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Stability and Convergence of the Canonical Euler Splitting Method for Nonlinear Composite Stiff Functional Differential-Algebraic Equations

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Abstract. A novel canonical Euler splitting method is proposed for nonlinear composite stiff functional differential-algebraic equations, the stability and convergence of the method is evidenced, theoretical results are further confirmed by some numerical experiments. Especially, the numerical method and its theories can be applied to special cases, such as delay differential-algebraic equations and integral differential-algebraic equations.

AMS subject classifications: 65L03, 65L04, 65L80

Key words: Canonical Euler splitting method, nonlinear composite stiff functional differential-algebraic equations, stability, convergence.

1 Introduction

Functional differential-algebraic equations (FDAEs) have been widely used in science problems in mechanics, control science, biology and other fields [1, 2]. The reference [3] has indicated that differential-algebraic equations (DAEs) are neither differential equations nor algebraic equations, they include the process of differentiation and the limitations of algebraic conditions, which change the behavior of the solution and lead to some difficulties of numerically solving FDAEs.

In recent years, there has been extensively studied on the numerical stability and convergence of delay differential-algebraic equations [4–15]. Further, we can refer to [16–19] for details on the numerical stability and convergence of integral differential-algebraic equations. However, most of the above studies are focused on theoretical and numerical analysis of linear or non-stiff problems, we can refer to [20–22] for the stability of the more

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general nonlinear stiff FDAEs and theirs numerical methods. For these stiff problems, since the right-side functions of the equations exhibit different stiffness at different stages of development, when we solve the stiff problems on the slow-varying interval, although the fast-varying interval has been attenuated to insignificance, but the fast-changing interference will still affect the stability and accuracy of the numerical solution. If it is solved numerically in the entire interval, it will increase the amount of calculation and fail to achieve high accuracy. As a consequence, some scholars have proposed some splitting methods, such as operator splitting method, symmetric weighted sequential splitting method, high-order splitting method, and Strang-Marchuk splitting method, but currently these splitting methods are mainly used to solve differential equations without algebraic constraints, such as the splitting methods for stiff differential equations [23–25], Schrödinger equations [26–33], nonlinear convection-diffusion-reaction equations [34,35], nonlinear delay differential equations and integral differential equations [36–40]. Nevertheless, the splitting methods and their theories mentioned in the above references are aimed at the nonlinear or stiff problems with some special structures, and still cannot be applied to the general nonlinear composite stiff functional differential equations [41].

The canonical Euler splitting method (CES) is proposed for solving nonlinear composite stiff evolution equations [41]. On this basis, in order to effectively overcome the difficulties caused by algebraic constraints, we further propose a new CES method to solve the nonlinear composite stiff FDAEs, and prove the stability and convergence of the CES method, and the numerical experiments verify the theoretical results of the method.

2 Canonical Euler splitting method for solving nonlinear composite stiff FDAEs

Consider the nonlinear composite stiff FDAEs

$$\begin{cases} y'(t)=f(t,y(t),y(\cdot),z(t),z(\cdot)), & t\in(0,T], \\ z(t)=g(y(t),y(\cdot),z(\cdot)), & t\in(0,T], \\ y(t)=\varphi_1(t), \quad z(t)=\varphi_2(t), & t\in[-\tau,0], \end{cases} \tag{2.1}$$

where $T>0$, $\tau\in[0,+\infty]$ are constants, and initial functions φ_1, φ_2 are given, $\mathbf{R}^{m_i}$ represents the m_i dimensional Euclidean space, $i=1,2$, the inner product is denoted as $\langle\cdot,\cdot\rangle$, and the corresponding norm is denoted as $\|\cdot\|$, the mappings

$$\begin{aligned} f:&\ [0,T]\times\mathbf{R}^{m_1}\times\mathbf{C}_{m_1}[-\tau,T]\times\mathbf{R}^{m_2}\times\mathbf{C}_{m_2}[-\tau,T] \ \rightarrow\ \mathbf{R}^{m_1}, \\ g:&\ \mathbf{R}^{m_1}\times\mathbf{C}_{m_1}[-\tau,T]\times\mathbf{C}_{m_2}[-\tau,T] \ \rightarrow\ \mathbf{R}^{m_2}, \end{aligned}$$

are given, and the mapping g satisfies the consistency condition at the point $t=0$: $z(0)=g(y(0),\varphi_1(0),\varphi_2(0))$, f can be divided into two sub-mappings

$$f(t,u,\psi(\cdot),v,\chi(\cdot))=f_1(t,u,\psi(\cdot),v,\chi(\cdot))+f_2(t,u,\psi(\cdot),v,\chi(\cdot)),$$

$\forall t\in(0,T]$, $u\in\mathbf{R}^{m_1}$, $v\in\mathbf{R}^{m_2}$, $\psi\in\mathbf{C}_{m_1}[-\tau,T]$, $\chi\in\mathbf{C}_{m_2}[-\tau,T]$, the mappings f_1, f_2 and g satisfy the conditions

$$\begin{cases} \|f_1(t,u_1,\psi_1(\cdot),v_1,\chi_1(\cdot))-f_1(t,u_2,\psi_2(\cdot),v_2,\chi_2(\cdot))\| \\ \le\alpha_1\|u_1-u_2\|+\beta_1\max\limits_{-\tau\le\xi\le t}\|\psi_1(\xi)-\psi_2(\xi)\|+\gamma_1\|v_1-v_2\| \\ \qquad +\sigma_1\max\limits_{-\tau\le\xi\le t}\|\chi_1(\xi)-\chi_2(\xi)\|, \\ \langle f_2(t,u_1,\psi(\cdot),v,\chi(\cdot))-f_2(t,u_2,\psi(\cdot),v,\chi(\cdot)),u_1-u_2\rangle\le\alpha_2\|u_1-u_2\|^2, \\ \|f_2(t,u,\psi_1(\cdot),v_1,\chi_1(\cdot))-f_2(t,u,\psi_2(\cdot),v_2,\chi_2(\cdot))\| \\ \le\beta_2\max\limits_{-\tau\le\xi\le t}\|\psi_1(\xi)-\psi_2(\xi)\|+\gamma_2\|v_1-v_2\|+\sigma_2\max\limits_{-\tau\le\xi\le t}\|\chi_1(\xi)-\chi_2(\xi)\|, \end{cases} \tag{2.2}$$

and

$$\begin{aligned} &\|g(u_1,\psi_1(\cdot),\chi_1(\cdot))-g(u_2,\psi_2(\cdot),\chi_2(\cdot))\| \\ \le& L_1\|u_1-u_2\|+L_2\max_{-\tau\le\xi\le t}\|\psi_1(\xi)-\psi_2(\xi)\|+L_3\max_{-\tau\le\xi\le t}\|\chi_1(\xi)-\chi_2(\xi)\|, \end{aligned} \tag{2.3}$$

where $u_1,u_2\in\mathbf{R}^{m_1}$, $v_1,v_2\in\mathbf{R}^{m_2}$, $\psi_1,\psi_2\in\mathbf{C}_{m_1}[-\tau,T]$, $\chi_1,\chi_2\in\mathbf{C}_{m_2}[-\tau,T]$, α_1, α_2, β_1, β_2, γ_1, γ_2, σ_1, σ_2, L_1, L_2, L_3 are all real constants and $L_3<1$, the constants $\alpha_1,\bar{\alpha}_2=\max\{\alpha_2,0\}$, β_1, β_2, γ_1, γ_2, σ_1, σ_2, L_1, L_2, L_3 and T are assumed to be of appropriate size.

Further, we assume that problem (2.1) has unique true solutions $y(t)$ and $z(t)$, and denote the problem class $\mathcal{S}(\alpha_1,\beta_1,\gamma_1,\sigma_1,\alpha_2,\beta_2,\gamma_2,\sigma_2,L_1,L_2,L_3)$ consisting of all the problems (2.1) with (2.2) and (2.3).

Secondly, if the mappings f and g are independent of the past values of the true solutions $y(t)$ and $z(t)$, it can be seen that problem (2.1) degenerates into nonlinear composite stiff DAEs

$$\begin{cases} y'(t)=f(t,y(t),z(t)):=f_1(t,y(t),z(t))+f_2(t,y(t),z(t)), & t\in(0,T], \\ z(t)=g(y(t)), & t\in(0,T], \\ y(0)=y_0, \quad z(0)=z_0, \end{cases} \tag{2.4}$$

where $y_0\in\mathbf{R}^{m_1}$, $z_0\in\mathbf{R}^{m_2}$, the mappings $f:[0,T]\times\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}\to\mathbf{R}^{m_1}$ and $g:\mathbf{R}^{m_1}\to\mathbf{R}^{m_2}$, and the sub-mappings f_1, f_2 and the mapping g satisfy the conditions

$$\begin{cases} \|f_1(t,u_1,v_1)-f_1(t,u_2,v_2)\|\le\alpha_1\|u_1-u_2\|+\gamma_1\|v_1-v_2\|, \\ \langle f_2(t,u_1,v)-f_2(t,u_2,v),u_1-u_2\rangle\le\alpha_2\|u_1-u_2\|^2, \\ \|f_2(t,u,v_1)-f_2(t,u,v_2)\|\le\gamma_2\|v_1-v_2\|, \end{cases} \tag{2.5}$$

and

$$\|g(y_1)-g(y_2)\|\le L_1\|y_1-y_2\|, \tag{2.6}$$

where $u_1,u_2\in\mathbf{R}^{m_1}$, $v_1,v_2\in\mathbf{R}^{m_2}$, we assume that the constants $\alpha_1,\bar{\alpha}_2:=\max\{\alpha_2,0\}$, γ_1, γ_2, and L_1 all have only appropriate size. We use the symbol $\mathcal{S}(\alpha_1,\gamma_1,\alpha_2,\gamma_2,L_1)$ to represent the problem class that contains all problems (2.4) with (2.5) and (2.6), and it can be seen as a special case of the problem class $\mathcal{S}(\alpha_1,\beta_1,\gamma_1,\sigma_1,\alpha_2,\beta_2,\gamma_2,\sigma_2,L_1,L_2,L_3)$.

