbol{u}_0, \boldsymbol{f}). \tag{44}$$

Combining (37) and (44), we complete the proof of (28). □

We denote by $\boldsymbol{u} = \mathcal{U}(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0)$ the unique solution to Problem 4.2 with data $(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0)$. Here

$$\mathcal{U}:\ L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d)) \times M \times V \to \mathcal{V},\ (\boldsymbol{f}_2, \mu, \boldsymbol{u}_0) \mapsto \boldsymbol{u} = \mathcal{U}(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0),$$

is the solution map of Problem 4.2. Now we present the continuous dependence result for the solution map.

Theorem 4.6. *If the assumptions $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mathcal{R})$, $H(\mathcal{C})$ and $H(\boldsymbol{f})$ hold, $\{\boldsymbol{f}_2^n\} \subset L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$, $\boldsymbol{f}_2^n \to \boldsymbol{f}_2$ weakly in $L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$, $\{\mu_n\} \subset M$, $\mu_n \to \mu$ weakly in $L^2(\Gamma_{3,2})$, $\{\boldsymbol{u}_0^n\} \subset V$, $\boldsymbol{u}_0^n \to \boldsymbol{u}_0$ weakly in V. Then*

$$\begin{aligned}\boldsymbol{u}_n &\to \boldsymbol{u} \quad \textit{weakly in } \mathcal{V}, \\ \boldsymbol{u}_n' &\to \boldsymbol{u}' \quad \textit{weakly in } \mathcal{V},\end{aligned}$$

where $\boldsymbol{u}_n = \mathcal{U}(\boldsymbol{f}_2^n, \mu_n, \boldsymbol{u}_0^n)$ and $\boldsymbol{u} = \mathcal{U}(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0)$.

Proof. Since $\boldsymbol{u}_n$ is the unique solution to Problem 4.2 with data $(\boldsymbol{f}_2^n, \mu_n, \boldsymbol{u}_0^n)$, we have

$$\boldsymbol{u}_n' + \mathcal{E}\boldsymbol{u}_n + \gamma_1^*\xi_1^n + \gamma_2^*\boldsymbol{\xi}_2^n + \gamma_3^*\xi_3^n = \boldsymbol{f}_n \quad \text{in } \mathcal{V}, \tag{45}$$

with

$$\xi_1^n(t) \in \partial\tilde{\varphi}_1(\gamma_1\boldsymbol{u}_n(t)), \quad \boldsymbol{\xi}_2^n(t) \in \partial\tilde{\varphi}_2(\mu_n, \gamma_2\boldsymbol{u}_n(t)), \quad \xi_3^n(t) \in \partial\tilde{\varphi}_3(\gamma_3\boldsymbol{u}_n(t)),$$

for a.e. $t \in (0,T)$ and

$$(\boldsymbol{f}_n(t), \boldsymbol{v})_V = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_2^n(t), \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)}, \quad \text{for all } v \in V,\ \text{a.e. } t \in (0,T).$$

According to Theorem 4.5, we have

$$\|\boldsymbol{u}_n\|_{\mathcal{V}} + \|\boldsymbol{u}_n'\|_{\mathcal{V}} \le c(\boldsymbol{u}_0^n, \boldsymbol{f}_n).$$

From the conditions that $\boldsymbol{f}_2^n \to \boldsymbol{f}_2$ weakly in $L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$ and $\boldsymbol{u}_0^n \to \boldsymbol{u}_0$ weakly in V, we deduce that $\{\boldsymbol{f}_2^n\}$ is bounded in $L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$ and $\{\boldsymbol{u}_0^n\}$ is bounded in V. Thus the sequences $\{\boldsymbol{u}_n\}$ and $\{\boldsymbol{u}_n'\}$ are bounded in $\mathcal{V}$. As a result, passing to a subsequence, if necessary, we have

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{weakly in } \mathcal{V}, \tag{46}$$

$$\boldsymbol{u}_n' \to \boldsymbol{u}' \quad \text{weakly in } \mathcal{V}. \tag{47}$$

