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Two-Grid Crank-Nicolson Finite Volume Element Method for the Time-Dependent Schrödinger Equation

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Abstract. In this paper, we construct a Crank-Nicolson finite volume element scheme and a two-grid decoupling algorithm for solving the time-dependent Schrödinger equation. Combining the idea of two-grid discretization, the decoupling algorithm involves solving a small coupling system on a coarse grid space and a decoupling system with two independent Poisson problems on a fine grid space, which can ensure the accuracy while the size of coarse grid is much coarser than that of fine grid. We further provide the optimal error estimate of these two schemes rigorously by using elliptic projection operator. Finally, numerical simulations are provided to verify the correctness of the theoretical analysis.

AMS subject classifications: 65N12, 65M60

Key words: Finite volume element method, two-grid method, Crank-Nicolson scheme, error estimates, Schrödinger equation.

1 Introduction

The time-dependent Schrödinger equation is one of the most important equations of mathematical physics. This model has been widely used in many fields, such as plasma physics, nonlinear optics, seismology, bimolecular dynamics and protein chemistry. In this article, we consider the initial boundary value problem of two-dimensional time-dependent linear Schrödinger equation as follows:

$$\begin{cases} iu_t = -\Delta u(x,t) + V(x)u(x,t) + f(x,t) & \text{in } \Omega\times(0,T], \\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T], \\ u(x,0)=u_0(x) & \text{in } \bar{\Omega}, \end{cases} \tag{1.1}$$

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where $x=(x_1,x_2)^T$, $\Omega\subset R^2$ is a bounded convex polygonal domain with smooth boundary $\partial\Omega$, the function $f(x,t)$ and unknown function $u(x,t)$ are complex-valued, $u_0(x)$ is a smooth known complex-valued function, the trapping potential function $V(x)$ is real-valued and non-negative bounded for all $x\in\Omega$, and $i=\sqrt{-1}$ is the complex unit.

At present, many scholars have studied the numerical solutions of the Schrödinger equation. For example, Akrivis et al. [1] used the standard Galerkin method in space and two implicit Crank-Nicholson schemes in time to approximate the solution of nonlinear Schrödinger equation. Jin et al. [2] studied the convergence of a finite element scheme for the two-dimensional time-dependent Schrödinger equation in a long strip. Antonopoulou et al. [3] investigated the generalized Schrödinger equation with mixed boundary conditions in a two-dimensional noncylindrical domain. Wang, Tian et al. [4,5] studied the superconvergence property of time-dependent nonlinear Schrödinger equations. It is well known that the finite volume element (FVE) method is not only simple in structure, but also can maintain the local conservation of physical quantities. Two-grid method was first proposed by professor Xu [6, 7]. The theoretical framework and basic tools of FVE method and two-grid method have developed rapidly [8–26]. In [27–29], a two-grid finite element scheme was proposed for solving the nonlinear Schrödinger equation.

For the problem (1.1), Zhang et al. [30] constructed semi-discrete two grid finite element scheme and performed the corresponding convergence analysis. Afterwards, Tian [31] and Wang [32] solved this problem by backward Euler and Crank-Nicolson finite element methods respectively and obtained the optimal error estimates. By now, the Schrödinger equation is mainly studied by spectral method, finite element method and finite difference method. Considering the advantages of FVE method and two grid method, we want to apply them to investigate the Schrödinger problem.

In this paper, we use the finite volume element method and Crank-Nicolson scheme in space and time respectively to solve the linear Schrödinger equation. Furthermore, the corresponding error estimates are analyzed by using elliptic projection operator. At the same time, we propose a two-grid finite volume element decoupling algorithm for the Schrödinger equation and derive the corresponding error estimate. With this decoupling algorithm, the solution of the original coupling problem is simplified to the solution of the same problem on a much coarser grid together with the solutions of two Poisson problems on the fine grid. It is worth noting that the two-grid algorithm can still maintain good approximation accuracy under the coarse grid which is much coarser than the fine grid.

The rest of this paper is organized as follows. In Section 2, we recall some notations and present finite volume element scheme for the time-dependent Schrödinger equation (1.1). Section 3 is devoted to provide the error estimates of FVE method by an elliptic projection operator. The two-grid FVE decoupling algorithm for the Schrödinger equation is developed in Section 4 and the optimal error estimates in the H^1 norm are also derived. Finally, two numerical examples are presented in Section 5 to validate the established theoretical findings.

Throughout this paper, the letter C or with its subscript denotes a generic positive constant independent of the mesh parameter and may be different at its different occurrences.

2 Finite volume element scheme

For $1\le p\le\infty$, let $L^p(\Omega)$ be the Banach space of complex-valued measurable functions defined on Ω with the norm $\|u\|_{L^p}$. We use $W^{s,p}(\Omega)$ to denote the standard Sobolev space of complex-valued measurable functions defined on Ω. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega}=\|u\|_{s,p}=\left(\int_\Omega\sum_{|\alpha|\le s}|D^\alpha u|^p dx\right)^{\frac{1}{p}}$$

with the standard modification for $p=\infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p=2$ and Ω whenever possible; i.e., $\|u\|_{s,2,\Omega}=\|u\|_{s,2}=\|u\|_s$. We denote $H_0^1(\Omega)$ by the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$.

For any two complex-valued functions $u(x),\varphi(x)\in L^2(\Omega)$, we set the inner product (u,φ) by

$$(u,\varphi)=\int_\Omega u(x)\overline{\varphi(x)}dx, \tag{2.1}$$

and with the corresponding L^2 norm

$$\|\varphi\|=\sqrt{(\varphi,\overline{\varphi})}, \tag{2.2}$$

where $\overline{\varphi}$ is the complex conjugate of function φ.

For any complex-valued function $v=v_1+iv_2$, let

$$\|v\|_{m,p}=(\|v_1\|_{m,p}^p+\|v_2\|_{m,p}^p)^{\frac{1}{p}}, \tag{2.3a}$$
$$\|v\|_{m,\infty}=\|v_1\|_{m,\infty}+\|v_2\|_{m,\infty}. \tag{2.3b}$$

For the polygonal domain Ω, we consider a quasi-uniform regular triangulation T_h consisting of closed triangle elements K such that $\bar{\Omega}=\cup_{K\in T_h}K$, and h = max h_K, where h_K is the diameter of the triangle $K\in T_h$.

Let V_h be the standard conforming finite element space of piecewise linear functions defined on the triangulation T_h,

$$V_h=\{v\in C(\Omega):\ v|_K \text{ is linear, } \forall K\in T_h;\ v|_{\partial\Omega}=0\}.$$

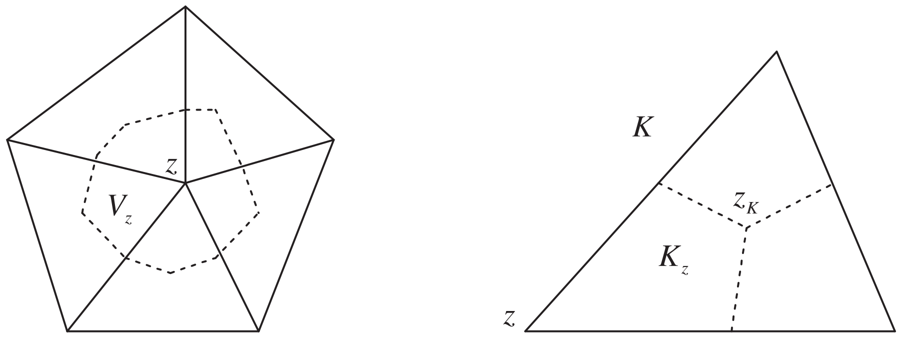

Figure 1: Left: Control volume V_z. Right: Triangular partition and its dual.

Then we introduce a dual mesh T_h^* corresponding to T_h. Let z_K be the barycenter of $K \in T_h$. We connect z_K with midpoints of each edge of K by line segments, thus partitioning K into three quadrilaterals K_z, $z \in \mathcal{Z}_h(K)$, where $\mathcal{Z}_h(K)$ are the vertices of K. Then with each vertex $z \in \mathcal{Z}_h = \cup_{K \in T_h} \mathcal{Z}_h(K)$ we structure a control volume V_z, which consists of the union of the subregions K_z, sharing the vertex z. Thus we finally obtain a group of control volumes covering the domain Ω, that is the dual partition T_h^* of the triangulation T_h. We denote the set of all interior vertices of $\mathcal{Z}_h$ by $\mathcal{Z}_h^0$ (see Fig. 1).

A control volume mesh T_h^* is quasi-uniform regular, if there exists a positive constant $C > 0$ such that

$$C^{-1} h^2 \leq \text{meas}(V_z) \leq C h^2, \quad \forall V_z \in T_h^*.$$

The dual volume element space V_h^* on T_h^* is defined by

$$V_h^* = \{ v \in L^2(\Omega) : \, v|_{V_z} \text{ is constant for all } V_z \in T_h^*; \, v|_{V_z} = 0, \text{ if } z \in \partial\Omega \}.$$

Then we obtain $V_h = \text{span}\{\phi_z(x) : z \in \mathcal{Z}_h^0\}$ and $V_h^* = \text{span}\{\psi_z(x) : z \in \mathcal{Z}_h^0\}$, where $\phi_z(x)$ is the standard nodal basis function associated with the node z, and $\psi_z(x)$ is the characteristic function of V_z.

For any $v_h \in H_0^1(\Omega) \cap H^2(\Omega)$, we define an interpolation operator $I_h : H_0^1(\Omega) \cap H^2(\Omega) \to V_h$, such that

$$I_h v_h = \sum_{z \in \mathcal{Z}_h^0} v_h(z) \phi_z = \sum_{z \in \mathcal{Z}_h^0} (v_{1h}(z) + i v_{2h}(z)) \phi_z,$$

where $v_{1h}(z) = Re\{v_h(z)\}$, $v_{2h}(z) = Im\{v_h(z)\}$, $\phi_z \in RV_h$, and RV_h is the space of the real-valued elements of V_h.