For the nonlinear composite stiff FDAEs (2.1), we construct the following numerical method

$$\begin{cases} y^h(t)=\Pi^h(t;\psi,y_1,y_2,\cdots,y_n), & t\in[-\tau,t_{n+1}],\\ z^h(t)=\hat{\Pi}^h(t;\chi,z_1,z_2,\cdots,z_n), & t\in[-\tau,t_{n+1}],\\ y_{n+1}=y_n+h_nf_1(t_{n+1},y_n,y^h(\cdot),z_n,z^h(\cdot))+h_nf_2(t_{n+1},y_{n+1},y^h(\cdot),z_{n+1},z^h(\cdot)),\\ z_{n+1}=g(y_{n+1},y^h(\cdot),z^h(\cdot)), \end{cases} \tag{2.7}$$

where $n=0,1,\cdots,N-1$, the grid $\Delta_h:=\{t_i:i=0,1,\cdots,N\}$, t_i $(i=0,1,\cdots,N)$ is the grid point satisfying $0=t_0<t_1<\cdots<t_N=T$, variable step size $h_i=t_{i+1}-t_i$ and $h=\max\limits_{0\le i\le N-1}h_i$. ψ and χ are approximations of the initial functions φ_1 and φ_2, $y_i\in\mathbf{R}^{m_1}$, $z_i\in\mathbf{R}^{m_2}$ are approximations of $y(t_i)$, $z(t_i)$, $i=0,1,\cdots,N$, and $y_0=\varphi_1(0)$, $z_0=\varphi_2(0)$, $y^h(t)$ and $z^h(t)$ are approximations of the true solutions $y(t)$ and $z(t)$, the piecewise constant or piecewise linear interpolation operator Π^h is constructed:

$$\Pi^h:\mathbf{C}_{m_1}[-\tau,0]\times\mathbf{R}^{m_1n} \;\rightarrow\; \mathbf{C}_{m_1}[-\tau,t_{n+1}]$$

and $\hat{\Pi}^h$ can be defined similarly.

In addition, we notice that for each time integration step $(t_n,\psi,y_1,y_2,\cdots,y_n)\rightarrow(t_{n+1},\psi,y_1,y_2,\cdots,y_{n+1})$ and $(t_n,\chi,z_1,z_2,\cdots,z_n)\rightarrow(t_{n+1},\chi,z_1,z_2,\cdots,z_{n+1})$, the method (2.7) for solving problem (2.1) can be done in two steps as follows:

First, we let

$$\begin{cases} \text{(a)}\;\; y^h(t)=\Pi^h(t;\psi,y_1,y_2,\cdots,y_n), & t\in[-\tau,t_{n+1}],\\ \text{(b)}\;\; z^h(t)=\hat{\Pi}^h(t;\chi,z_1,z_2,\cdots,z_n), & t\in[-\tau,t_{n+1}],\\ \text{(c)}\;\; \bar{y}_{n+1}=y_n+h_nf_1(t_{n+1},y_n,y^h(\cdot),z_n,z^h(\cdot)). \end{cases} \tag{2.8}$$

Here $\bar{y}_{n+1}$ can be regarded as the result obtained by explicit Euler method for solving the non-stiff sub-problem in nonlinear FDAEs

$$\begin{cases} \bar{y}'(t)=f_1(t,\bar{y}(t),y^h(\cdot),\bar{z}(t),z^h(\cdot)), & t\in(t_n,t_{n+1}),\\ \bar{y}(t_n)=y_n,\;\; \bar{z}(t_n)=z_n, \end{cases} \tag{2.9}$$

where $y^h(t)$ and $z^h(t)$ are defined by the formulas (2.8a) and (2.8b) respectively, the method (2.8) is called the generalized explicit Euler method. For the advantages of this method, please refer to the literature [42].

Second, we make

$$\begin{cases} y_{n+1}=\bar{y}_{n+1}+h_nf_2(t_{n+1},y_{n+1},y^h(\cdot),z_{n+1},z^h(\cdot)),\\ z_{n+1}=g(y_{n+1},y^h(\cdot),z^h(\cdot)). \end{cases} \tag{2.10}$$

Here y_{n+1} and z_{n+1} can be regarded as the results obtained by the implicit Euler method for solving the stiff sub-problem in nonlinear FDAEs

$$\begin{cases} \hat{y}'(t) = f_2(t,\hat{y}(t),y^h(\cdot),\hat{z}(t),z^h(\cdot)), \quad t\in(t_n,t_{n+1}), \\ \hat{z}(t) = g(\hat{y}(t),y^h(\cdot),z^h(\cdot)), \\ \hat{y}(t_n) = \bar{y}_{n+1}, \end{cases} \tag{2.11}$$

similarly, the method (2.10) is called the generalized implicit Euler method.

It can be seen that the method (2.7) for solving the original problem (2.1) can be transformed into the methods (2.8) and (2.10) for solving the sub-problems (2.9) and (2.11) in turn. We call the method (2.7) (that is, (2.8) and (2.10)) as the canonical Euler splitting method (CES).

In particular, for the nonlinear composite stiff DAEs (2.4) $\in \mathcal{S}(\alpha_1,\gamma_1,\alpha_2,\gamma_2,L_1)$, it can be seen that this is a special case, and the CES method (2.7) degenerates to

$$\begin{cases} y_{n+1} = y_n + h_n f_1(t_{n+1},y_n,z_n) + h_n f_2(t_{n+1},y_{n+1},z_{n+1}), \\ z_{n+1} = g(y_{n+1}), \end{cases} \tag{2.12}$$

for each time integration step from t_n to t_{n+1}, the method (2.12) can be divided into two steps for solving Eq. (2.4), that is, we first use the generalized explicit Euler method

$$\bar{y}_{n+1} = y_n + h_n f_1(t_{n+1},y_n,z_n) \tag{2.13}$$

to solve the non-stiff sub-problem of the nonlinear DAEs

$$\begin{cases} \bar{y}'(t) = f_1(t,\bar{y}(t),\bar{z}(t)), \quad t\in(t_n,t_{n+1}), \\ \bar{y}(t_n) = y_n, \quad \bar{z}(t_n) = z_n, \end{cases} \tag{2.14}$$

then employ the generalized implicit Euler method

$$\begin{cases} \hat{y}_{n+1} = \bar{y}_{n+1} + h_n f_2(t_{n+1},\hat{y}_{n+1},\hat{z}_{n+1}), \\ \hat{z}_{n+1} = g(\hat{y}_{n+1}), \end{cases} \tag{2.15}$$

to solve the stiff sub-problem of nonlinear DAEs

$$\begin{cases} \hat{y}'(t) = f_2(t,\hat{y}(t),\hat{z}(t)), \quad t\in(t_n,t_{n+1}), \\ \hat{z}(t) = g(\hat{y}(t)), \\ \hat{y}(t_n) = \bar{y}_{n+1}, \end{cases} \tag{2.16}$$

and let

$$y_{n+1} = \hat{y}_{n+1}, \quad z_{n+1} = \hat{z}_{n+1}.$$

3 Stability analysis of CES method for solving nonlinear composite stiff FDAEs

We first perform canonical analysis on the interpolation operators Π^h and $\hat{\Pi}^h$ in the CES method (2.7), and then establish the stability theory of CES method for nonlinear composite stiff FDAEs (2.1).

Lemma 3.1 ([41–43])**.** *The interpolation operators Π^h and $\hat{\Pi}^h$ in* (2.7) *satisfy*

$$\begin{cases} \max\limits_{-\tau\le t\le t_{n+1}} \left\|\Pi^h(t;\psi,y_1,\cdots,y_n)-\Pi^h(t;\tilde{\psi},\tilde{y}_1,\cdots,\tilde{y}_n)\right\| \\ \quad \le \max\left\{\max\limits_{1\le i\le n}\|y_i-\tilde{y}_i\|, \max\limits_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|\right\}, \\ \max\limits_{-\tau\le t\le t_{n+1}} \left\|\hat{\Pi}^h(t;\chi,z_1,\cdots,z_n)-\hat{\Pi}^h(t;\tilde{\chi},\tilde{z}_1,\cdots,\tilde{z}_n)\right\| \\ \quad \le \max\left\{\max\limits_{1\le i\le n}\|z_i-\tilde{z}_i\|, \max\limits_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\right\}, \end{cases} \tag{3.1}$$

$\forall\psi,\tilde{\psi}\in\mathbf{C}_{m_1}[-\tau,0]$, $\chi,\tilde{\chi}\in\mathbf{C}_{m_2}[-\tau,0]$, $y_i,\tilde{y}_i\in\mathbf{R}^{m_1}$, $z_i,\tilde{z}_i\in\mathbf{R}^{m_2}$, $i=1,2,\cdots,n$.