By direct calculation of convex subdifferentials, we can find that

$$\xi_1^n(\boldsymbol{x},t) = F(\boldsymbol{x}), \tag{48}$$

$$\boldsymbol{\xi}_2^n(\boldsymbol{x},t) \in \mu_n(\boldsymbol{x})F(\boldsymbol{x})\partial\|\boldsymbol{u}_{n,\tau}'(\boldsymbol{x},t)\|, \tag{49}$$

for a.e. $(\boldsymbol{x},t) \in \Gamma_{3,2}\times(0,T)$. From (48), we get $\xi_1^n \to \xi_1$ in $L^2(0,T;L^2(\Gamma_{3,2}))$, with $\xi_1(t) = F$ and

$$\xi_1(t) \in \partial\tilde{\varphi}_1(\gamma_1\boldsymbol{u}(t)),$$

for a.e. $t \in (0,T)$. In fact, $\partial\tilde{\varphi}_1$ is a single-valued map such as $\partial\tilde{\varphi}_1(v) = F$ for all $v \in L^2(\Gamma_{3,2})$. By (48) and (49), we can also find that $\{\xi_1^n\}$ and $\{\boldsymbol{\xi}_2^n\}$ are bounded in $L^2(0,T;L^2(\Gamma_{3,2}))$ and $L^2(0,T;L^2(\Gamma_{3,2};\mathbb{R}^d))$, respectively. From (45), we have

$$\gamma_3^*\xi_3^n = \boldsymbol{f}_n - (\boldsymbol{u}_n' + \mathcal{E}\boldsymbol{u}_n + \gamma_1^*\xi_1^n + \gamma_2^*\boldsymbol{\xi}_2^n) \quad \text{in } \mathcal{V}.$$

Since $\mathcal{E}$, γ_1^* and γ_2^* are all bounded operators, all terms in the right side of the above equality are bounded. Thus $\{\xi_3^n\}$ is bounded in $L^2(0,T;L^2(\Gamma_{3,1}))$. Passing to a subsequence, if necessary, we have

$$\boldsymbol{\xi}_2^n \to \boldsymbol{\xi}_2 \quad \text{weakly in } L^2(0,T;L^2(\Gamma_{3,2};\mathbb{R}^d)),$$
$$\xi_3^n \to \xi_3 \quad \text{weakly in } L^2(0,T;L^2(\Gamma_{3,1})).$$

In what follows, we will prove that $\boldsymbol{u} \in H^1(0,T;V)$ is the solution of Problem 4.2 with data $(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0)$.

We first prove the initial condition. From (46) and (47), by Lemma 2.55(i) of [9], we have $\boldsymbol{u}_n(0) \to \boldsymbol{u}(0)$ weakly in V. Since $\boldsymbol{u}_n(0) = \boldsymbol{u}_0^n \to \boldsymbol{u}_0$ weakly in V, we obtain $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

Then we prove that $\boldsymbol{u} \in H^1(0,T;V)$ satisfies the equation. Under the assumptions $H(\mathcal{B})$ and $H(\mathcal{R})$, the operator $\mathcal{E}$ is linear and bounded, thus it is weakly continuous. From (46), we have

$$\mathcal{E}\boldsymbol{u}_n \to \mathcal{E}\boldsymbol{u} \quad \text{weakly in } \mathcal{V}.$$

By Theorem 2.18 of [1],

$$\gamma_2\boldsymbol{u}_n \to \gamma_2\boldsymbol{u} \quad \text{in } L^2(0,T;L^2(\Gamma_{3,2};\mathbb{R}^d)),$$
$$\gamma_3\boldsymbol{u}_n \to \gamma_3\boldsymbol{u} \quad \text{in } L^2(0,T;L^2(\Gamma_{3,1})).$$

Passing to a subsequence, if necessary, we have for a.e. $t \in (0,T)$,

$$\gamma_2\boldsymbol{u}_n(t) \to \gamma_2\boldsymbol{u}(t) \quad \text{in } L^2(\Gamma_{3,2};\mathbb{R}^d),$$
$$\gamma_3\boldsymbol{u}_n(t) \to \gamma_3\boldsymbol{u}(t) \quad \text{in } L^2(\Gamma_{3,1}).$$

According to Lemma 10 in [4], $\partial\varphi_2$ is upper semicontinuous as a multifunction from $M_w \times L^2(\Gamma_{3,2};\mathbb{R}^d)$ to $L^2_w(\Gamma_{3,2};\mathbb{R}^d)$. We use Lemma 2.4 to conclude that

$$\boldsymbol{\xi}_2(t) \in \partial\tilde{\varphi}_2(\mu, \gamma_2\boldsymbol{u}(t)),$$

for a.e. $t \in (0,T)$. Moreover, we refer to Lemma 3.5 (P3) in [22], the graph of $\partial\tilde{\varphi}_3$ is closed in the topology of $L^2(\Gamma_{3,2}) \times L^2_w(\Gamma_{3,2})$. Thus

$$\xi_3(t) \in \partial\tilde{\varphi}_3(\gamma_3\boldsymbol{u}(t)),$$

for a.e. $t \in (0,T)$. On the other hand, it is immediate to see that

$$\gamma_1^*\xi_1^n \to \gamma_1^*\xi_1, \quad \gamma_2^*\boldsymbol{\xi}_2^n \to \gamma_2^*\boldsymbol{\xi}_2, \quad \gamma_3^*\xi_3^n \to \gamma_3^*\xi_3 \quad \text{weakly in } \mathcal{V}.$$