By the interpolation theorem in Sobolev space, we have

$$\|v - I_h v\|_m \leq C h^{2-m} \|v\|_2, \quad m = 0,1. \tag{2.4}$$

It is easy to see that

$$\|I_h v_h\|_1 \le C\|v_h\|_1. \tag{2.5}$$

For any $v_h \in V_h$, we define another interpolation operator $I_h^*: V_h \to V_h^*$, such that

$$I_h^* v_h = \sum_{z\in\mathcal{Z}_h^0} v_h(z)\psi_z = \sum_{z\in\mathcal{Z}_h^0} (v_{1h}(z)+iv_{2h}(z))\psi_z,$$

where $\psi_z \in RV_h^*$ and RV_h^* denotes the space of the real-valued of V_h^*.

Notice that

$$\overline{I_h^* v_h} = \overline{\sum_{z\in\mathcal{Z}_h^0} v_h(z)\psi_z} = \sum_{z\in\mathcal{Z}_h^0} \overline{(v_{1h}(z)+iv_{2h}(z))\psi_z} = \sum_{z\in\mathcal{Z}_h^0} (v_{1h}(z)-iv_{2h}(z))\psi_z = I_h^*\overline{v_h}, \tag{2.6a}$$

$$\|\overline{I_h^* v_h}\| = \|I_h^* v_h\|. \tag{2.6b}$$

It has been shown in [17,19] that

$$\|v_h - I_h^* v_h\|_{0,p} \le Ch^s\|v_h\|_{s,p}, \qquad 0\le s\le 1, \tag{2.7a}$$

$$\|I_h^* v_h\|_{0,p} \le C\|v_h\|_{0,p}, \qquad p>1. \tag{2.7b}$$

Based on above notations, the weak solution $u(x,t)$ of the problem (1.1) can be defined as follows: for all $t\in[0.T]$, find the complex-valued function $u(x,t)\in H_0^1(\Omega)$ such that

$$\begin{cases} i(u_t, I_h^* v) = a(u, I_h^* v) + (f, I_h^* v), & \forall v \in H_0^1(\Omega), \quad 0\le t\le T, \\ u(x,0) = u_0(x), & \forall x\in\Omega, \end{cases} \tag{2.8}$$

where

$$a(u, I_h^* v) = -\int_{\partial\Omega} (\nabla u)\cdot \mathbf{n}\overline{I_h^* v} ds + \int_\Omega V(x) u \overline{I_h^* v} dx.$$

Now we formulate FVE method for problem (1.1) as follows. Given a vertex $z\in\mathcal{Z}_h$, integrating (1.1) over the associated control volume V_z and using the Green formula, we obtain

$$i\int_{V_z} u_t(x,t)dx = -\int_{\partial V_z} (\nabla u(x,t))\cdot\mathbf{n}ds + \int_{V_z} (V(x)u(x,t)+f(x,t))dx, \tag{2.9}$$

where $\mathbf{n}$ denotes the unit outward normal on ∂V_z. It should be noted that the above formulation is a way of stating that we have an integral conservation form on the dual element V_z.

We consider a time step Δt and approximate the solution at time $t^n = n\Delta t$, $n=0,1,\cdots,N$; $\Delta t = T/N$. Then the Backward Euler fully-discrete FVE method for (1.1) reads as: Find $u_h\in V_h$, such that

$$\begin{cases} i\Big(\dfrac{u_h^n - u_h^{n-1}}{\Delta t}, I_h^* v_h\Big) = a_h(u_h^{n-\frac{1}{2}}, I_h^* v_h) + (f^{n-\frac{1}{2}}, I_h^* v_h), & \forall v_h\in V_h, \\ u_h^0 = P_h u_0, \end{cases} \tag{2.10}$$

where

$$u_h^{n-\frac{1}{2}}=\frac{u_h^n+u_h^{n-1}}{2},\quad f^{n-\frac{1}{2}}=\frac{f^n+f^{n-1}}{2},$$

and $a_h(\cdot,I_h^*\cdot)$ is defined by, for any $u_h,v_h\in V_h$,

$$\begin{aligned}a_h(u_h(t),I_h^*v_h)=&-\sum_{z\in\mathcal{Z}_h^0}\int_{\partial V_z}(\nabla u_h(t))\cdot\mathbf{n}\overline{I_h^*v_h}ds+\sum_{z\in\mathcal{Z}_h^0}\int_{V_z}V(x)u_h(t)\overline{I_h^*v_h}dx\\=&-\sum_{z\in\mathcal{Z}_h^0}\overline{v_h(z)}\int_{\partial V_z}(\nabla u_h(t))\cdot\mathbf{n}ds+\sum_{z\in\mathcal{Z}_h^0}\overline{v_h(z)}\int_{V_z}V(x)u_h(t)dx.\end{aligned}$$

Furthermore the corresponding discrete norm of $(u_h,I_h^*v_h)$ is equivalent to the L^2 norm, i.e., that there exist two positive constants $C_*,C^*>0$, independent of h such that

$$C_*\|u_h\|\le|\|u_h\||_0\le C^*\|u_h\|,\quad\forall u_h\in V_h,\tag{2.11}$$

with $|\|u_h\||_0=(u_h,I_h^*u_h)^{1/2}$.

Let $P_h:H^2(\Omega)\cap H_0^1(\Omega)\to V_h$ be an elliptic operator, which defined by

$$a_h(P_hu,I_h^*v_h)=a_h(u,I_h^*v_h),\quad\forall v_h\in V_h.\tag{2.12}$$

From Chen and Wu [14] and Chou and Li [15], it's shown that

$$\|u-P_hu\|_1\le Ch\|u\|_2,\qquad\forall u\in H^2(\Omega)\cap H_0^1(\Omega),\tag{2.13a}$$

$$\|u-P_hu\|_{0,p}\le Ch^2\|u\|_{3,q},\qquad\forall u\in W^{3,q}(\Omega)\cap H_0^1(\Omega),\tag{2.13b}$$

where $q>1$, if $p=2$ and $q=2p/(p+2)$, if $p\ge2$.

3 Error analysis for FVE method

To show error estimates of the finite volume element method (2.10) for the Schrödinger equation, we firstly introduce some valuable lemmas. The following lemmas have been proved in [14,15], which indicate that the bilinear form $a_h(\cdot,I_h^*\cdot)$ is continuous and coercive on RV_h.

Lemma 3.1. *For h sufficiently small, there exist two positive constants $M_1,M_2>0$ such that, for all $u_h,v_h\in RV_h$, the coercive property*

$$a_h(u_h,I_h^*u_h)\ge M_1\|u_h\|_1^2$$

and the boundedness property

$$|a_h(u_h,I_h^*v_h)|\le M_2\|u_h\|_1\|v_h\|_1$$

hold true.

Lemma 3.2. *I_h^* is self-adjoint with respect to the L^2 inner product, then*

$$(u_h, I_h^* v_h)_0 = (v_h, I_h^* u_h)_0, \quad \forall u_h, v_h \in RV_h, \tag{3.1}$$

where

$$(u_h, I_h^* v_h)_0 = \int_\Omega u_h I_h^* v_h dx.$$

Lemma 3.3. *For any $u_h, v_h \in RV_h$, we have*

$$a_h^*(u_h, I_h^* v_h) = a_c(u_h, v_h), \tag{3.2}$$

where

$$a_h^*(u_h, I_h^* v_h) = -\sum_{z \in \mathcal{Z}_h^0} \int_{\partial V_z} (\nabla u_h) \cdot \mathbf{n} I_h^* v_h ds, \tag{3.3a}$$

$$a_c(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h dx. \tag{3.3b}$$

Remark 3.1. From Lemma 3.3, we know the bilinear form $a_h^*(u_h, I_h^* v_h)$ is symmetric.

Lemma 3.4. *For any $u_h, v_h \in V_h$, we have*

$$(u_h, I_h^* v_h) = (I_h^* u_h, v_h), \tag{3.4}$$

where

$$(u_h, I_h^* v_h) = \int_\Omega u_h \overline{I_h^* v_h} dx.$$

Firstly, we present the optimal error estimate of the FVE method (2.10) in L^2 norm.

Theorem 3.1. *Let u^n and u_h^n be the solutions of* (2.8) *and* (2.10)*, respectively. Assume $u \in L^\infty(W^{3,q}(\Omega), u_t \in L^2(W^{3,q}(\Omega))$, $u_{ttt} \in L^2(L^2(\Omega))$. For Δt small enough, for all $t^n \le T$ we have*

$$\|u^n - u_h^n\| \le \mathcal{C}(\Delta t^2 + h^2), \tag{3.5}$$

where $\mathcal{C} = C(\Omega, \|u\|_{L^\infty(W^{3,q})}, \|u_t\|_{L^2(W^{3,q})}, \|u_{ttt}\|_{L^2(L^2)})$ is independent of h and Δt.