Proof. When Π^h is a piecewise constant interpolation operator, there is

$$\Pi^h(t;\psi,y_1,\cdots,y_n)=\begin{cases} \psi(t) & \text{for } t\in[-\tau,0], \\ y_i & \text{for } t\in(t_{i-1},t_i], \quad i=1,2,\cdots,n, \end{cases}$$

we obviously have

$$\max_{-\tau\le t\le t_{n+1}} \left\|\Pi^h(t;\psi,y_1,\cdots,y_n)-\Pi^h(t;\tilde{\psi},\tilde{y}_1,\cdots,\tilde{y}_n)\right\| \le\max\left\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|, \max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|\right\}.$$

When Π^h is a piecewise linear interpolation operator, there is

$$\Pi^h(t;\psi,y_1,\cdots,y_n)=\begin{cases} \psi(t) & \text{for } t\in[-\tau,0], \\ \dfrac{(t_i-t)y_{i-1}+(t-t_{i-1})y_i}{t_i-t_{i-1}} & \text{for } t\in(t_{i-1},t_i], \quad i=1,2,\cdots,n, \end{cases}$$

it is easily proved that

$$\left\|\frac{(t_i-t)y_{i-1}+(t-t_{i-1})y_i}{t_i-t_{i-1}}-\frac{(t_i-t)\tilde{y}_{i-1}+(t-t_{i-1})\tilde{y}_i}{t_i-t_{i-1}}\right\| \le\max\{\|y_{i-1}-\tilde{y}_{i-1}\|,\|y_i-\tilde{y}_i\|\},$$

where $t\in(t_{i-1},t_i]$, $i=1,2,\cdots,n$, this leads to

$$\begin{aligned}&\max_{-\tau\le t\le t_{n+1}}\left\|\Pi^h(t;\psi,y_1,\cdots,y_n)-\Pi^h(t;\tilde{\psi},\tilde{y}_1,\cdots,\tilde{y}_n)\right\|\\ \le&\max\Big\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|,\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|\Big\}.\end{aligned}$$

$\hat{\Pi}^h$ can be similarly proved. □

Theorem 3.1. *Suppose the CES method* (2.7) *is used to solve the nonlinear composite stiff problem* (2.1) $\in\mathcal{S}(\alpha_1,\beta_1,\gamma_1,\sigma_1,\alpha_2,\beta_2,\gamma_2,\sigma_2,L_1,L_2,L_3)$ *on a given grid* Δ_h*, for the starting functions* $\{\psi(t),\chi(t)\}$ *of method* (2.7) *and the starting functions* $\{\tilde{\psi}(t),\tilde{\chi}(t)\}$ *of the corresponding method* (3.2)*, let* $\{y_n,z_n\}$*,* $\{\tilde{y}_n,\tilde{z}_n\}$ *denote the approximate sequences generated by the CES method* (2.7) *for solving the nonlinear composite stiff problem of the form* (2.1) *under these two sets of starting functions, then for any four parallel integration steps*

$$\begin{aligned}(t_n,\psi,y_1,y_2,\cdots,y_n)&\rightarrow(t_{n+1},\psi,y_1,y_2,\cdots,y_{n+1}),\\(t_n,\chi,z_1,z_2,\cdots,z_n)&\rightarrow(t_{n+1},\chi,z_1,z_2,\cdots,z_{n+1}),\\(t_n,\tilde{\psi},\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_n)&\rightarrow(t_{n+1},\tilde{\psi},\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_{n+1}),\\(t_n,\tilde{\chi},\tilde{z}_1,\tilde{z}_2,\cdots,\tilde{z}_n)&\rightarrow(t_{n+1},\tilde{\chi},\tilde{z}_1,\tilde{z}_2,\cdots,\tilde{z}_{n+1}),\end{aligned}$$

where the first and second integration steps are determined by (2.7)*, and the third and fourth integration steps are determined by*

$$\begin{cases}\tilde{y}^h(t)=\Pi^h(t;\tilde{\psi},\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_n), & t\in[-\tau,t_{n+1}],\\ \tilde{z}^h(t)=\hat{\Pi}^h(t;\tilde{\chi},\tilde{z}_1,\tilde{z}_2,\cdots,\tilde{z}_n), & t\in[-\tau,t_{n+1}],\\ \tilde{y}_{n+1}=\tilde{y}_n+h_nf_1(t_{n+1},\tilde{y}_n,\tilde{y}^h(\cdot),\tilde{z}_n,\tilde{z}^h(\cdot))+h_nf_2(t_{n+1},\tilde{y}_{n+1},\tilde{y}^h(\cdot),\tilde{z}_{n+1},\tilde{z}^h(\cdot)),\\ \tilde{z}_{n+1}=g(\tilde{y}_{n+1},\tilde{y}^h(\cdot),\tilde{z}^h(\cdot)),\end{cases}\tag{3.2}$$

then, we have the stability inequalities

$$\|y_n-\tilde{y}_n\|\le\exp(c_1t_n)\max\Big\{\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},\tag{3.3a}$$

$$\|z_n-\tilde{z}_n\|\le c_2\exp(c_1t_n)\max\Big\{\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},\tag{3.3b}$$

where $\max\limits_{0\le i\le n-1}h_i\le\bar{h}$*,* $n=1,2,\cdots,N$*, the constants*

$$c_1=\begin{cases}2(\bar{c}_1+\alpha_2+\gamma_2L_1), & \alpha_2+\gamma_2L_1>0,\\ \max\{\bar{c}_1+\alpha_2+\gamma_2L_1,0\}, & \alpha_2+\gamma_2L_1\le 0,\end{cases}\qquad c_2=\max\Big\{\frac{L_1+L_2}{1-L_3},1\Big\},\tag{3.4}$$

and

$$\bar{c}_1=\alpha_1+\gamma_1L_1+\beta_1+\beta_2+\gamma_1L_2+\gamma_2L_2+(\sigma_1+\sigma_2+\gamma_1L_3+\gamma_2L_3)c_2,$$

and

$$\bar{h}=\begin{cases}\dfrac{1}{2(\alpha_2+\gamma_2 L_1)}, & \alpha_2+\gamma_2 L_1>0,\\ h^*, & \alpha_2+\gamma_2 L_1\le 0,\end{cases} \tag{3.5}$$

here, $h^*>0$ is any given constant, it can be seen that the constants c_1, c_2, $\bar{c}_1$, $\bar{h}^{-1}$ are of appropriate size.

Proof. From (2.7) and (3.2) we have

$$\begin{aligned} y_{n+1}-\tilde{y}_{n+1}=&y_n-\tilde{y}_n+h_n[f_1(t_{n+1},y_n,y^h(\cdot),z_n,z^h(\cdot))-f_1(t_{n+1},\tilde{y}_n,\tilde{y}^h(\cdot),\tilde{z}_n,\tilde{z}^h(\cdot))]\\ &+h_n[f_2(t_{n+1},y_{n+1},y^h(\cdot),z_{n+1},z^h(\cdot))-f_2(t_{n+1},\tilde{y}_{n+1},y^h(\cdot),z_{n+1},z^h(\cdot))\\ &+f_2(t_{n+1},\tilde{y}_{n+1},y^h(\cdot),z_{n+1},z^h(\cdot))-f_2(t_{n+1},\tilde{y}_{n+1},\tilde{y}^h(\cdot),\tilde{z}_{n+1},\tilde{z}^h(\cdot))], \end{aligned}$$

therefore, according to the Lipschitz conditions (2.2)

$$\begin{aligned} &\|y_{n+1}-\tilde{y}_{n+1}\|^2\\ =&\langle y_n-\tilde{y}_n,y_{n+1}-\tilde{y}_{n+1}\rangle\\ &+h_n\langle f_1(t_{n+1},y_n,y^h(\cdot),z_n,z^h(\cdot))-f_1(t_{n+1},\tilde{y}_n,\tilde{y}^h(\cdot),\tilde{z}_n,\tilde{z}^h(\cdot)),y_{n+1}-\tilde{y}_{n+1}\rangle\\ &+h_n\langle f_2(t_{n+1},y_{n+1},y^h(\cdot),z_{n+1},z^h(\cdot))-f_2(t_{n+1},\tilde{y}_{n+1},y^h(\cdot),z_{n+1},z^h(\cdot)),y_{n+1}-\tilde{y}_{n+1}\rangle\\ &+h_n\langle f_2(t_{n+1},\tilde{y}_{n+1},y^h(\cdot),z_{n+1},z^h(\cdot))-f_2(t_{n+1},\tilde{y}_{n+1},\tilde{y}^h(\cdot),\tilde{z}_{n+1},\tilde{z}^h(\cdot)),y_{n+1}-\tilde{y}_{n+1}\rangle\\ \le&\|y_n-\tilde{y}_n\|\|y_{n+1}-\tilde{y}_{n+1}\|+h_n\alpha_1\|y_n-\tilde{y}_n\|\|y_{n+1}-\tilde{y}_{n+1}\|\\ &+h_n\beta_1\max_{-\tau\le\xi\le t_{n+1}}\|y^h(\xi)-\tilde{y}^h(\xi)\|\|y_{n+1}-\tilde{y}_{n+1}\|+h_n\gamma_1\|z_n-\tilde{z}_n\|\|y_{n+1}-\tilde{y}_{n+1}\|\\ &+h_n\sigma_1\max_{-\tau\le\xi\le t_{n+1}}\|z^h(\xi)-\tilde{z}^h(\xi)\|\|y_{n+1}-\tilde{y}_{n+1}\|\\ &+h_n\alpha_2\|y_{n+1}-\tilde{y}_{n+1}\|^2+h_n\beta_2\max_{-\tau\le\xi\le t_{n+1}}\|y^h(\xi)-\tilde{y}^h(\xi)\|\|y_{n+1}-\tilde{y}_{n+1}\|\\ &+h_n\gamma_2\|z_{n+1}-\tilde{z}_{n+1}\|\|y_{n+1}-\tilde{y}_{n+1}\|+h_n\sigma_2\max_{-\tau\le\xi\le t_{n+1}}\|z^h(\xi)-\tilde{z}^h(\xi)\|\|y_{n+1}-\tilde{y}_{n+1}\|, \end{aligned}$$