Moreover, from $\boldsymbol{f}_2^n \to \boldsymbol{f}_2$ weakly in $L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$, we can easily deduce that

$$\boldsymbol{f}_n \to \boldsymbol{f} \quad \text{weakly in } \mathcal{V}.$$

Now we pass to the limit in the inequality (45) to obtain

$$\boldsymbol{u}' + \mathcal{E}\boldsymbol{u} + \gamma_1^*\xi_1 + \gamma_2^*\boldsymbol{\xi}_2 + \gamma_3^*\xi_3 = \boldsymbol{f} \quad \text{in } \mathcal{V},$$

which completes the proof. □

5. An optimal control problem

In this section, we consider an optimal control problem for the problem we studied in Section 3 and Section 4, say Problem 3.3 or its equivalent form Problem 4.2. The control variables contain boundary and initial conditions, and the objective cost functional combines observations within the domain, on the boundary and at the terminal time. Our goal is to prove that the optimal control problem has at least one solution.

To begin with, the control variables are given by $\boldsymbol{f}_2$, μ and $\boldsymbol{u}_0$, and the admissible control set, denoted by U_{ad}, is defined by

$$U_{ad} = F_{ad} \times M \times V_{ad},$$

where F_{ad} and V_{ad} are convex, closed, bounded in $L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$ and V, respectively.

Then we define a cost functional $J : L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d)) \times M \times V \times \mathcal{V} \to \mathbb{R}$ as follows

$$\begin{aligned} J(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0, \boldsymbol{u}) &= \rho_1\|\boldsymbol{u}_0 - \bar{\boldsymbol{u}}_0\|_V^2 + \rho_2\|\boldsymbol{u}(T) - \bar{\boldsymbol{u}}_T\|_V^2 \\ &+ \rho_3\|\mu - \bar{\mu}\|^2_{L^2(\Gamma_{3,2})} + \int_0^T \rho_4(t)\|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|_V^2 dt \\ &+ \int_0^T \Big(\rho_5(t)\|\boldsymbol{u}(t) - \bar{\boldsymbol{u}}(t)\|_H^2 + \rho_6(t)\|\boldsymbol{f}_2(t) - \bar{\boldsymbol{f}}_2(t)\|^2_{L^2(\Gamma_2;\mathbb{R}^d)}\Big)\, dt, \end{aligned}$$

with the following assumptions of the given data

$\underline{H_{op}}$. $\rho_i \in \mathbb{R}, \rho_i \geq 0$, for $i = 1,2,3$. $\rho_j \in L^\infty(0,T), \rho_j \geq 0$, for $j = 4,5,6$. $\bar{\boldsymbol{u}}_0 \in V$, $\bar{\boldsymbol{u}}_T \in V$, $\bar{\mu} \in L^2(\Gamma_{3,2})$, $\tilde{\boldsymbol{u}} \in \mathcal{V}$, $\bar{\boldsymbol{u}} \in \mathcal{H}$, $\bar{\boldsymbol{f}}_2 \in L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$.

The optimal control problem we are concerned with is

$$\text{minimize } J(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0, \boldsymbol{u}) \text{ subject to } (\boldsymbol{f}_2, \mu, \boldsymbol{u}_0) \in U_{ad} \text{ and } \boldsymbol{u} = \mathcal{U}(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0). \tag{50}$$

Theorem 5.1. *Under the assumptions $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mathcal{R})$, $H(\mathcal{C})$, $H(\boldsymbol{f})$ and H_{op}, the optimal control problem* (50) *has a solution $(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0, \boldsymbol{u}) \in L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d)) \times M \times V \times \mathcal{V}$.*

Proof. We set

$$m = \inf\{J(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0, \boldsymbol{u}) \mid (\boldsymbol{f}_2, \mu, \boldsymbol{u}_0) \in U_{ad},\ \boldsymbol{u} = \mathcal{U}(\boldsymbol{f}_2, \mu, \boldsymbol{u}_0)\}.$$

Let $(\boldsymbol{f}_2^n, \mu_n, \boldsymbol{u}_0^n, \boldsymbol{u}_n)$ be a minimizing sequence for the cost functional J, i.e.,

$$\lim_{n\to\infty} J(\boldsymbol{f}_2^n, \mu_n, \boldsymbol{u}_0^n, \boldsymbol{u}_n) = m \text{ with } \boldsymbol{u}_n = \mathcal{U}(\boldsymbol{f}_2^n, \mu_n, \boldsymbol{u}_0^n).$$