Proof. For convenience, we set

$$u^n - u_h^n = (u^n - P_h u^n) - (u_h^n - P_h u^n) =: \xi^n - \eta^n. \tag{3.6}$$

Denote

$$\partial_t \eta^n = \frac{\eta^n - \eta^{n-1}}{\Delta t}, \quad u^n = u(x, t^n), \quad \eta^{n-\frac{1}{2}} = \frac{\eta^n + \eta^{n-1}}{2}.$$

Taking $t=n-\frac{1}{2}$ in (2.8), then making (2.8) subtracts (2.10), and using (2.12), we get the following error equation

$$\begin{aligned}&i\Big(u_t^{n-\frac{1}{2}}-\frac{\eta^n-\eta^{n-1}}{\Delta t},I_h^*v_h\Big)=i(u_t^{n-\frac{1}{2}}-\partial_t u_h^n,I_h^*v_h)\\=&a_h(u^{n-\frac{1}{2}}-u_h^{n-\frac{1}{2}},I_h^*v_h)=a_h(\xi^{n-\frac{1}{2}}-\eta^{n-\frac{1}{2}},I_h^*v_h)\\=&a_h(u^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}},I_h^*v_h)-a_h(\eta^{n-\frac{1}{2}},I_h^*v_h)\\=&-a_h(\eta^{n-\frac{1}{2}},I_h^*v_h).\end{aligned}\tag{3.7}$$

For the first term, we have

$$i(u_t^{n-\frac{1}{2}}-\partial_t u_h^n,I_h^*v_h)=i(u_t^{n-\frac{1}{2}}-\partial_t u^n,I_h^*v_h)+i(\partial_t\xi^n,I_h^*v_h)-i(\partial_t\eta^n,I_h^*v_h),\quad\forall v_h\in V_h.\tag{3.8}$$

Then we get

$$i(u_t^{n-\frac{1}{2}}-\partial_t u^n,I_h^*v_h)+i(\partial_t\xi^n,I_h^*v_h)-i(\partial_t\eta^n,I_h^*v_h)=-a_h(\eta^{n-\frac{1}{2}},I_h^*v_h).\tag{3.9}$$

Setting $v_h=-\eta^{n-\frac{1}{2}}$ and taking the imaginary part of (3.9), one finds

$$\begin{aligned}&Re\{(\partial_t\eta^n,I_h^*\eta^{n-\frac{1}{2}})\}\\=&Im\{a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})+Re\{(u_t^{n-\frac{1}{2}}-\partial_t u^n,I_h^*\eta^{n-\frac{1}{2}})\}+Re\{(\partial_t\xi^n,I_h^*\eta^{n-\frac{1}{2}})\}.\end{aligned}\tag{3.10}$$

Now we estimate (3.10), for the left-hand terms of (3.10), it holds

$$\begin{aligned}&(\partial_t\eta^n,I_h^*\eta^{n-\frac{1}{2}})=\frac{1}{2\Delta t}(\eta^n-\eta^{n-1},I_h^*(\eta^n+\eta^{n-1}))\\=&\frac{1}{2\Delta t}(|||\eta^n|||_0^2-|||\eta^{n-1}|||_0^2+(\eta^n,I_h^*\eta^{n-1})-(\eta^{n-1},I_h^*\eta^n)).\end{aligned}\tag{3.11a}$$

$$\overline{(\partial_t\eta^n,I_h^*\eta^{n-\frac{1}{2}})}=\frac{1}{2\Delta t}(|||\eta^n|||_0^2-|||\eta^{n-1}|||_0^2+(I_h^*\eta^{n-1},\eta^n)-(I_h^*\eta^n,\eta^{n-1})).\tag{3.11b}$$

Combining (2.11) with (3.11a)-(3.11b), using Lemma 3.4, we have

$$\begin{aligned}&Re\{(\partial_t\eta^n,I_h^*\eta^{n-\frac{1}{2}})\}=\frac{(\partial_t\eta^n,I_h^*\eta^{n-\frac{1}{2}})+\overline{(\partial_t\eta^n,I_h^*\eta^{n-\frac{1}{2}})}}{2}\\=&\frac{1}{2\Delta t}(|||\eta^n|||_0^2-|||\eta^{n-1}|||_0^2)\geq\frac{C}{2\Delta t}(\|\eta^n\|^2-\|\eta^{n-1}\|^2).\end{aligned}\tag{3.12}$$

We can treat the right-hand terms of (3.10) as follows

$$a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})+\overline{a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})}=2\Big[a_h(\eta_{1h}^{n-\frac{1}{2}},I_h^*\eta_{1h}^{n-\frac{1}{2}})+a_h(\eta_{2h}^{n-\frac{1}{2}},I_h^*\eta_{2h}^{n-\frac{1}{2}})\Big],\tag{3.13a}$$

$$a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})-\overline{a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})}=2i\Big[a_h(\eta_{2h}^{n-\frac{1}{2}},I_h^*\eta_{1h}^{n-\frac{1}{2}})-a_h(\eta_{1h}^{n-\frac{1}{2}},I_h^*\eta_{2h}^{n-\frac{1}{2}})\Big].\tag{3.13b}$$

Thanks to (3.13a) and Lemma 3.1, we arrive at

$$Re\{a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})\}\geq[M_1\|\eta_{1h}^{n-\frac{1}{2}}\|_1^2+M_1\|\eta_{2h}^{n-\frac{1}{2}}\|_1^2]=M_1\|\eta^{n-\frac{1}{2}}\|_1^2. \tag{3.14}$$

With the help of Lemma 3.3, (3.13b) and the Hölder inequality, one gets

$$\begin{aligned}
&Im\{a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})\}=\frac{a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})-\overline{a_h(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})}}{2i}\\
=&-\sum_{z\in\mathcal{Z}_h^0}\int_{\partial V_z}(\nabla\eta_{2h}^{n-\frac{1}{2}})\cdot\mathbf{n}I_h^*\eta_{1h}^{n-\frac{1}{2}}ds+\sum_{z\in\mathcal{Z}_h^0}\int_{\partial V_z}(\nabla\eta_{1h}^{n-\frac{1}{2}})\cdot\mathbf{n}I_h^*\eta_{2h}^{n-\frac{1}{2}}ds\\
&+\sum_{z\in\mathcal{Z}_h^0}\int_{V_z}\big(-V\eta_{1h}^{n-\frac{1}{2}}I_h^*\eta_{2h}^{n-\frac{1}{2}}+V\eta_{2h}^{n-\frac{1}{2}}I_h^*\eta_{1h}^{n-\frac{1}{2}}\big)dx\\
\leq&C\|\eta_{2h}^{n-\frac{1}{2}}\|\|\eta_{1h}^{n-\frac{1}{2}}\|\leq C\|\eta^{n-\frac{1}{2}}\|^2.
\end{aligned} \tag{3.15}$$

Then, by the Hölder inequality and (2.7b), we obtain

$$\begin{aligned}
&Re\{(u_t^{n-\frac{1}{2}}-\partial_tu^n,I_h^*\eta^{n-\frac{1}{2}})\}\leq|(u_t^{n-\frac{1}{2}}-\partial_tu^n,I_h^*\eta^{n-\frac{1}{2}})|\leq\|u_t^{n-\frac{1}{2}}-\partial_tu^n\|\|I_h^*\eta^{n-\frac{1}{2}}\|\\
&\leq C\|u_t^{n-\frac{1}{2}}-\partial_tu^n\|\|\eta^{n-\frac{1}{2}}\|\leq C_1\|u_t^{n-\frac{1}{2}}-\partial_tu^n\|^2+C_1^*\|\eta^{n-\frac{1}{2}}\|^2,
\end{aligned} \tag{3.16a}$$

$$\begin{aligned}
&\|u_t^{n-\frac{1}{2}}-\partial_tu^n\|^2=\frac{1}{4(\Delta t)^2}\Big\|\int_{t^{n-\frac{1}{2}}}^{t^n}(t^n-t)^2u_{ttt}(\cdot,t)dt+\int_{t^{n-1}}^{t^{n-\frac{1}{2}}}(t^{n-1}-t)^2u_{ttt}(\cdot,t)dt\Big\|^2\\
&\leq\frac{1}{4(\Delta t)^2}\int_\Omega\Big|\int_{t^{n-1}}^{t^n}\Big(\frac{\Delta t}{2}\Big)^2u_{ttt}(\cdot,t)dt\Big|^2dx\leq C(\Delta t)^2\int_\Omega\Big|\int_{t^{n-1}}^{t^n}u_{ttt}(\cdot,t)dt\Big|^2dx\\
&\leq C(\Delta t)^2\int_\Omega\Big|\Big(\int_{t_{n-1}}^{t_n}|u_{ttt}(\cdot,t)|^2dt\Big)^{\frac{1}{2}}\Big(\int_{t_{n-1}}^{t_n}1^2dt\Big)^{\frac{1}{2}}\Big|^2dx\\
&=C(\Delta t)^3\int_{t^{n-1}}^{t^n}\|u_{ttt}(\cdot,t)\|^2dt.
\end{aligned} \tag{3.16b}$$

Combining (3.16a) with (3.16b), it holds that

$$Re\{(u_t^{n-\frac{1}{2}}-\partial_tu^n,I_h^*\eta^{n-\frac{1}{2}})\}\leq C_1(\Delta t)^3\int_{t^{n-1}}^{t^n}\|u_{ttt}\|^2dt+C_1^*\|\eta^{n-\frac{1}{2}}\|^2. \tag{3.17}$$

Now, we estimate the last term of (3.10), by the Hölder inequality and (2.7b), (2.13b), we get

$$\begin{aligned}
Re\{(\partial_t\xi^n,I_h^*\eta^{n-\frac{1}{2}})\}&\leq|(\partial_t\xi^n,I_h^*\eta^{n-\frac{1}{2}})|\leq\|\partial_t\xi^n\|\|I_h^*\eta^{n-\frac{1}{2}}\|\\
&\leq C\|\partial_t\xi^n\|\|\eta^{n-\frac{1}{2}}\|\leq C_2\|\partial_t\xi^n\|^2+C_2^*\|\eta^{n-\frac{1}{2}}\|^2,
\end{aligned} \tag{3.18a}$$

$$\begin{aligned}
\|\partial_t\xi^n\|^2&=\Big\|\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\xi_tdt\Big\|^2\leq\frac{1}{(\Delta t)^2}\int_\Omega\bigg(\Big(\int_{t^{n-1}}^{t^n}|\xi_t|^2dt\Big)^{\frac{1}{2}}\Big(\int_{t^{n-1}}^{t^n}1^2dt\Big)^{\frac{1}{2}}\bigg)^2dx\\
&=\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\int_\Omega|\xi_t|^2dxdt=\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\|\xi_t\|^2dt\leq\frac{C}{\Delta t}h^4\int_{t^{n-1}}^{t^n}\|u_t\|_{3,q}^2dt.
\end{aligned} \tag{3.18b}$$

From (3.18a) and (3.18b), one obtains

$$Re\{(\partial_t\xi^n, I_h^*\eta^{n-\frac{1}{2}})\} \le C_2\frac{h^4}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t\|_{3,q}^2dt + C_2^*\|\eta^{n-\frac{1}{2}}\|^2. \tag{3.19}$$