and together with (2.3), we have

$$\begin{aligned} \|y_{n+1}-\tilde{y}_{n+1}\|\le&\|y_n-\tilde{y}_n\|+h_n\alpha_1\|y_n-\tilde{y}_n\|+h_n\beta_1\max_{-\tau\le\xi\le t_{n+1}}\|y^h(\xi)-\tilde{y}^h(\xi)\|+h_n\gamma_1\|z_n-\tilde{z}_n\|\\ &+h_n\sigma_1\max_{-\tau\le\xi\le t_{n+1}}\|z^h(\xi)-\tilde{z}^h(\xi)\|+h_n\alpha_2\|y_{n+1}-\tilde{y}_{n+1}\|+h_n\gamma_2\|z_{n+1}-\tilde{z}_{n+1}\|\\ &+h_n\beta_2\max_{-\tau\le\xi\le t_{n+1}}\|y^h(\xi)-\tilde{y}^h(\xi)\|+h_n\sigma_2\max_{-\tau\le\xi\le t_{n+1}}\|z^h(\xi)-\tilde{z}^h(\xi)\|\\ \le&\|y_n-\tilde{y}_n\|+h_n\alpha_1\|y_n-\tilde{y}_n\|+h_n\alpha_2\|y_{n+1}-\tilde{y}_{n+1}\|\\ &+(\beta_1+\beta_2)h_n\max_{-\tau\le\xi\le t_{n+1}}\|y^h(\xi)-\tilde{y}^h(\xi)\| \end{aligned}$$

$$
\begin{aligned}
&+(\sigma_1+\sigma_2)h_n \max_{-\tau\le\xi\le t_{n+1}} \|z^h(\xi)-\tilde{z}^h(\xi)\| + h_n\gamma_1[L_1\|y_n-\tilde{y}_n\| \\
&+L_2 \max_{-\tau\le\xi\le t_n} \|y^h(\xi)-\tilde{y}^h(\xi)\| + L_3 \max_{-\tau\le\xi\le t_n} \|z^h(\xi)-\tilde{z}^h(\xi)\|] \\
&+h_n\gamma_2[L_1\|y_{n+1}-\tilde{y}_{n+1}\| + L_2 \max_{-\tau\le\xi\le t_{n+1}} \|y^h(\xi)-\tilde{y}^h(\xi)\| \\
&+L_3 \max_{-\tau\le\xi\le t_{n+1}} \|z^h(\xi)-\tilde{z}^h(\xi)\|],
\end{aligned} \tag{3.6}
$$

according to canonical conditions (3.1), this leads to

$$
\begin{aligned}
&[1-(\alpha_2+\gamma_2L_1)h_n]\|y_{n+1}-\tilde{y}_{n+1}\| \\
\le&[1+(\alpha_1+\gamma_1L_1)h_n]\|y_n-\tilde{y}_n\| \\
&+(\beta_1+\beta_2+\gamma_1L_2+\gamma_2L_2)h_n \max_{-\tau\le\xi\le t_{n+1}} \|y^h(\xi)-\tilde{y}^h(\xi)\| \\
&+(\sigma_1+\sigma_2+\gamma_1L_3+\gamma_2L_3)h_n \max_{-\tau\le\xi\le t_{n+1}} \|z^h(\xi)-\tilde{z}^h(\xi)\| \\
\le&[1+(\alpha_1+\gamma_1L_1)h_n]\|y_n-\tilde{y}_n\| \\
&+(\beta_1+\beta_2+\gamma_1L_2+\gamma_2L_2)h_n \max\Big\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|, \max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|\Big\} \\
&+(\sigma_1+\sigma_2+\gamma_1L_3+\gamma_2L_3)h_n \max\Big\{\max_{1\le i\le n}\|z_i-\tilde{z}_i\|, \max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}.
\end{aligned} \tag{3.7}
$$

Next, we will prove

$$
\begin{aligned}
&\max\Big\{\max_{1\le i\le n}\|z_i-\tilde{z}_i\|, \max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\} \\
\le& c_2\max\Big\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|, \max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|, \max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},
\end{aligned} \tag{3.8}
$$

where $c_2=\max\{\frac{L_1+L_2}{1-L_3},1\}$. The following three cases are considered to prove the formula (3.8).

The first case: when

$$
\max\Big\{\max_{1\le i\le n}\|z_i-\tilde{z}_i\|, \max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\} = \|z_n-\tilde{z}_n\|,
$$

there is

$$
\begin{aligned}
\|z_n-\tilde{z}_n\| =& \|g(y_n,y^h(\cdot),z^h(\cdot))-g(\tilde{y}_n,\tilde{y}^h(\cdot),\tilde{z}^h(\cdot))\| \\
\le& L_1\|y_n-\tilde{y}_n\| + L_2 \max_{-\tau\le\xi\le t_{n+1}} \|y^h(\xi)-\tilde{y}^h(\xi)\| + L_3 \max_{-\tau\le\xi\le t_{n+1}} \|z^h(\xi)-\tilde{z}^h(\xi)\| \\
\le& L_1\|y_n-\tilde{y}_n\| + L_2\max\Big\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|, \max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|\Big\} \\
&+L_3\max\Big\{\max_{1\le i\le n}\|z_i-\tilde{z}_i\|, \max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},
\end{aligned}
$$

since $L_3<1$, we have

$$\|z_n-\tilde{z}_n\|\leq\frac{L_1+L_2}{1-L_3}\max\Big\{\max_{1\leq i\leq n}\|y_i-\tilde{y}_i\|,\max_{-\tau\leq t\leq 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},$$

so we can get

$$\begin{aligned}&\max\Big\{\max_{1\leq i\leq n}\|z_i-\tilde{z}_i\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}\\ \leq&c_2\max\Big\{\max_{1\leq i\leq n}\|y_i-\tilde{y}_i\|,\max_{-\tau\leq t\leq 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}.\end{aligned}$$

The second case: when

$$\max\Big\{\max_{1\leq i\leq n}\|z_i-\tilde{z}_i\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}=\|z_k-\tilde{z}_k\|,$$

where $k\in[1,n-1]$ and $k\in\mathbb{Z}$, from (2.3) and (3.1) we can get

$$\begin{aligned}\|z_k-\tilde{z}_k\|=&\|g(y_k,y^h(\cdot),z^h(\cdot))-g(\tilde{y}_k,\tilde{y}^h(\cdot),\tilde{z}^h(\cdot))\|\\ \leq&L_1\|y_k-\tilde{y}_k\|+L_2\max_{-\tau\leq\xi\leq t_k}\|y^h(\xi)-\tilde{y}^h(\xi)\|+L_3\max_{-\tau\leq\xi\leq t_k}\|z^h(\xi)-\tilde{z}^h(\xi)\|\\ \leq&L_1\|y_k-\tilde{y}_k\|+L_2\max\Big\{\max_{1\leq i\leq k-1}\|y_i-\tilde{y}_i\|,\max_{-\tau\leq t\leq 0}\|\psi(t)-\tilde{\psi}(t)\|\Big\}\\ &+L_3\max\Big\{\max_{1\leq i\leq k-1}\|z_i-\tilde{z}_i\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}\\ \leq&(L_1+L_2)\max\Big\{\max_{1\leq i\leq k}\|y_i-\tilde{y}_i\|,\max_{-\tau\leq t\leq 0}\|\psi(t)-\tilde{\psi}(t)\|\Big\}+L_3\|z_k-\tilde{z}_k\|,\end{aligned}$$

due to $L_3<1$, it is easily obtained

$$\begin{aligned}\|z_k-\tilde{z}_k\|\leq&\frac{L_1+L_2}{1-L_3}\max\Big\{\max_{1\leq i\leq k}\|y_i-\tilde{y}_i\|,\max_{-\tau\leq t\leq 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}\\ \leq&\frac{L_1+L_2}{1-L_3}\max\Big\{\max_{1\leq i\leq n}\|y_i-\tilde{y}_i\|,\max_{-\tau\leq t\leq 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},\end{aligned}$$

where $k\in[1,n-1]$ and $k\in\mathbb{Z}$, so we can get

$$\begin{aligned}&\max\Big\{\max_{1\leq i\leq n}\|z_i-\tilde{z}_i\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}\\ \leq&c_2\max\Big\{\max_{1\leq i\leq n}\|y_i-\tilde{y}_i\|,\max_{-\tau\leq t\leq 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}.\end{aligned}$$

The third case: when

$$\max\Big\{\max_{1\leq i\leq n}\|z_i-\tilde{z}_i\|,\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}=\max_{-\tau\leq t\leq 0}\|\chi(t)-\tilde{\chi}(t)\|,$$

since $c_2=\max\{\frac{L_1+L_2}{1-L_3},1\}$, there is a special case of (3.8) obviously.