Since F_{ad} and V_{ad} are convex, closed, bounded subsets in $L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$ and V, respectively, there exists a subsequence of $\{(\boldsymbol{f}_2^n, \boldsymbol{u}_0^n)\}$, denoted by the same notation, such that

$$\boldsymbol{f}_2^n \to \boldsymbol{f}_2^* \quad \text{weakly in } L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d)), \tag{51}$$

$$\boldsymbol{u}_0^n \to \boldsymbol{u}_0^* \quad \text{weakly in } V. \tag{52}$$

Note that M is a convex, closed, bounded subset of $L^2(\Gamma_{3,2})$, then we may also find a subsequence of $\{\mu_n\}$, denoted by the same notation, such that

$$\mu_n \to \mu^* \quad \text{weakly in } L^2(\Gamma_{3,2}). \tag{53}$$

It is well-known that a convex subset of a Banach space is closed if and only if it is weakly closed, thus

$$(\boldsymbol{f}_2^*, \mu^*, \boldsymbol{u}_0^*) \in U_{ad}.$$

From Theorem 4.6, we have

$$\boldsymbol{u}_n \to \boldsymbol{u}^* \quad \text{weakly in } \mathcal{V}, \tag{54}$$

$$\boldsymbol{u}_n' \to \boldsymbol{u}^{*'} \quad \text{weakly in } \mathcal{V}, \tag{55}$$

where $\boldsymbol{u}^* = \mathcal{U}(\boldsymbol{f}_2^*, \mu^*, \boldsymbol{u}_0^*)$ is the unique solution to Problem 4.2. We note that the norm $\|\cdot\|_V$ is weak lower semicontinuous, from (52), we have

$$\liminf_{n\to\infty} \|\boldsymbol{u}_0^n - \bar{\boldsymbol{u}}_0\|_V^2 \geq \|\boldsymbol{u}_0^* - \bar{\boldsymbol{u}}_0\|_V^2. \tag{56}$$

From (54) and (55), by Lemma 2.55(i) of [9], we get

$$\boldsymbol{u}_n(t) \to \boldsymbol{u}^*(t) \quad \text{weakly in } V, \quad \text{for all } t \in [0,T]. \tag{57}$$

Specifically,

$$\boldsymbol{u}_n(T) \to \boldsymbol{u}^*(T) \quad \text{weakly in } V,$$

thus,

$$\liminf_{n\to\infty} \|\boldsymbol{u}_n(T) - \bar{\boldsymbol{u}}_T\|_V^2 \geq \|\boldsymbol{u}^*(T) - \bar{\boldsymbol{u}}_T\|_V^2. \tag{58}$$

Since the norm $\|\cdot\|_{L^2(\Gamma_{3,2})}$ is also weak lower semicontinuous, we deduce the following inequality from (53)

$$\liminf_{n\to\infty} \|\mu_n - \bar{\mu}\|_{L^2(\Gamma_{3,2})}^2 \geq \|\mu^* - \bar{\mu}\|_{L^2(\Gamma_{3,2})}^2. \tag{59}$$

Applying Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \int_0^T \rho_4(t)\|\boldsymbol{u}_n(t) - \tilde{\boldsymbol{u}}(t)\|_V^2 dt \\
&\geq \int_0^T \liminf_{n\to\infty} \rho_4(t)\|\boldsymbol{u}_n(t) - \tilde{\boldsymbol{u}}(t)\|_V^2 dt \\
&\geq \int_0^T \rho_4(t)\|\boldsymbol{u}^*(t) - \tilde{\boldsymbol{u}}(t)\|_V^2 dt.
\end{aligned} \tag{60}$$

Since the embedding $V \subset H$ is compact, from (57), we have

$$\boldsymbol{u}_n(t) \to \boldsymbol{u}^*(t) \quad \text{in } H, \quad \text{for all } t \in [0,T],$$

Combined with the fact $\boldsymbol{u}_n, \boldsymbol{u}^* \in C([0,T];H)$, the above convergence gives

$$\boldsymbol{u}_n \to \boldsymbol{u}^* \quad \text{in } \mathcal{H}. \tag{61}$$

Now we define an operator $L : (0,T) \times H \times L^2(\Gamma_2;\mathbb{R}^d) \to \mathbb{R}$ by

$$L(t,\boldsymbol{v},\boldsymbol{w}) = \rho_5(t)\|\boldsymbol{v} - \bar{\boldsymbol{u}}(t)\|_H^2 + \rho_6(t)\|\boldsymbol{w} - \bar{\boldsymbol{f}}_2(t)\|^2_{L^2(\Gamma_2;\mathbb{R}^d)}.$$