Combining (3.12), (3.15), (3.17) with (3.19), we arrive at

$$\begin{aligned}&\frac{C}{2\Delta t}(\|\eta^n\|^2-\|\eta^{n-1}\|^2)\\ \le& C_3\|\eta^{n-\frac{1}{2}}\|^2 + C_1(\Delta t)^3\int_{t^{n-1}}^{t^n}\|u_{ttt}(\cdot,t)\|^2dt + C_2\frac{h^4}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t\|_{3,q}^2dt.\end{aligned} \tag{3.20}$$

Multiplying by $2\Delta t$ and summing over n from 1 to l $(1\le l\le N)$ at both sides of (3.20), since $\eta^0=0$, we have

$$C\|\eta^l\|^2 \le C_1(\Delta t)^4\int_0^{t^l}\|u_{ttt}\|^2dt + C_2h^4\int_0^{t^l}\|u_t\|_{3,q}^2dt + \sum_{n=1}^{l}C_3\|\eta^{n-\frac{1}{2}}\|^2\Delta t. \tag{3.21}$$

Applying the discrete Gronwall Lemma [13], for small Δt, we obtain

$$\|\eta^l\|^2 \le C_1(\Delta t)^4\int_0^{t^l}\|u_{ttt}\|^2dt + C_2h^4\int_0^{t^l}\|u_t\|_{3,q}^2dt \le C\Delta t^4 + Ch^4. \tag{3.22}$$

Thanks to (2.13b) we finally get

$$\|u^n-u_h^n\| \le \mathcal{C}(\Delta t^2+h^2). \tag{3.23}$$

This completes the proof. □

Lemma 3.5. *Let u and u_h be the solutions of* (2.8) *and* (2.10)*, respectively. Assume $u_t,u_{tt}\in L^\infty(W^{3,q}(\Omega))$, $u_{ttt}\in L^\infty(H^2(\Omega))$, $f_{ttt}\in L^\infty(L^2(\Omega))$. Then the time-difference of error $\eta=u_h-P_hu$ has high order error*

$$\|\eta^n-\eta^{n-1}\| \le \mathcal{C}\Delta t(\Delta t^2+h^2), \tag{3.24}$$

where $\mathcal{C}=C(\Omega,\|u_t\|_{L^\infty(W^{3,q})},\|u_{tt}\|_{L^\infty(W^{3,q})},\|u_{ttt}\|_{L^\infty(H^2)},\|f_{ttt}\|_{L^\infty(L^2)})$ is independent of h and Δt.

Proof. It follows from (2.8) that

$$\int_0^t i(u_t,I_h^*v) - a(u,I_h^*v) - (f,I_h^*v)dt = 0, \quad \forall v\in H_0^1(\Omega). \tag{3.25}$$

Integrating (3.25) in $[t^{n-1},t^n]$ and using the Taylor formula, we have

$$\int_{t^{n-1}}^{t^n} i(u_t,I_h^*v)-a(u,I_h^*v)-(f,I_h^*v)dt=0, \tag{3.26a}$$

$$\int_{t^{n-1}}^{t^n} i(u_t,I_h^*v)dt=i\int_\Omega\int_{t^{n-1}}^{t^n} u_t\overline{I_h^*v}dtdx=i(u^n-u^{n-1},I_h^*v), \tag{3.26b}$$

$$\begin{aligned}\int_{t^{n-1}}^{t^n} a(u,I_h^*v)dt=&a(u^{n-\frac{1}{2}},I_h^*v)(\Delta t)\\&+\int_{t^{n-1}}^{t^n} a\Big(\frac{u_{tt}(\gamma)}{2}(t-t^{n-\frac{1}{2}})^2,I_h^*v\Big)dt,\quad \gamma\in(t^{n-\frac{1}{2}},t^n),\end{aligned} \tag{3.26c}$$

$$\int_{t^{n-1}}^{t^n}(f,I_h^*v)dt=(f^{n-\frac{1}{2}},I_h^*v)(\Delta t)+\int_{t^{n-1}}^{t^n}\Big(\frac{f_{tt}(\gamma)}{2}(t-t^{n-\frac{1}{2}})^2,I_h^*v\Big)dt,\quad \gamma\in(t^{n-\frac{1}{2}},t^n). \tag{3.26d}$$

Combining (3.26b)-(3.26d) with (3.26a), we obtain

$$\begin{aligned}&i(u^n-u^{n-1},I_h^*v)-a(u^{n-\frac{1}{2}},I_h^*v)(\Delta t)\\=&(f^{n-\frac{1}{2}},I_h^*v)(\Delta t)+r_1^n(I_h^*v)+r_2^n(I_h^*v),\quad \forall v\in H_0^1(\Omega),\end{aligned} \tag{3.27}$$

where

$$r_1^n(I_h^*v)=\int_{t^{n-1}}^{t^n}\Big(\frac{f_{tt}(\gamma)}{2}(t-t^{n-\frac{1}{2}})^2,I_h^*v\Big)dt,\qquad \gamma\in(t^{n-\frac{1}{2}},t^n), \tag{3.28a}$$

$$r_2^n(I_h^*v)=\int_{t^{n-1}}^{t^n}a\Big(\frac{u_{tt}(\gamma)}{2}(t-t^{n-\frac{1}{2}})^2,I_h^*v\Big)dt,\qquad \gamma\in(t^{n-\frac{1}{2}},t^n). \tag{3.28b}$$

From (2.10), we get

$$i(u_h^n-u_h^{n-1},I_h^*v_h)-a_h(u_h^{n-\frac{1}{2}},I_h^*v_h)(\Delta t)=(f^{n-\frac{1}{2}},I_h^*v_h)(\Delta t),\quad \forall v_h\in V_h. \tag{3.29}$$

For all $v_h\in V_h$, the following error equation holds by combining (3.27) with (3.29)

$$i(u^n-u_h^n-(u^{n-1}-u_h^{n-1}),I_h^*v_h)-a_h(u^{n-\frac{1}{2}}-u_h^{n-\frac{1}{2}},I_h^*v_h)\Delta t=r_1^n(I_h^*v_h)+r_2^n(I_h^*v_h). \tag{3.30}$$

From (3.6) and (2.12), we have

$$i(\eta^n-\eta^{n-1},I_h^*v_h)=(\Delta t)a_h(\eta^{n-\frac{1}{2}},I_h^*v_h)+ir_3^n(I_h^*v_h)-r_1^n(I_h^*v_h)-r_2^n(I_h^*v_h), \tag{3.31}$$

where $r_3^n(I_h^*v_h)=(\xi^n-\xi^{n-1},I_h^*v_h)$.

Substituting n by $n-1$ in (3.31), one gets

$$i(\eta^{n-1}-\eta^{n-2},I_h^*v_h)=(\Delta t)a_h(\eta^{n-\frac{3}{2}},I_h^*v_h)+ir_3^{n-1}(I_h^*v_h)-r_1^{n-1}(I_h^*v_h)-r_2^{n-1}(I_h^*v_h). \tag{3.32}$$

Let $\delta^n=\eta^n-\eta^{n-1}$, from (3.31) and (3.32), we obtain

$$\begin{aligned}i(\delta^n-\delta^{n-1},I_h^*v_h)=&(\Delta t)a_h(\eta^{n-\frac{1}{2}}-\eta^{n-\frac{3}{2}},I_h^*v_h)+i(r_3^n(I_h^*v_h)-r_3^{n-1}(I_h^*v_h))\\&+(r_1^{n-1}(I_h^*v_h)-r_1^n(I_h^*v_h))+(r_2^{n-1}(I_h^*v_h)-r_2^n(I_h^*v_h)),\quad \forall v_h\in V_h,\end{aligned} \tag{3.33}$$

where

$$a_h(\eta^{n-\frac{1}{2}}-\eta^{n-\frac{3}{2}},I_h^*v_h)=a_h\Big(\frac{\eta^n+\eta^{n-1}}{2}-\frac{\eta^{n-1}+\eta^{n-2}}{2},I_h^*v_h\Big)=\frac{1}{2}a_h(\delta^n+\delta^{n-1},I_h^*v_h). \tag{3.34}$$

Choosing $v_h=\delta^n+\delta^{n-1}$ in (3.33) and taking the imaginary part of (3.33), by Lemma 3.3, we have

$$\begin{aligned}
&Re\{(\delta^n-\delta^{n-1},I_h^*(\delta^n+\delta^{n-1}))\}\\
=&\frac{\Delta t}{2}Im\{a_h(\delta^n+\delta^{n-1},I_h^*(\delta^n+\delta^{n-1}))\}+Re\{r_3^n(I_h^*(\delta^n+\delta^{n-1}))-r_3^{n-1}(I_h^*(\delta^n+\delta^{n-1}))\}\\
&+Im\{r_1^{n-1}(I_h^*(\delta^n+\delta^{n-1}))-r_1^n(I_h^*(\delta^n+\delta^{n-1}))\}\\
&+Im\{r_2^{n-1}(I_h^*(\delta^n+\delta^{n-1}))-r_2^n(I_h^*(\delta^n+\delta^{n-1}))\}\\
\le& C\Delta t\|\delta^n+\delta^{n-1}\|^2+|r_1^{n-1}-r_1^n|+|r_2^{n-1}-r_2^n|+|r_3^n-r_3^{n-1}|\\
=&C\Delta t\|\delta^n+\delta^{n-1}\|^2+\sum_{i=1}^3|T_i|.
\end{aligned} \tag{3.35}$$