Therefore, according to the analysis of the above three cases, we can obtain

$$\begin{aligned}&\max\Big\{\max_{1\le i\le n}\|z_i-\tilde{z}_i\|,\max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\}\\ \le& c_2\max\Big\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|,\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},\end{aligned}$$

where $c_2=\max\{\frac{L_1+L_2}{1-L_3},1\}$, on this basis, substitute (3.8) into (3.7), we get

$$\begin{aligned}&[1-(\alpha_2+\gamma_2L_1)h_n]\|y_{n+1}-\tilde{y}_{n+1}\|\\ \le&(1+\bar{c}_1h_n)\max\Big\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|,\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},\end{aligned}\tag{3.9}$$

where

$$\bar{c}_1=\alpha_1+\gamma_1L_1+\beta_1+\beta_2+\gamma_1L_2+\gamma_2L_2+(\sigma_1+\sigma_2+\gamma_1L_3+\gamma_2L_3)c_2,\tag{3.10}$$

when $\alpha_2+\gamma_2L_1\le 0$, it is not difficult to check that

$$\frac{1+\bar{c}_1h_n}{1-(\alpha_2+\gamma_2L_1)h_n}\le 1+\max\{\bar{c}_1+\alpha_2+\gamma_2L_1,0\}h_n,\tag{3.11}$$

when $\alpha_2+\gamma_2L_1>0$, let $h_n<\frac{1}{2(\alpha_2+\gamma_2L_1)}$, we have

$$0<\frac{1}{1-(\alpha_2+\gamma_2L_1)h_n}\le 1+2(\alpha_2+\gamma_2L_1)h_n,$$

thus, we can get

$$\frac{1+\bar{c}_1h_n}{1-(\alpha_2+\gamma_2L_1)h_n}\le 1+2(\bar{c}_1+\alpha_2+\gamma_2L_1)h_n,\tag{3.12}$$

according to the relationships (3.9)-(3.12), we infer that

$$\begin{aligned}&\|y_{n+1}-\tilde{y}_{n+1}\|\\ \le&(1+c_1h_n)\max\Big\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|,\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},\end{aligned}\tag{3.13}$$

where c_1 is defined by (3.4). Let

$$X_n=\max\Big\{\max_{1\le i\le n}\|y_i-\tilde{y}_i\|,\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|,\max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},$$

thus from the inequality (3.13), we have

$$X_n\le(1+c_1h_{n-1})X_{n-1},\quad h_{n-1}\le\bar{h},\tag{3.14}$$

therefore, through further iteration

$$
\begin{aligned}
\|y_n-\tilde{y}_n\| \le X_n \le \prod_{i=0}^{n-1}(1+c_1h_i)X_0 \le \prod_{i=0}^{n-1}\exp(c_1h_i)X_0 \\
\le \exp(c_1t_n)\max\Big\{\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|, \max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},
\end{aligned}
\tag{3.15}
$$

where

$$\max_{0\le i\le n-1} h_i \le \bar{h}, \quad n=1,2,\cdots,N,$$

the constant c_1 is defined by (3.4), so we can get the stability inequality (3.3a). On the other hand, from (3.8) and (3.15) we know

$$
\begin{aligned}
\|z_n-\tilde{z}_n\| \le \max\Big\{\max_{1\le i\le n}\|z_i-\tilde{z}_i\|, \max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\} \le c_2X_n \\
\le c_2\exp(c_1t_n)\max\Big\{\max_{-\tau\le t\le 0}\|\psi(t)-\tilde{\psi}(t)\|, \max_{-\tau\le t\le 0}\|\chi(t)-\tilde{\chi}(t)\|\Big\},
\end{aligned}
\tag{3.16}
$$

where

$$\max_{0\le i\le n-1} h_i \le \bar{h}, \quad n=1,2,\cdots,N,$$

c_2 and $\bar{h}$ is defined by (3.4) and (3.5) respectively, thus the stability inequality (3.3b) is also obtained. The proof of Theorem 3.1 is completed. ☐

Corollary 3.1. *Suppose the CES method* (2.12) *is used to solve the nonlinear composite stiff problem* (2.4) $\in \mathcal{S}(\alpha_1,\gamma_1,\alpha_2,\gamma_2,L_1)$ *on any given grid* Δ_h*, for the starting values* $\{y_0,z_0\}$ *and* $\{\tilde{y}_0,\tilde{z}_0\}$*, and then for any parallel integration steps* $(t_n,y_1,y_2,\cdots,y_n)\to(t_{n+1},y_1,y_2,\cdots,y_{n+1})$ *and* $(t_n,z_1,z_2,\cdots,z_n)\to(t_{n+1},z_1,z_2,\cdots,z_{n+1})$ *defined by* (2.12)*, and the parallel integration steps* $(t_n,\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_n)\to(t_{n+1},\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_{n+1})$ *and* $(t_n,\tilde{z}_1,\tilde{z}_2,\cdots,\tilde{z}_n)\to(t_{n+1},\tilde{z}_1,\tilde{z}_2,\cdots,\tilde{z}_{n+1})$ *defined by*

$$
\begin{cases}
\tilde{y}_{n+1}=\tilde{y}_n+h_nf_1(t_{n+1},\tilde{y}_n,\tilde{z}_n)+h_nf_2(t_{n+1},\tilde{y}_{n+1},\tilde{z}_{n+1}),\\
\tilde{z}_{n+1}=g(\tilde{y}_{n+1}),
\end{cases}
\tag{3.17}
$$

we have stability inequalities

$$\|y_n-\tilde{y}_n\| \le \exp(c_1t_n)\|y_0-\tilde{y}_0\|, \tag{3.18}$$

and

$$\|z_n-\tilde{z}_n\| \le L_1\exp(c_1t_n)\|y_0-\tilde{y}_0\|, \tag{3.19}$$

where

$$\max_{0\le i\le n-1} h_i \le \bar{h}, \quad n=1,2,\cdots,N,$$

and

$$
c_1=\begin{cases}
2(\bar{c}_1+\alpha_2+\gamma_2L_1), & \alpha_2+\gamma_2L_1>0,\\
\max\{\bar{c}_1+\alpha_2+\gamma_2L_1,0\}, & \alpha_2+\gamma_2L_1\le 0,
\end{cases}
\tag{3.20}
$$

here, $\bar{c}_1=\alpha_1+\gamma_1L_1$*,* $\bar{h}$ *is defined by* (3.5)*.*

Proof. Using the CES method (2.12) to solve nonlinear composite stiff problem (2.4) $\in \mathcal{S}(\alpha_1,\gamma_1,\alpha_2,\gamma_2,L_1)$, we have $\beta_1=\beta_2=\sigma_1=\sigma_2=L_2=L_3=0$, so the inequality (3.7) degenerates to

$$\begin{aligned}&[1-(\alpha_2+\gamma_2L_1)h_n]\,\|y_{n+1}-\tilde{y}_{n+1}\|\\ \le&[1+(\alpha_1+\gamma_1L_1)h_n]\|y_n-\tilde{y}_n\|\\ \le&(1+\bar{c}_1h_n)\|y_n-\tilde{y}_n\|,\quad h_n\le\bar{h},\end{aligned}$$

where $\bar{c}_1=\alpha_1+\gamma_1L_1$, thus

$$\begin{aligned}\|y_n-\tilde{y}_n\|\le&(1+c_1h_{n-1})\|y_{n-1}-\tilde{y}_{n-1}\|\\ \le&\prod_{i=0}^{n-1}(1+c_1h_i)\|y_0-\tilde{y}_0\|\le\prod_{i=0}^{n-1}\exp(c_1h_i)\|y_0-\tilde{y}_0\|\\ \le&\exp(c_1t_n)\|y_0-\tilde{y}_0\|,\quad \max_{0\le i\le n-1}h_i\le\bar{h},\quad n=1,2,\cdots,N,\end{aligned}\tag{3.21}$$

the formula (3.18) is obtained. On the other hand, by (2.3) and (3.18) there are

$$\begin{aligned}\|z_n-\tilde{z}_n\|\le&L_1\|y_n-\tilde{y}_n\|\le L_1(1+c_1h_{n-1})\|y_{n-1}-\tilde{y}_{n-1}\|\\ \le&L_1\exp(c_1t_n)\|y_0-\tilde{y}_0\|,\quad \max_{0\le i\le n-1}h_i\le\bar{h},\quad n=1,2,\cdots,N,\end{aligned}$$

where c_1 and $\bar{h}$ are defined by (3.20) and (3.5), respectively, thus we can get formula (3.19). □

4 Convergence analysis of CES method for solving nonlinear composite stiff FDAEs

In this section, we mainly perform the convergence analysis of the CES method (2.7) to solve the nonlinear composite stiff FDAEs (2.1), and prove that the CES method is consistent of order 1. Based on this, we further obtain that the CES method is convergent of order 1. To prove this conclusion, we first note

$$\psi(t)-\varphi_1(t)=0,\quad \chi(t)-\varphi_2(t)=0,\quad -\tau\le t\le 0.$$

In addition, it should be noted that in order to match the convergence order of the method, Π^h and $\hat{\Pi}^h$ used in this section are linear interpolation operators.