In fact, the operator $L : (0,T) \times H \times L^2(\Gamma_2;\mathbb{R}^d) \to \mathbb{R}$ satisfies

(i) $L(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $H \times L^2(\Gamma_2;\mathbb{R}^d)$, for a.e. $t \in (0,T)$;
(ii) $L(t,\boldsymbol{v},\cdot)$ is convex on $L^2(\Gamma_2;\mathbb{R}^d)$, for all $\boldsymbol{v} \in H$, a.e. $t \in (0,T)$;
(iii) There is $N > 0$ and $\psi \in L^1(0,T)$ such that

$$L(t,\boldsymbol{v},\boldsymbol{w}) \geq \psi(t) - N\left(\|\boldsymbol{v}\|_H + \|\boldsymbol{w}\|_{L^2(\Gamma_2;\mathbb{R}^d)}\right),$$

for all $\boldsymbol{v} \in H$, $\boldsymbol{w} \in L^2(\Gamma_2;\mathbb{R}^d)$, a.e. $t \in (0,T)$.

We give the proof of (iii). For $a \in \mathbb{R}$, the simple inequality holds: $a^2 \geq 2a - 1$, then we have

$$\begin{aligned}
L(t,\boldsymbol{v},\boldsymbol{w}) &\geq 2\sqrt{\rho_5(t)}\|\boldsymbol{v} - \bar{\boldsymbol{u}}(t)\|_H + 2\sqrt{\rho_6(t)}\|\boldsymbol{w} - \bar{\boldsymbol{f}}_2(t)\|_{L^2(\Gamma_2;\mathbb{R}^d)} - 2\\
&\geq -2\sqrt{\rho_5(t)}\left(\|\boldsymbol{v}\|_H + \|\bar{\boldsymbol{u}}(t)\|_H\right)\\
&\quad - 2\sqrt{\rho_6(t)}\left(\|\boldsymbol{w}\|_{L^2(\Gamma_2;\mathbb{R}^d)} + \|\bar{\boldsymbol{f}}_2(t)\|_{L^2(\Gamma_2;\mathbb{R}^d)}\right) - 2\\
&\geq -2\sqrt{\rho_5(t)}\|\bar{\boldsymbol{u}}(t)\|_H - 2\sqrt{\rho_6(t)}\|\bar{\boldsymbol{f}}_2(t)\|_{L^2(\Gamma_2;\mathbb{R}^d)} - 2\\
&\quad - 2\rho\left(\|\boldsymbol{v}\|_H + \|\boldsymbol{w}\|_{L^2(\Gamma_2;\mathbb{R}^d)}\right),
\end{aligned}$$

where

$$\rho = \max\{\sqrt{\|\rho_5\|_{L^\infty(0,T)}}, \sqrt{\|\rho_6\|_{L^\infty(0,T)}}\}.$$

We complete the proof of (iii) by taking $N = 2\rho$ and

$$\psi(t) = -2\sqrt{\rho_5(t)}\|\bar{\boldsymbol{u}}(t)\|_H - 2\sqrt{\rho_6(t)}\|\bar{\boldsymbol{f}}_2(t)\|_{L^2(\Gamma_2;\mathbb{R}^d)} - 2.$$

According to the claims (i)-(iii) related to the operator L, using Theorem 2.1 of [15], the integral

$$I = \int_0^T L(t,\boldsymbol{u}(t),\boldsymbol{f}_2(t))dt$$

is sequentially strong-weak lower semicontinuous on $\mathcal{H} \times L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d))$. Then we use (51) and (61) to obtain

$$\liminf_{n\to\infty} \int_0^T L(t,\boldsymbol{u}_n(t),\boldsymbol{f}_2^n(t))dt \geq \int_0^T L(t,\boldsymbol{u}^*(t),\boldsymbol{f}_2^*(t))dt. \tag{62}$$

Finally, from (56), (58), (59), (60) and (62), we have

$$J(\boldsymbol{f}_2^*,\mu^*,\boldsymbol{u}_0^*,\boldsymbol{u}^*) \leq \liminf_{n\to\infty} J(\boldsymbol{f}_2^n,\mu_n,\boldsymbol{u}_0^n,\boldsymbol{u}_n) = m,$$

which means $(\boldsymbol{f}_2^*,\mu^*,\boldsymbol{u}_0^*,\boldsymbol{u}^*)$ is a solution to the optimal control problem (50). $\square$

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