By the Hölder inequality and (2.13b), we obtain

$$\begin{aligned}
|T_1|=&\Big|\int_{t^{n-1}}^{t^n}\Big(\frac{f_{tt}(\gamma)}{2}(t-t^{n-\frac{1}{2}})^2,I_h^*(\delta^n+\delta^{n-1})\Big)dt\\
&\quad-\int_{t^{n-2}}^{t^{n-1}}\Big(\frac{f_{tt}(\theta)}{2}(t-t^{n-\frac{3}{2}})^2,I_h^*(\delta^n+\delta^{n-1})\Big)dt\Big|\\
=&\Big|\frac{(\Delta t)^3}{24}\int_\Omega(f_{tt}(\gamma)-f_{tt}(\theta))\overline{I_h^*(\delta^n+\delta^{n-1})}dx\Big|\\
=&\Big|\frac{(\Delta t)^3}{24}\int_\Omega\int_\theta^\gamma f_{ttt}(\tau)d\tau\overline{I_h^*(\delta^n+\delta^{n-1})}dx\Big|\\
\le& C(\Delta t)^3\int_{t^{n-2}}^{t^n}\|f_{ttt}\|\|\delta^n+\delta^{n-1}\|d\tau,
\end{aligned} \tag{3.36a}$$

$$\begin{aligned}
|T_2|=&\Big|\int_{t^{n-1}}^{t^n}a\Big(\frac{u_{tt}(\gamma)}{2}(t-t^{n-\frac{1}{2}})^2,I_h^*(\delta^n+\delta^{n-1})\Big)dt\\
&\quad-\int_{t^{n-2}}^{t^{n-1}}a\Big(\frac{u_{tt}(\theta)}{2}(t-t^{n-\frac{3}{2}})^2,I_h^*(\delta^n+\delta^{n-1})\Big)dt\Big|\\
=&\Big|\frac{(\Delta t)^3}{24}a(u_{tt}(\gamma),I_h^*(\delta^n+\delta^{n-1}))-\frac{(\Delta t)^3}{24}a(u_{tt}(\theta),I_h^*(\delta^n+\delta^{n-1}))\Big|\\
=&\frac{(\Delta t)^3}{24}\Big|\int_\Omega\big(-[\Delta u_{tt}(\gamma)-\Delta u_{tt}(\theta)]+V[u_{tt}(\gamma)-u_{tt}(\theta)]\big)\overline{I_h^*(\delta^n+\delta^{n-1})}dx\Big|\\
=&\frac{(\Delta t)^3}{24}\Big|\int_\Omega\int_\theta^\gamma(-\Delta u_{ttt}(\tau)+Vu_{ttt}(\tau))\overline{I_h^*(\delta^n+\delta^{n-1})}d\tau dx\Big|\\
\le& C(\Delta t)^3\int_{t^{n-2}}^{t^n}(\|u_{ttt}\|_2+\|Vu_{ttt}\|)\|\delta^n+\delta^{n-1}\|d\tau.
\end{aligned} \tag{3.36b}$$

Thanks to the Lagrange mean value Theorem, Hölder inequality and (2.13b), one finds

$$\begin{aligned}|T_3|=&\Big|\int_\Omega(\xi^n-\xi^{n-1})\overline{I_h^*(\delta^n+\delta^{n-1})}dx-\int_\Omega(\xi^{n-1}-\xi^{n-2})\overline{I_h^*(\delta^n+\delta^{n-1})}dx\Big|\\=&\Big|\int_\Omega\xi_t(\rho)(t^n-t^{n-1})\overline{I_h^*(\delta^n+\delta^{n-1})}-\int_\Omega\xi_t(\iota)(t^{n-1}-t^{n-2})\overline{I_h^*(\delta^n+\delta^{n-1})}dx\Big|\\\le&(\Delta t)\int_\Omega\int_\iota^\rho|\xi_{tt}(\tau)\overline{I_h^*(\delta^n+\delta^{n-1})}|d\tau dx\le(\Delta t)\int_{t^{n-2}}^{t^n}\|\xi_{tt}\|\|(\delta^n+\delta^{n-1})\|d\tau\\\le&C(\Delta t)h^2\int_{t^{n-2}}^{t^n}\|u_{tt}\|_{3,q}\|(\delta^n+\delta^{n-1})\|d\tau.\end{aligned}\tag{3.37}$$

Similar to (3.12), we get

$$Re\{(\delta^n-\delta^{n-1},I_h^*(\delta^n+\delta^{n-1}))\}=|\|\delta^n\||_0^2-|\|\delta^{n-1}\||_0^2\ge C(\|\delta^n\|^2-\|\delta^{n-1}\|^2).\tag{3.38}$$

Combining (3.35) with (3.36a)-(3.38), we obtain

$$\begin{aligned}C(\|\delta^n\|^2-\|\delta^{n-1}\|^2)\le& C\Delta t\|\delta^n+\delta^{n-1}\|^2+C(\Delta t)h^2\int_{t^{n-2}}^{t^n}\|u_{tt}\|_{3,q}\|\delta^n+\delta^{n-1}\|d\tau\\&+C(\Delta t)^3\int_{t^{n-2}}^{t^n}(\|u_{ttt}\|_2+\|Vu_{ttt}\|+\|f_{ttt}\|)\|\delta^n+\delta^{n-1}\|d\tau.\end{aligned}\tag{3.39}$$

Without loss of generality, we assume that there is an integer $1\le k\le N$, such that

$$\|\delta^k\|=\max_{1\le n\le N}\|\delta^n\|.\tag{3.40}$$

Then we obtain

$$\begin{aligned}C(\|\delta^n\|^2-\|\delta^{n-1}\|^2)\le& C\Delta t\|\delta^n+\delta^{n-1}\|\|\delta^k\|+C(\Delta t)h^2\|\delta^k\|\int_{t^{n-2}}^{t^n}\|u_{tt}\|_{3,q}d\tau\\&+C(\Delta t)^3\|\delta^k\|\int_{t^{n-2}}^{t^n}(\|u_{ttt}\|_2+\|Vu_{ttt}\|+\|f_{ttt}\|)d\tau.\end{aligned}\tag{3.41}$$

Summing up for n from 2 to k in (3.41) and combining (3.40), we have

$$\begin{aligned}C\|\delta^k\|\le& C\|\delta^1\|+\sum_{n=2}^k C(\|\delta^n\|+\|\delta^{n-1}\|)\Delta t+C(\Delta t)h^2\int_0^{t^k}\|u_{tt}\|_{3,q}d\tau\\&+C(\Delta t)^3\int_0^{t^k}(\|u_{ttt}\|_2+\|Vu_{ttt}\|+\|f_{ttt}\|)d\tau.\end{aligned}\tag{3.42}$$

Since $\eta^0=0$, we have $\|\delta^1\|=\|\eta^1\|$.

Let $n=1$ in (3.10), we get

$$Re\{(\partial_t\eta^1,I_h^*\eta^{\frac12})\}=Im\{a_h(\eta^{\frac12},I_h^*\eta^{\frac12})+Re\{(u_t^{\frac12}-\partial_t u^1,I_h^*\eta^{\frac12})\}+Re\{(\partial_t\xi^1,I_h^*\eta^{\frac12})\}.\tag{3.43}$$

Combining $\eta^0=0$, (3.12), (3.15), (3.16a) and (3.18a), it holds that

$$C\|\eta^1\|\le C(\|u_t^{\frac{1}{2}}-\partial_t u^1\|+\|\partial_t\xi^1\|)\Delta t, \tag{3.44}$$

where we have used $2\|\eta^{\frac{1}{2}}\|=\|\eta^1\|$.

With the help of (3.16b) and (3.18b), we obtain

$$\begin{aligned}
&\|u_t^{\frac{1}{2}}-\partial_t u^1\|\le C(\Delta t)^{\frac{3}{2}}\Big(\|u_{ttt}(\cdot,t)\|^2_{L^\infty(0,t^1,L^2)}\int_0^{t^1}1dt\Big)^{\frac{1}{2}}\le C\Delta t^2,\\
&\|\partial_t\xi^1\|\le C(\Delta t)^{-\frac{1}{2}}h^2\Big(\|u_t\|^2_{L^\infty(0,t^1,W^{3,q})}\int_0^{t^1}1dt\Big)^{\frac{1}{2}}\le Ch^2.
\end{aligned}$$

Combining above inequalities with (3.44), one gets

$$\|\eta^1\|\le C(\Delta t)(\Delta t^2+h^2). \tag{3.45}$$

Substituting (3.45) into (3.42) and applying the discrete Gronwall lemma [13], for small Δt, we have

$$C\|\delta^k\|\le C(\Delta t)(\Delta t^2+h^2). \tag{3.46}$$

This completes the proof. $\square$

Now we show the error estimate in the H^1-norm for the FVE method (2.10).

Theorem 3.2. *Let u and u_h be the solutions of* (2.8) *and* (2.10), *respectively. Assume that $u\in L^\infty(H^2(\Omega))$, $u_t,u_{tt}\in L^\infty(W^{3,q}(\Omega))$, $u_{ttt}\in L^\infty(H^2(\Omega))$, $f_{ttt}\in L^\infty(L^2(\Omega))$. For Δt small enough, for all $t^n\le T$ we have*

$$\|u^n-u_h^n\|_1\le \mathcal{C}(\Delta t^2+h), \tag{3.47}$$

where $\mathcal{C}=C(\Omega,\|u\|_{L^\infty(H^2)},\|u_t\|_{L^\infty(W^{3,q})},\|u_{tt}\|_{L^\infty(W^{3,q})}\|u_{ttt}\|_{L^\infty(H^2)},\|f_{ttt}\|_{L^\infty(L^2)})$ is independent of h and Δt.