Theorem 4.1. *The CES method* (2.7) *is consistent of order 1 for solving the nonlinear composite stiff problem* (2.1)$\in\mathcal{S}(\alpha_1,\beta_1,\gamma_1,\sigma_1,\alpha_2,\beta_2,\gamma_2,\sigma_2,L_1,L_2,L_3)$ *on any given grid* Δ_h*, for any fictitious integration steps* $(\varphi_1,y(t_1),y(t_2),\cdots,y(t_n))\to(\varphi_1,y(t_1),y(t_2),\cdots,y(t_n),\check{y}_{n+1})$ *and*

$(\varphi_1,z(t_1),z(t_2),\cdots,z(t_n)) \to (\varphi_1,z(t_1),z(t_2),\cdots,z(t_n),\check{z}_{n+1})$ *defined by*

$$\begin{cases} \check{y}^h(t)=\Pi^h(t;\varphi_1(t),y(t_1),y(t_2),\cdots,y(t_n)), \qquad t\in[-\tau,t_{n+1}], & (4.1a)\\ \check{z}^h(t)=\hat{\Pi}^h(t;\varphi_2(t),z(t_1),z(t_2),\cdots,z(t_n)), \qquad t\in[-\tau,t_{n+1}], & (4.1b)\\ \check{y}_{n+1}=y(t_n)+h_nf_1(t_{n+1},y(t_n),\check{y}^h(\cdot),z(t_n),\check{z}^h(\cdot)) & \\ \qquad\qquad +h_nf_2(t_{n+1},\check{y}_{n+1},\check{y}^h(\cdot),\check{z}_{n+1},\check{z}^h(\cdot)), & (4.1c)\\ \check{z}_{n+1}=g(\check{y}_{n+1},\check{y}^h(\cdot),\check{z}^h(\cdot)), & (4.1d)\end{cases}$$

then, we have

$$\|\check{y}_{n+1}-y(t_{n+1})\| \le c_3\Big(\max_{0\le i\le n} h_i\Big)^2, \tag{4.2a}$$

$$\|\check{z}_{n+1}-z(t_{n+1})\| \le c_4\Big(\max_{0\le i\le n} h_i\Big)^2, \tag{4.2b}$$

where

$$\max_{0\le i\le n} h_i \le \bar{h}, \quad n=1,2,\cdots,N-1,$$

in addition, we always assume that symbols M_1, M_2, $\hat{M}_1$ and $\hat{M}_2$ denote boundaries of certain derivatives of the true solutions $y(t)$ and $z(t)$ respectively, that is

$$\Big\|\frac{dy(t)}{dt}\Big\| \le M_1, \quad \Big\|\frac{d^2y(t)}{dt^2}\Big\| \le M_2, \quad \Big\|\frac{dz(t)}{dt}\Big\| \le \hat{M}_1, \quad \Big\|\frac{d^2z(t)}{dt^2}\Big\| \le \hat{M}_2, \quad t\in[0,T],$$

here, c_3 and c_4 depend on Lipschitz constants α_1, β_1, β_2, γ_1, γ_2, σ_1, σ_2, L_1, L_2, L_3 and boundaries M_1, M_2, $\hat{M}_1$, $\hat{M}_2$, and

$$c_3=\begin{cases} 2\bar{c}_2M+M_2, & \alpha_2+\gamma_2L_1>0,\\ \bar{c}_2M+\dfrac{M_2}{2}, & \alpha_2+\gamma_2L_1\le 0,\end{cases} \qquad c_4=L_1c_3+L_2M_2+L_3\hat{M}_2, \tag{4.3a}$$

$$M=\max\{M_1,M_2,\hat{M}_1,\hat{M}_2\}, \tag{4.3b}$$

$$\bar{c}_2=\alpha_1+(\beta_1+\beta_2+\gamma_2L_2)\bar{h}+\gamma_1+(\sigma_1+\sigma_2+\gamma_2L_3)\bar{h}, \tag{4.3c}$$

$\bar{h}$ is defined by (3.5), *it can be seen that c_3, c_4 and $\bar{h}^{-1}$ are of appropriate size.*

Proof. For the Taylor expansion of $y(t_n)$ at t_{n+1}, we have

$$y(t_n)=y(t_{n+1})-h_ny'(t_{n+1})+R,$$

where

$$R=\frac{y''(\xi)}{2!}h_n^2, \quad \xi\in(t_n,t_{n+1}), \quad \|R\|\le\frac{M_2}{2}h_n^2,$$

so there is

$$\begin{aligned} y(t_{n+1})=&y(t_n)+h_nf_1(t_{n+1},y(t_{n+1}),y(\cdot),z(t_{n+1}),z(\cdot))\\ &+h_nf_2(t_{n+1},y(t_{n+1}),y(\cdot),z(t_{n+1}),z(\cdot))-R, \end{aligned} \tag{4.4}$$

it can be concluded from Eqs. (4.1c) and (4.4)

$$\begin{aligned}
&\check{y}_{n+1}-y(t_{n+1})\\
=&h_n[f_1(t_{n+1},y(t_n),\check{y}^h(\cdot),z(t_n),\check{z}^h(\cdot))-f_1(t_{n+1},y(t_{n+1}),y(\cdot),z(t_{n+1}),z(\cdot))]\\
&\quad+h_n[f_2(t_{n+1},\check{y}_{n+1},\check{y}^h(\cdot),\check{z}_{n+1},\check{z}^h(\cdot))-f_2(t_{n+1},y(t_{n+1}),\check{y}^h(\cdot),\check{z}_{n+1},\check{z}^h(\cdot))\\
&+f_2(t_{n+1},y(t_{n+1}),\check{y}^h(\cdot),\check{z}_{n+1},\check{z}^h(\cdot))-f_2(t_{n+1},y(t_{n+1}),y(\cdot),z(t_{n+1}),z(\cdot))]+R,
\end{aligned}$$

so

$$\begin{aligned}
&\|\check{y}_{n+1}-y(t_{n+1})\|^2\\
=&h_n\langle f_1(t_{n+1},y(t_n),\check{y}^h(\cdot),z(t_n),\check{z}^h(\cdot))-f_1(t_{n+1},y(t_{n+1}),y(\cdot),z(t_{n+1}),z(\cdot)),\\
&\quad\check{y}_{n+1}-y(t_{n+1})\rangle+h_n\langle f_2(t_{n+1},\check{y}_{n+1},\check{y}^h(\cdot),\check{z}_{n+1},\check{z}^h(\cdot))\\
&\quad-f_2(t_{n+1},y(t_{n+1}),\check{y}^h(\cdot),\check{z}_{n+1},\check{z}^h(\cdot)),\check{y}_{n+1}-y(t_{n+1})\rangle\\
&\quad+h_n\langle f_2(t_{n+1},y(t_{n+1}),\check{y}^h(\cdot),\check{z}_{n+1},\check{z}^h(\cdot))-f_2(t_{n+1},y(t_{n+1}),y(\cdot),z(t_{n+1}),z(\cdot)),\\
&\quad\check{y}_{n+1}-y(t_{n+1})\rangle+\langle R,\check{y}_{n+1}-y(t_{n+1})\rangle,
\end{aligned}$$

according to the Lipschitz conditions (2.2), there is

$$\begin{aligned}
&\|\check{y}_{n+1}-y(t_{n+1})\|\\
\le&h_n[\alpha_1\|y(t_n)-y(t_{n+1})\|+\beta_1\max_{-\tau\le\xi\le t_{n+1}}\|\check{y}^h(\xi)-y(\xi)\|+\gamma_1\|z(t_n)-z(t_{n+1})\|\\
&\quad+\sigma_1\max_{-\tau\le\xi\le t_{n+1}}\|\check{z}^h(\xi)-z(\xi)\|+\alpha_2\|\check{y}_{n+1}-y(t_{n+1})\|\\
&\quad+\beta_2\max_{-\tau\le\xi\le t_{n+1}}\|\check{y}^h(\xi)-y(\xi)\|+\gamma_2\|\check{z}_{n+1}-z(t_{n+1})\|\\
&\quad+\sigma_2\max_{-\tau\le\xi\le t_{n+1}}\|\check{z}^h(\xi)-z(\xi)\|]+\|R\|\\
\le&h_n[\alpha_1\|y(t_n)-y(t_{n+1})\|+\alpha_2\|\check{y}_{n+1}-y(t_{n+1})\|\\
&\quad+\gamma_1\|z(t_n)-z(t_{n+1})\|+\gamma_2\|\check{z}_{n+1}-z(t_{n+1})\|\\
&\quad+(\beta_1+\beta_2)\max_{-\tau\le\xi\le t_{n+1}}\|\Pi^h(\xi;\varphi_1(\xi),y(t_1),y(t_2),\cdots,y(t_n))-y(\xi)\|\\
&\quad+(\sigma_1+\sigma_2)\max_{-\tau\le\xi\le t_{n+1}}\|\tilde{\Pi}^h(\xi;\varphi_2(\xi),z(t_1),z(t_2),\cdots,z(t_n))-z(\xi)\|]+\|R\|, \qquad (4.5)
\end{aligned}$$

using differential mean value theorem and Lipschitz condition (2.3) yield that

$$\|y(t_n)-y(t_{n+1})\|\le M_1h_n,\quad \|z(t_n)-z(t_{n+1})\|\le\hat{M}_1h_n, \qquad (4.6a)$$

$$\begin{aligned}
\|\check{z}_{n+1}-z(t_{n+1})\|\le L_1\|\check{y}_{n+1}-y(t_{n+1})\|+L_2\max_{-\tau\le\xi\le t_{n+1}}\|\check{y}^h(\xi)-y(\xi)\|\\
+L_3\max_{-\tau\le\xi\le t_{n+1}}\|\check{z}^h(\xi)-z(\xi)\|, \qquad (4.6b)
\end{aligned}$$

because Π^h and $\hat{\Pi}^h$ are piecewise linear interpolation operators, so we have

$$\max_{-\tau\leq\xi\leq t_{n+1}}\|\Pi^h(\xi,\varphi_1,y(t_1),y(t_2),\cdots,y(t_n))-y(\xi)\|\leq M_2\Big(\max_{0\leq i\leq n}h_i\Big)^2, \tag{4.7a}$$

$$\max_{-\tau\leq\xi\leq t_{n+1}}\|\hat{\Pi}^h(\xi,\varphi_2,z(t_1),z(t_2),\cdots,z(t_n))-z(\xi)\|\leq \hat{M}_2\Big(\max_{0\leq i\leq n}h_i\Big)^2, \tag{4.7b}$$

according to (4.6a), (4.6b) and (4.7), we can get from (4.5)

$$\begin{aligned}
&[1-(\alpha_2+\gamma_2L_1)h_n]\,\|\check{y}_{n+1}-y(t_{n+1})\|\\
\leq&\alpha_1M_1h_n^2+(\beta_1+\beta_2+\gamma_2L_2)M_2h_n\Big(\max_{0\leq i\leq n}h_i\Big)^2\\
&\quad+\gamma_1\hat{M}_1h_n^2+(\sigma_1+\sigma_2+\gamma_2L_3)\hat{M}_2h_n\Big(\max_{0\leq i\leq n}h_i\Big)^2+\frac{M_2}{2}h_n^2\\
\leq&\Big(\bar{c}_2M+\frac{M_2}{2}\Big)\Big(\max_{0\leq i\leq n}h_i\Big)^2,
\end{aligned} \tag{4.8}$$

where M and $\bar{c}_2$ are defined by (4.3b) and (4.3c) respectively, so we can get

$$\|\check{y}_{n+1}-y(t_{n+1})\|\leq c_3\Big(\max_{0\leq i\leq n}h_i\Big)^2,\quad h_n\leq\bar{h},\quad n=1,2,\cdots,N-1, \tag{4.9}$$

where c_3 is defined by (4.3a), that is, the formula (4.2a) is obtained. On the other hand, from the Lipschitz condition (2.3) and (4.7), we have

$$\begin{aligned}
\|\check{z}_{n+1}-z(t_{n+1})\|\leq&L_1\|\check{y}_{n+1}-y(t_{n+1})\|+L_2\max_{-\tau\leq\xi\leq t_{n+1}}\|\check{y}^h(\xi)-y(\xi)\|\\
&+L_3\max_{-\tau\leq\xi\leq t_{n+1}}\|\check{z}^h(\xi)-z(\xi)\|\\
\leq&L_1c_3(\max_{0\leq i\leq n}h_i)^2+L_2M_2\Big(\max_{0\leq i\leq n}h_i\Big)^2+L_3\hat{M}_2\Big(\max_{0\leq i\leq n}h_i\Big)^2\\
\leq&c_4\Big(\max_{0\leq i\leq n}h_i\Big)^2,\quad \max_{0\leq i\leq n}h_i\leq\bar{h},\quad n=1,2,\cdots,N-1,
\end{aligned}$$

where c_4 and $\bar{h}$ are defined by (4.3a) and (3.5) respectively, that is, the formula (4.2b) is obtained. The proof of Theorem 4.1 is completed. □

Theorem 4.2. *The CES method* (2.7) *is convergent of order 1 for solving the nonlinear composite stiff problem* (2.1)$\in\mathcal{S}(\alpha_1,\beta_1,\gamma_1,\sigma_1,\alpha_2,\beta_2,\gamma_2,\sigma_2,L_1,L_2,L_3)$ *on any given grid* Δ_h, *let* $\{y_n,z_n\}$ *denote the approximate sequences generated by the CES method* (2.7) *applied to the nonlinear composite stiff FDAEs* (2.1), *then, we have*

$$\|y(t_n)-y_n\|\leq C_1(t_n)\max_{0\leq i\leq n-1}h_i, \tag{4.10a}$$

$$\|z(t_n)-z_n\|\leq C_2(t_n)\max_{0\leq i\leq n-1}h_i, \tag{4.10b}$$

where

$$\max_{0\le i\le n-1} h_i \le \bar{h}, \quad n=1,2,\cdots,N,$$

$C_1(t)$ and $C_2(t)$ depend on Lipschitz constants α_1, β_1, β_2, γ_1, γ_2, σ_1, σ_2, L_1, L_2, L_3 and boundaries M_1, M_2, $\hat{M}_1$, $\hat{M}_2$, $\bar{h}$ is defined by (3.5), *and*

$$C_1(t)=Kc_3\exp(c_1t), \quad C_2(t)=c_2C_1(t), \tag{4.11}$$

here, c_1, c_2 and c_3 are defined by (3.4) *and* (4.3a) *respectively,*

$$n\max_{0\le i\le n-1} h_i \le K, \quad n=1,2,\cdots,N,$$

here, K is a constant of appropriate size, it can be seen that $C_1(t)$, $C_2(t)$ and $\bar{h}^{-1}$ are of appropriate size.

Proof. Considering Theorem 4.1, for any fictitious integration step

$$(t_n,\varphi_1,y(t_1),y(t_2),\cdots,y(t_n)) \ \rightarrow \ (t_{n+1},\varphi_1,y(t_1),y(t_2),\cdots,y(t_n),\breve{y}_{n+1})$$

defined by (4.1), we have consistency inequality (4.2a). In addition, from formulas (3.9)-(3.12) of Theorem 3.1, we have

$$\|\breve{y}_n-y_n\|\le(1+c_1h_{n-1})\max_{1\le i\le n-1}\|y(t_i)-y_i\|, \tag{4.12}$$

where c_1 is defined by (3.4), the inequalities (4.2a) and (4.12) imply that

$$\begin{aligned}\|y(t_n)-y_n\|\le&\|y(t_n)-\breve{y}_n\|+\|\breve{y}_n-y_n\|\\ \le& c_3\Big(\max_{0\le i\le n-1}h_i\Big)^2+\exp(c_1h_{n-1})\max_{1\le i\le n-1}\|y(t_i)-y_i\|, \quad h_{n-1}\le\bar{h}.\end{aligned} \tag{4.13}$$

Let

$$X_n=\max_{1\le i\le n}\|y(t_i)-y_i\|, \quad X_0=0,$$

we get from (4.13)

$$X_n\le c_3\Big(\max_{0\le i\le n-1}h_i\Big)^2+\exp(c_1h_{n-1})X_{n-1},$$

so through further iteration, we have

$$\begin{aligned}\|y(t_n)-y_n\| &\le X_n \le c_3\Big(\max_{0\le i\le n-1} h_i\Big)^2 + \exp(c_1 h_{n-1}) X_{n-1}\\ &\le \exp(c_1 h_{n-1})\Big[c_3\Big(\max_{0\le i\le n-1} h_i\Big)^2 + X_{n-1}\Big]\\ &\le \exp(c_1(h_{n-1}+h_{n-2}))\Big[2c_3\Big(\max_{0\le i\le n-1} h_i\Big)^2 + X_{n-2}\Big] \le \cdots\\ &\le \exp\Big(c_1\sum_{i=0}^{n-1} h_i\Big)\Big(nc_3\Big(\max_{0\le i\le n-1} h_i\Big)^2 + X_0\Big)\\ &\le nc_3\exp(c_1 t_n)\Big(\max_{0\le i\le n-1} h_i\Big)^2\\ &\le C_1(t_n)\max_{0\le i\le n-1} h_i, \quad \max_{0\le i\le n-1} h_i \le \bar{h},\end{aligned}$$

where $n=1,2,\cdots,N$, $C_1(t)=Kc_3\exp(c_1 t)$, $n\max_{0\le i\le n-1} h_i\le K$. On the other hand, we can get

$$\begin{aligned}\|z(t_n)-z_n\| &\le \max_{1\le i\le n}\|z(t_i)-z_i\|\\ &\le c_2\max_{1\le i\le n}\|y(t_i)-y_i\|\\ &\le c_2 C_1(t_n)\max_{0\le i\le n-1} h_i\\ &\le C_2(t_n)\max_{0\le i\le n-1} h_i, \quad \max_{0\le i\le n-1} h_i\le\bar{h},\end{aligned}$$

where $n=1,2,\cdots,N$, $C_2(t)=c_2 C_1(t)$, $C_1(t)$ and $C_2(t)$ are defined by (4.11). The proof of Theorem 4.2 is completed. □

Corollary 4.1. *The CES method* (2.12) *is consistent of order 1 for solving the nonlinear composite stiff problem* (2.4) $\in\mathcal{S}(\alpha_1,\gamma_1,\alpha_2,\gamma_2,L_1)$ *on any given grid* Δ_h*, for any fictitious integration steps* $(t_n,y(t_1),y(t_2),\cdots,y(t_n))\to(t_{n+1},y(t_1),y(t_2),\cdots,y(t_n),\check{y}_{n+1})$ *and* $(t_n,z(t_1),z(t_2),\cdots,z(t_n))\to(t_{n+1},z(t_1),z(t_2),\cdots,z(t_n),\check{z}_{n+1})$ *defined by*

$$\begin{cases}\check{y}_{n+1}=y(t_n)+h_n f_1(t_{n+1},y(t_n),z(t_n))+h_n f_2(t_{n+1},\check{y}_{n+1},\check{z}_{n+1}),\\ \check{z}_{n+1}=g(\check{y}_{n+1}),\end{cases}$$

then, we have

$$\|y(t_{n+1})-\check{y}_{n+1}\|\le c_3\Big(\max_{0\le i\le n} h_i\Big)^2, \tag{4.14a}$$

$$\|z(t_{n+1})-\check{z}_{n+1}\|\le L_1 c_3\Big(\max_{0\le i\le n} h_i\Big)^2, \tag{4.14b}$$

where

$$\max_{0\le i\le n} h_i \le \bar{h}, \quad n=0,1,\cdots,N-1,$$

c_3 depends on Lipschitz constants α_1, γ_1, and boundaries M_1, M_2, $\hat{M}_1$, $\bar{h}$ is defined by (3.5), *and*

$$c_3=\begin{cases} 2\bar{c}_2M+M_2, & \alpha_2+\gamma_2L_1>0, \\ \bar{c}_2M+\dfrac{M_2}{2}, & \alpha_2+\gamma_2L_1\le 0, \end{cases} \tag{4.15}$$

here, $\bar{c}_2=\alpha_1+\gamma_1$, $M=\max\{M_1,\hat{M}_1\}$, it can be seen that c_3 and $\bar{h}^{-1}$ are of appropriate size.