Proof. Let $v_h=-\partial_t\eta^n$ and taking the real part of (3.9), we can obtain

$$\begin{aligned}
&Re\{a_h(\eta^{n-\frac{1}{2}},I_h^*\partial_t\eta^n)\}+Im\{(\partial_t\eta^n,I_h^*\partial_t\eta^n)\}\\
=&Im\{(u_t^{n-\frac{1}{2}}-\partial_t u^n,I_h^*\partial_t\eta^n)\}+Im\{(\partial_t\xi^n,I_h^*\partial_t\eta^n)\}.
\end{aligned} \tag{3.48}$$

For the second term of the left-hand side of (3.48), it holds that

$$Im\{(\partial_t\eta^n,I_h^*\partial_t\eta^n)\}=Im\{|\|\partial_t\eta^n\||_0^2\}=0. \tag{3.49}$$

By the Hölder inequality, Young inequalities, (2.7b), (3.16b) and (3.18b) we obtain

$$\begin{aligned}
&Im\{(u_t^{n-\frac{1}{2}}-\partial_t u^n, I_h^*\partial_t\eta^n)\}\le|(u_t^{n-\frac{1}{2}}-\partial_t u^n, I_h^*\partial_t\eta^n)|\\
&\qquad\qquad\le C_1\Delta t^3\int_{t^{n-1}}^{t^n}\|u_{ttt}\|^2dt+C_1^*\|\partial_t\eta^n\|^2, \qquad (3.50a)\\
&Im\{(\partial_t\xi^n, I_h^*\partial_t\eta^n)\}\le|(\partial_t\xi^n, I_h^*\partial_t\eta^n)|\le C_2\|\partial_t\xi^n\|^2+C_2^*\|\partial_t\eta^n\|^2\\
&\qquad\qquad\le C_2\frac{h^4}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t\|_{3,q}^2dt+C_2^*\|\partial_t\eta^n\|^2, \qquad (3.50b)\\
&a_h(\eta^{n-\frac{1}{2}}, I_h^*\partial_t\eta^n)=\frac{1}{2\Delta t}(a_h(\eta^n, I_h^*\eta^n)-a_h(\eta^n, I_h^*\eta^{n-1})\\
&\qquad\qquad+a_h(\eta^{n-1}, I_h^*\eta^n)-a_h(\eta^{n-1}, I_h^*\eta^{n-1})). \qquad (3.50c)
\end{aligned}$$

From (3.13a), it is easy to check that

$$\begin{aligned}
&a_h(\eta^{n-1}, I_h^*\eta^{n-1})+\overline{a_h(\eta^{n-1}, I_h^*\eta^{n-1})}=2[a_h(\eta_{1h}^{n-1}, I_h^*\eta_{1h}^{n-1})+a_h(\eta_{2h}^{n-1}, I_h^*\eta_{2h}^{n-1})], \qquad (3.51a)\\
&a_h(\eta^{n-1}, I_h^*\eta^n)=a_h(\eta_{1h}^{n-1}, I_h^*\eta_{1h}^n)-ia_h(\eta_{1h}^{n-1}, I_h^*\eta_{2h}^n)\\
&\qquad\qquad+ia_h(\eta_{2h}^{n-1}, I_h^*\eta_{1h}^n)+a_h(\eta_{2h}^{n-1}, I_h^*\eta_{2h}^n), \qquad (3.51b)\\
&\overline{a_h(\eta^{n-1}, I_h^*\eta^n)}=a_h(\eta_{1h}^{n-1}, I_h^*\eta_{1h}^n)+ia_h(\eta_{1h}^{n-1}, I_h^*\eta_{2h}^n)\\
&\qquad\qquad-ia_h(\eta_{2h}^{n-1}, I_h^*\eta_{1h}^n)+a_h(\eta_{2h}^{n-1}, I_h^*\eta_{2h}^n), \qquad (3.51c)\\
&a_h(\eta^{n-1}, I_h^*\eta^n)+\overline{a_h(\eta^{n-1}, I_h^*\eta^n)}=2[a_h(\eta_{1h}^{n-1}, I_h^*\eta_{1h}^n)+a_h(\eta_{2h}^{n-1}, I_h^*\eta_{2h}^n)]. \qquad (3.51d)
\end{aligned}$$

As a consequence, it arrives at

$$a_h(\eta^n, I_h^*\eta^{n-1})+\overline{a_h(\eta^n, I_h^*\eta^{n-1})}=2[a_h(\eta_{1h}^n, I_h^*\eta_{1h}^{n-1})+a_h(\eta_{2h}^n, I_h^*\eta_{2h}^{n-1})]. \qquad (3.52)$$

Combining (3.13a), (3.50c) with (3.51d)-(3.52) and using Lemma 3.1, we obtain

$$\begin{aligned}
&Re\{a_h(\eta^{n-\frac{1}{2}}, I_h^*\partial_t\eta^n)\}\\
=&\frac{1}{2\Delta t}[a_h(\eta_{1h}^n, I_h^*\eta_{1h}^n)+a_h(\eta_{2h}^n, I_h^*\eta_{2h}^n)-a_h(\eta_{1h}^{n-1}, I_h^*\eta_{1h}^{n-1})-a_h(\eta_{2h}^{n-1}, I_h^*\eta_{2h}^{n-1})\\
&+a_h(\eta_{1h}^{n-1}, I_h^*\eta_{1h}^n)+a_h(\eta_{2h}^{n-1}, I_h^*\eta_{2h}^n)-a_h(\eta_{1h}^n, I_h^*\eta_{1h}^{n-1})-a_h(\eta_{2h}^n, I_h^*\eta_{2h}^{n-1})]\\
\ge&\frac{1}{2\Delta t}[M_1\|\eta^n\|_1^2-M_1\|\eta^{n-1}\|_1^2]-\frac{1}{2\Delta t}[a_h(\eta_{1h}^n, I_h^*\eta_{1h}^{n-1})-a_h(\eta_{1h}^{n-1}, I_h^*\eta_{1h}^n)]\\
&-\frac{1}{2\Delta t}[a_h(\eta_{2h}^n, I_h^*\eta_{2h}^{n-1})-a_h(\eta_{2h}^{n-1}, I_h^*\eta_{2h}^n)]. \qquad (3.53)
\end{aligned}$$

Thanks to (3.48)-(3.50b) and (3.53), we get

$$\begin{aligned}
&\frac{M_1}{2\Delta t}[\|\eta^n\|_1^2-\|\eta^{n-1}\|_1^2] \\
\leq& C_1(\Delta t)^3\int_{t^{n-1}}^{t^n}\|u_{ttt}\|^2dt+C_2\frac{h^4}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t\|_{3,q}^2dt \\
&+(C_1^*+C_2^*)\|\partial_t\eta^n\|^2+\frac{1}{2\Delta t}[a_h(\eta_{1h}^n,I_h^*\eta_{1h}^{n-1})-a_h(\eta_{1h}^{n-1},I_h^*\eta_{1h}^n)]. \\
&+\frac{1}{2\Delta t}[a_h(\eta_{2h}^n,I_h^*\eta_{2h}^{n-1})-a_h(\eta_{2h}^{n-1},I_h^*\eta_{2h}^n)].
\end{aligned}\tag{3.54}$$

Now, we estimate the last two terms of (3.54). By Lemma 3.3 and the Young inequalities, one gets

$$\begin{aligned}
&\frac{1}{2\Delta t}[a_h(\eta_{1h}^n,I_h^*\eta_{1h}^{n-1})-a_h(\eta_{1h}^{n-1},I_h^*\eta_{1h}^n)] \\
=&\frac{1}{2}[a_h(\partial_t\eta_{1h}^n,I_h^*\eta_{1h}^{n-1})-a_h(\eta_{1h}^{n-1},I_h^*\partial_t\eta_{1h}^n)] \\
\leq&\frac{C}{2}\|\partial_t\eta_{1h}^n\|\|\eta_{1h}^{n-1}\|\leq\frac{C_3^*}{2}\|\partial_t\eta^n\|^2+\frac{C_3}{2}\|\eta^{n-1}\|^2.
\end{aligned}\tag{3.55}$$

In the same way as treating (3.15) one gets

$$\frac{1}{2\Delta t}[a_h(\eta_{2h}^n,I_h^*\eta_{2h}^{n-1})-a_h(\eta_{2h}^{n-1},I_h^*\eta_{2h}^n)]\leq\frac{C_3^*}{2}\|\partial_t\eta^n\|^2+\frac{C_3}{2}\|\eta^{n-1}\|^2. \tag{3.56}$$

Substituting (3.55)-(3.56) into (3.54), we have

$$\begin{aligned}
\frac{M_1}{2\Delta t}[\|\eta^n\|_1^2-\|\eta^{n-1}\|_1^2]\leq& C_1(\Delta t)^3\int_{t^{n-1}}^{t^n}\|u_{ttt}\|^2dt+C_2\frac{h^4}{\Delta t}\int_{t^{n-1}}^{t^n}\|u_t\|_{3,q}^2dt \\
&+C_3\|\eta^{n-1}\|_1^2+C_4\|\partial_t\eta^n\|^2.
\end{aligned}\tag{3.57}$$

Multiplying by $2\Delta t$ and summing over n from 1 to l $(1\leq l\leq N)$ at both sides of (3.57), since $\eta^0=0$, it holds

$$\begin{aligned}
\|\eta^l\|_1^2\leq& C_1(\Delta t)^4\int_0^{t^l}\|u_{ttt}\|^2dt+C_2h^4\int_0^{t^l}\|u_t\|_{3,q}^2dt \\
&+\sum_{n=1}^{l}C_3\|\eta^{n-1}\|_1^2\Delta t+\sum_{n=1}^{l}C_4\|\partial_t\eta^n\|^2\Delta t.
\end{aligned}\tag{3.58}$$

Thanks to Lemma 3.5, we can get

$$\sum_{n=1}^{l}C_4\|\partial_t\eta^n\|^2\Delta t\leq C(\Delta t^2+h^2)^2. \tag{3.59}$$

By (3.59) and the discrete Gronwall lemma [13], for small Δt, we have

$$\|\eta^l\|_1^2\leq C(\Delta t^2+h^2)^2. \tag{3.60}$$

With help of (2.13a), this yields (3.47). □

4 Error analysis for two-grid FVE method

In order to present two-grid FVE method for the time-dependent Schrödinger equation (1.1), we introduce two quasi-uniform triangulations T_H and T_h of Ω, with two different mesh sizes H and h $(H>h)$. The corresponding finite element spaces V_H and V_h which satisfy $V_H\subset V_h$ were also proposed. They will be called the coarse-grid and fine-grid space, respectively. The idea of the two-grid method is solving the time-dependent Schrödinger equation on the coarse space to obtain a rough approximation of the solution and on the basis of this approximate solution solving a decoupling system to get a more accurate solution on the fine space. We present the two-grid FVE method as two steps:

Algorithm 4.1.

Step 1. On the coarse grid T_H, find $u_H^n\in V_H$ $(n=1,2,\cdots)$, such that

$$\begin{cases} i\Big(\dfrac{u_H^n-u_H^{n-1}}{\Delta t},I_H^*v_H\Big)=\overline{a_h}(u_H^{n-\frac{1}{2}},I_H^*v_H)+(Vu_H^{n-\frac{1}{2}},I_H^*v_H) \\ \qquad\qquad\qquad\qquad +(f^{n-\frac{1}{2}},I_H^*v_H), \qquad\qquad \forall v_H\in V_H, \\ u_H^0=P_Hu_0, \end{cases} \tag{4.1}$$

where

$$\overline{a_h}(u_H^n,I_H^*v_H)=-\sum_{z\in\mathcal{Z}_h^0}\int_{\partial V_z}(\nabla u_H^n)\cdot\mathbf{n}\overline{I_H^*v_H}ds.$$

Step 2. On the fine grid T_h, find $u_h^n\in V_h(n=1,2,\cdots)$, such that

$$\begin{cases} \overline{a_h}(u_h^{n-\frac{1}{2}},I_h^*v_h)=i\Big(\dfrac{u_H^n-u_H^{n-1}}{\Delta t},I_h^*v_h\Big)-(Vu_H^{n-\frac{1}{2}},I_h^*v_h) \\ \qquad\qquad\qquad\qquad -(f^{n-\frac{1}{2}},I_h^*v_h), \qquad\qquad \forall v_h\in V_h, \\ u_h^0=P_hu_0. \end{cases} \tag{4.2}$$

For the H^1-norm error estimate of two-grid FVE algorithm (4.1)-(4.2), we have

Theorem 4.1. *Let u and u_h be the solutions of* (2.8) *and the two-grid FVE method* (4.1)-(4.2), *respectively. Assume that $u,u_{tt}\in L^\infty(H^2(\Omega)\bigcap W^{3,q}(\Omega))$, $u_t\in L^\infty(W^{3,q}(\Omega))$, $f_{tt}\in L^\infty(L^2(\Omega))$. For Δt small enough, if $u_h^0=P_hu_0$ with P_h defined by* (2.12), *then for $t^n\le T$ we have*

$$\|u^n-u_h^n\|_1\le C(\Delta t+h+H^2), \tag{4.3}$$

where $C=C(\Omega,\|u\|_{L^\infty(H^2\bigcap W^{3,q})},\|u_t\|_{L^\infty(W^{3,q})},\|u_{tt}\|_{L^\infty(H^2\bigcap W^{3,q})},\|f_{tt}\|_{L^\infty(L^2)})$ is independent of h and Δt.

Proof. The following error equation can be obtained by combining (2.8) with (4.2)

$$\overline{a_h}(u^{n-\frac{1}{2}}-u_h^{n-\frac{1}{2}},I_h^*v_h)=i(u_t^{n-\frac{1}{2}}-\partial_t u_H^n,I_h^*v_h)-(V(u^{n-\frac{1}{2}}-u_H^{n-\frac{1}{2}}),I_h^*v_h). \tag{4.4}$$

Denote

$$u^{n-\frac{1}{2}}-u_h^{n-\frac{1}{2}}=(u^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}})-(u_h^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}})=:\xi^{n-\frac{1}{2}}-\eta^{n-\frac{1}{2}}.$$

Thanks to

$$a_h(\xi^{n-\frac{1}{2}},I_h^*v_h)=\overline{a_h}(\xi^{n-\frac{1}{2}},I_h^*v_h)+(V\xi^{n-\frac{1}{2}},I_h^*v_h)=0,$$

then it holds

$$-\overline{a_h}(\eta^{n-\frac{1}{2}},I_h^*v_h)=i(u_t^{n-\frac{1}{2}}-\partial_t u_H^n,I_h^*v_h)+(V(u_H^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}}),I_h^*v_h). \tag{4.5}$$

Choosing $v_h=-\eta^{n-\frac{1}{2}}$ in (4.5), we obtain

$$\begin{aligned}\overline{a_h}(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})=&i(\partial_t u_H^n-\partial_t P_hu^n,I_h^*\eta^{n-\frac{1}{2}})-i(u_t^{n-\frac{1}{2}}-\partial_t u^n,I_h^*\eta^{n-\frac{1}{2}})\\&-i(\partial_t\xi^n,I_h^*\eta^{n-\frac{1}{2}})-(V(u_H^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}}),I_h^*\eta^{n-\frac{1}{2}}).\end{aligned} \tag{4.6}$$

Taking the real part of (4.6) and using the Hölder inequality, we have

$$\begin{aligned}Re\{\overline{a_h}(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})\}=&-Im\{(\partial_t(u_H^n-P_hu^n),I_h^*\eta^{n-\frac{1}{2}})\}+Im\{(u_t^{n-\frac{1}{2}}-\partial_t u^n,I_h^*\eta^{n-\frac{1}{2}})\}\\&+Im\{(\partial_t\xi^n,I_h^*\eta^{n-\frac{1}{2}})\}-Re\{(V(u_H^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}}),I_h^*\eta^{n-\frac{1}{2}})\}\\\le&|(\partial_t(u_H^n-P_hu^n),I_h^*\eta^{n-\frac{1}{2}})|+|(u_t^{n-\frac{1}{2}}-\partial_t u^n,I_h^*\eta^{n-\frac{1}{2}})|\\&+|(\partial_t\xi^n,I_h^*\eta^{n-\frac{1}{2}})|+|(V(u_H^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}}),I_h^*\eta^{n-\frac{1}{2}})|\\\le&C(\|\partial_t(u_H^n-P_hu^n)\|+\|u_t^{n-\frac{1}{2}}-\partial_t u^n\|)\|\eta^{n-\frac{1}{2}}\|\\&+(\|\partial_t\xi^n\|+\|u_H^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}}\|)\|\eta^{n-\frac{1}{2}}\|.\end{aligned} \tag{4.7}$$

For the right hand side term of (4.7), by the same tricks as used in Theorem 3.1, one gets

$$Re\{\overline{a_h}(\eta^{n-\frac{1}{2}},I_h^*\eta^{n-\frac{1}{2}})\}\ge M_1\|\eta^{n-\frac{1}{2}}\|_1^2. \tag{4.8}$$

Thanks to (3.16b), (3.18b), (2.13b) and Theorem 3.1, we obtain

$$\|u_t^{n-\frac{1}{2}}-\partial_t u^n\|\le C(\Delta t)^{\frac{3}{2}}\Big(\|u_{ttt}(\cdot,t)\|^2_{L^\infty(t^{n-1},t^n,L^2)}\int_{t^{n-1}}^{t^n}1dt\Big)^{\frac{1}{2}}\le C(\Delta t)^2, \tag{4.9a}$$

$$\|\partial_t\xi^n\|\le C(\Delta t)^{-\frac{1}{2}}h^2\Big(\|u_t\|^2_{L^\infty(t^{n-1},t^n,W^{3,q})}\int_{t^{n-1}}^{t^n}1dt\Big)^{\frac{1}{2}}\le Ch^2, \tag{4.9b}$$

$$\|u_H^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}}\|\le\|u_H^{n-\frac{1}{2}}-u^{n-\frac{1}{2}}\|+\|u^{n-\frac{1}{2}}-P_hu^{n-\frac{1}{2}}\|\le C(H^2+\Delta t^2+h^2). \tag{4.9c}$$

With the help of Lemma 3.5 and (4.9b), it holds that

$$\begin{aligned}\|\partial_t(u_H^n - P_h u^n)\| \le & \|\partial_t(u_H^n - P_H u^n)\| + \|\partial_t(u^n - P_H u^n)\| + \|\partial_t(u^n - P_h u^n)\| \\ \le & C(H^2 + \Delta t^2 + h^2). \end{aligned} \tag{4.10}$$

Since $\|\eta^{n-\frac{1}{2}}\| \le \|\eta^{n-\frac{1}{2}}\|_1$ and together with (4.8)-(4.10), we obtain

$$\|\eta^{n-\frac{1}{2}}\|_1 \le C(\Delta t^2 + H^2 + h^2). \tag{4.11}$$

Noting that $\eta^{\frac{1}{2}} = \frac{\eta^1 - \eta^0}{2} = \frac{\eta^1}{2}$ holds as $n=1$, as a consequence, one finds

$$\|\eta^1\|_1 = 2\|\eta^{\frac{1}{2}}\|_1 \le C(H^2 + h^2 + \Delta t^2), \tag{4.12}$$

Furthermore, we have

$$\|\eta^2\|_1 = \left\|2\frac{\eta^2 + \eta^1}{2} - \eta^1\right\|_1 \le \|2\eta^{\frac{3}{2}}\|_1 + \|\eta^1\|_1 \le C(H^2 + h^2 + \Delta t^2), \tag{4.13a}$$

$$\|\eta^3\|_1 = \left\|2\frac{\eta^3 + \eta^2}{2} - \eta^2\right\|_1 \le \|2\eta^{\frac{5}{2}}\|_1 + \|\eta^2\|_1 \le C(H^2 + h^2 + \Delta t^2). \tag{4.13b}$$

Hence, by triangular inequality, it yields

$$\|\eta^n\|_1 = \left\|2\frac{\eta^n + \eta^{n-1}}{2} - \eta^{n-1}\right\|_1 \le \|2\eta^{n-\frac{1}{2}}\|_1 + \|\eta^{n-1}\|_1 \le C(H^2 + h^2 + \Delta t^2). \tag{4.14}$$

Together with (2.13a), we complete the proof of (4.3). □

5 Numerical experiments

In this section, we present some numerical examples to illustrate the theoretical results presented in the previous sections. We consider the following the Schrödinger problem:

Example 5.1. In model (1.1), taking $V(x)$=1, Ω=$[-1,1]^2$, and the function $f(x,t)$ is chosen corresponding to the exact solution

$$u(x,t) = 2t^4(1-x^2)(1-y^2) + ie^t \sin(\pi(1+x))\sin(\pi(1+y)).$$

We use the standard finite volume element method to solve the coupled partial differential equation and obtain the data at time $T=1$ is presented in Table 1. It can be seen that the orders of L^2 and H^1 errors are consistent with the established theoretical findings. The computational results of the two grid finite volume element decoupling algorithm at time $T=1$ are presented in Table 2, from which we see that the order of H^1 error is also consistent with Theorem 4.1. Compared with Tables 1 and 2, it is obviously that the two-grid decoupling algorithm can keep the optimal convergence accuracy as the coarse grid is much coarser than the fine grid.

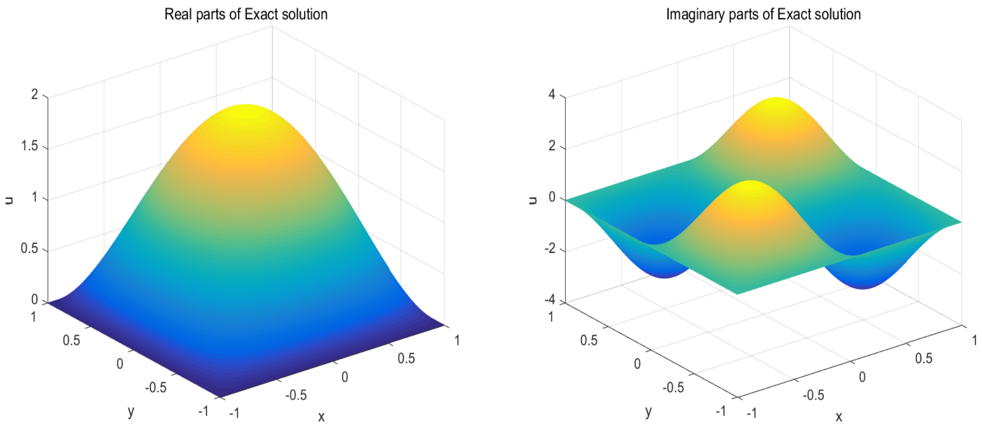

Figure 2: Exact solution of Example 5.1.

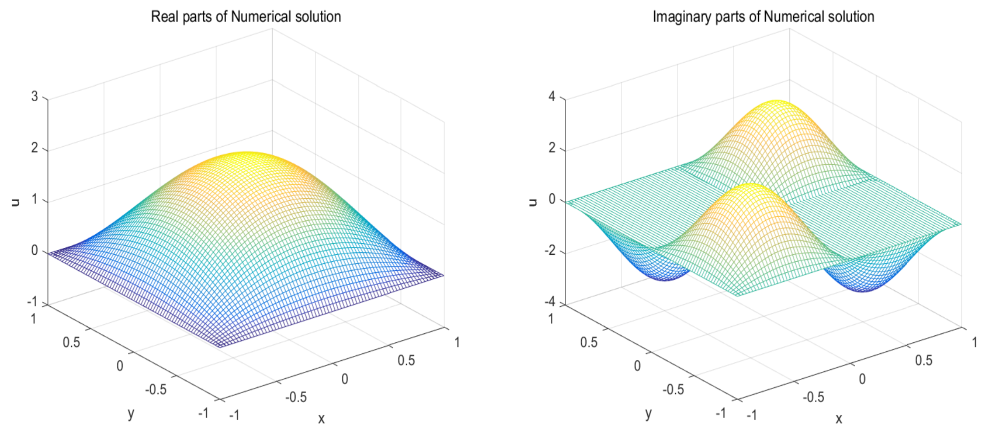

Figure 3: Two-grid numerical solution of Example 5.1.

Example 5.2. In this test, we choose $V(x)=1$, $\Omega=[-1,1]^2$ in model (1.1) and the function $f(x,t)$ is determined by the exact solution

$$u(x,t)=(1+i)e^t(1+x)(1+y)\sin(1-x)\sin(1-y).$$

The computational results of standard FVE method and two grid FVE decoupling algorithm for the Schrödinger equation are presented in Tables 3 and 4 with time $T=1$. From these data, we can see that the convergence orders of numerical solutions in both L^2 and H^1-norms are consistent with the established convergence analysis well.

Table 1: L^2 and H^1 errors of standard finite volume element method (2.10) at $T=1$, $\Delta t=h$.

h	$L^2 error$	Rate	$H^1 error$	Rate	CPU time
1/2	1.2404		8.6268		0.91s
1/4	0.3565	1.7988	4.6985	0.8766	4.52s
1/9	0.0724	1.9658	2.1391	0.9703	22.60s
1/16	0.0231	1.9855	1.2081	0.9930	83.88s
1/25	0.0093	2.0386	0.7741	0.9988	244.67s
1/36	0.0045	1.9908	0.5378	0.9988	651.95s
1/64	0.0014	2.0293	0.3026	0.9995	3024.5s

Table 2: H^1 error of two grid finite volume element algorithm (4.1)-(4.2) at $T=1$, $\Delta t=h$.

h	H	$H^1 error$	Rate	CPU time
1/4	1/2	4.9251		2.72s
1/9	1/3	2.2439	0.9694	18.26s
1/16	1/4	1.2718	0.9868	86.33s
1/25	1/5	0.8194	0.9851	305.97s
1/36	1/6	0.5703	0.9939	955.78s
1/64	1/8	0.3230	0.9881	5169.1s

Table 3: L^2 and H^1 errors of standard finite volume element method (2.10) at $T=1$, $\Delta t=h$.

h	$L^2 error$	Rate	$H^1 error$	Rate	CPU time
h=1/2	0.3662		2.6857		1.00s
h=1/4	0.0949	1.9482	1.3605	0.9812	3.91s
h=1/9	0.0188	1.9964	0.6068	0.9957	24.17s
h=1/16	0.0059	2.0142	0.3416	0.9986	107.58s
h=1/36	0.0012	1.9009	0.1518	1.0001	737.45s
h=1/64	3.6754e-04	2.0565	0.0854	0.9997	2987.4s

Table 4: H^1 error of two grid finite volume element algorithm (4.1)-(4.2) at $T=1$, $\Delta t=h$.

h	H	$H^1 error$	Rate	CPU time
1/4	1/2	1.4307		2.65s
1/9	1/3	0.7233	0.8411	17.96s
1/16	1/4	0.3603	1.2112	84.79s
1/36	1/6	0.1609	0.9941	891.17s
1/64	1/8	0.0909	0.9925	4841.9s

6 Conclusions

In this paper, combining with the finite volume element method, we propose a time second-order two-grid Crank-Nicolson decoupling algorithm for the time-dependent

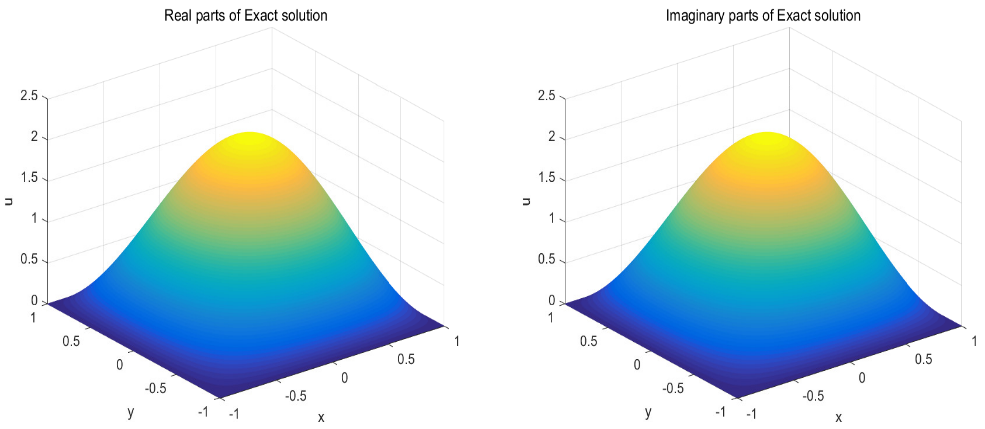

Figure 4: Exact solution of Example 5.2.

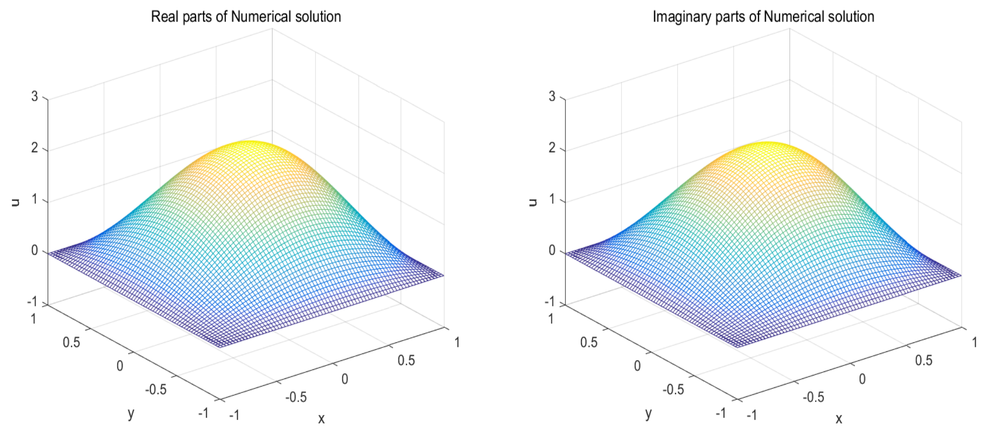

Figure 5: Two-grid numerical solution of Example 5.2.

Schrödinger equation and present the corresponding convergence analysis and numerical examples. Our two-grid decoupling technique includes solving the original coupling system in the coarse grid space and the decoupling system with two independent Poisson problems in the fine grid space, and the two-grid algorithm can keep the original accuracy. Obviously, the decoupling technique can be easily extended to other coupled differential equation systems. Numerical examples are provided to verify the established theoretical findings very well. In our future research, we will construct and prove the higher accuracy two-grid algorithm for the the nonlinear Schrödinger equation.

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