Proof. By solving the nonlinear composite stiff problem (2.4) $\in \mathcal{S}(\alpha_1,\gamma_1,\alpha_2,\gamma_2,L_1)$ with CES method (2.12), we have $\beta_1=\beta_2=\sigma_1=\sigma_2=L_2=L_3=0$, so the inequality (4.8) degenerates to

$$\begin{aligned} &[1-(\alpha_2+\gamma_2L_1)h_n]\,\|\check{y}_{n+1}-y(t_{n+1})\| \\ \le&\alpha_1M_1h_n^2+\gamma_1\hat{M}_1h_n^2+\frac{M_2}{2}h_n^2 \\ \le&\Big(\bar{c}_2M+\frac{M_2}{2}\Big)\Big(\max_{0\le i\le n} h_i\Big)^2, \end{aligned}$$

so there is

$$\|\check{y}_{n+1}-y(t_{n+1})\|\le c_3\Big(\max_{0\le i\le n} h_i\Big)^2, \quad \max_{0\le i\le n} h_i\le\bar{h}, \quad n=1,2,\cdots,N-1,$$

Thus, the formula (4.14a) is obtained. On the other hand, by the Lipschitz condition (2.6) and (4.14a) we have

$$\begin{aligned} \|\check{z}_{n+1}-z(t_{n+1})\| \le& L_1\|\check{y}_{n+1}-y(t_{n+1})\| \\ \le& L_1c_3\Big(\max_{0\le i\le n} h_i\Big)^2, \quad h_n\le\bar{h}, \quad n=1,2,\cdots,N-1, \end{aligned}$$

where c_3 and $\bar{h}$ are defined by (4.15) and (3.5) respectively. Thus, the formula (4.14b) is obtained. □

Corollary 4.2. *The CES method* (2.12) *is convergent of order 1 for solving the nonlinear composite stiff problem* (2.4)$\in \mathcal{S}(\alpha_1,\gamma_1,\alpha_2,\gamma_2,L_1)$ *on any given grid Δ_h, let $\{y_n,z_n\}$ denote the approximate sequences generated by the CES method* (2.12) *applied to the nonlinear composite stiff FDAEs* (2.4), *we have*

$$\|y(t_n)-y_n\|\le C_1(t_n)\max_{0\le i\le n-1} h_i, \tag{4.16a}$$

$$\|z(t_n)-z_n\|\le L_1C_1(t_n)\max_{0\le i\le n-1} h_i, \tag{4.16b}$$

where

$$\max_{0\le i\le n-1} h_i \le \bar{h}, \quad n=1,2,\cdots,N,$$

$C_1(t)$ depends on Lipschitz constants α_1, α_2, γ_1, γ_2, L_1 and boundaries M_1, M_2, $\hat{M}_1$, $\bar{h}$ is defined by (3.5), *and*

$$C_1(t)=Kc_3\exp(c_1 t), \tag{4.17}$$

here, c_1 and c_3 are defined (3.20) *and* (4.15) *respectively, $C_1(t)$ and $\bar{h}^{-1}$ are of appropriate size.*

Proof. By solving the nonlinear composite stiff problem (2.4) $\in \mathcal{S}(\alpha_1,\gamma_1,\alpha_2,\gamma_2,L_1)$ with CES method (2.12), we have $\beta_1=\beta_2=\sigma_1=\sigma_2=L_2=L_3=0$, so it's easy to see that the inequalities (4.10a) and (4.10b) degenerate to (4.16a) and (4.16b) respectively. □

5 Numerical results

Example 5.1. Consider differential-algebraic equations

$$\begin{cases} y'(t)=z(t)-10^4(2z(t)\cdot y(t)-2\cos t-\sin 2t), & t\in\left[0,\dfrac{\pi}{2}\right], \\ z^2(t)(y(t)-1)+(y(t)-1)^3-\sin t=0, & t\in\left[0,\dfrac{\pi}{2}\right], \\ y(0)=1, \quad z(0)=1, \end{cases} \tag{5.1}$$

this equations has a unique true solution $y(t)=\sin t+1$, $z(t)=\cos t$. For each time integration step from t_n to t_{n+1}, we split the differential equation of the problem (5.1) into two sub-problems, that is, the non-stiff sub-problem

$$\begin{cases} \bar{y}'(t)=\bar{z}(t)+2\cdot 10^4\cos t+10^4\sin 2t, & t\in(t_n,t_{n+1}), \\ \bar{y}(t_n)=y_n, \quad \bar{z}(t_n)=z_n, \end{cases} \tag{5.2}$$

and stiff sub-problem

$$\begin{cases} \tilde{y}'(t)=-2\cdot 10^4\cdot\tilde{y}(t)\tilde{z}(t), \\ \tilde{z}^2(t)(\tilde{y}(t)-1)+(\tilde{y}(t)-1)^3-\sin t=0, & t\in(t_n,t_{n+1}), \\ \tilde{y}(t_n)=\bar{y}_{n+1}, \end{cases} \tag{5.3}$$

where the symbols $\bar{y}_{n+1}$ and $\{\tilde{y}_{n+1},\tilde{z}_{n+1}\}$ represent the numerical solutions generated by solving problems (5.2) and (5.3) with methods (2.13) and (2.15) respectively, and let $y_{n+1}=\tilde{y}_{n+1}$, $z_{n+1}=\tilde{z}_{n+1}$, where $y_0=y(0)$, $z_0=z(0)$. In order to test the stability theory of CES method established in this paper, we use the method (2.1) to solve equations (5.1) with different initial values, we let $h=0.01$, and first let initial values $\{y^a(0)=1,\ z^a(0)=1\}$ equal the true solution of Eqs. (5.1), and then let the different initial values be $\{y^b(0)=0.5,\ z^b(0)=0.5\}$ and $\{y^c(0)=2,\ z^c(0)=2\}$, respectively. The numerical solutions corresponding to the initial values $\{y^a(0),\ z^a(0)\}$, $\{y^b(0),\ z^b(0)\}$ and

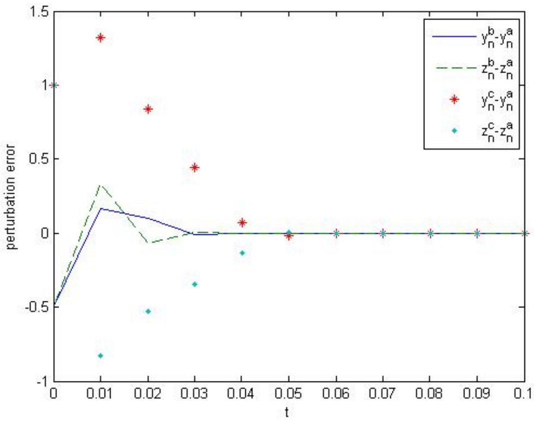

Figure 1: The difference between the perturbed solution and the unperturbed solution as t changes.

$\{y^c(0), z^c(0)\}$ are denoted by $\{y_n^a, z_n^a\}$, $\{y_n^b, z_n^b\}$ and $\{y_n^c, z_n^c\}$, respectively, the difference between the disturbed solution and the undisturbed solution is shown in the Fig. 1.

It can be seen from the Fig. 1 that the difference between the perturbed solution and the undisturbed solution will approach zero with the increase of t for two different initial value perturbations.

Example 5.2. Consider nonlinear delay differential-algebraic equations

$$\begin{cases} (a)\ \dfrac{\partial u}{\partial t} = t^4 \dfrac{\partial^2 u}{\partial x^2} + 2u(x,t)z(x,t) \\ \qquad\quad + 3u\Big(x,t-\dfrac{\pi}{2}\Big) z\Big(x,t-\dfrac{\pi}{2}\Big) + G(x,t), \quad x\in(0,1), \quad t\in[0,\pi], \\ (b)\ z(x,t) = u(x,t)u\Big(x,t-\dfrac{\pi}{2}\Big) + \dfrac{1}{2} z\Big(x,t-\dfrac{\pi}{2}\Big) + x\cos t \\ \qquad\quad -\dfrac{1}{2}x\sin t + 8x^2(1-x)^2\sin 2t, \\ (c)\ u(0,t)=u(1,t)=z(0,t)=0, \quad z(1,t)=\cos t, \quad t\in[0,\pi], \\ (d)\ u(x,t)=4x(1-x)\sin t, \quad z(x,t)=x\cos t, \quad x\in(0,1), \quad t\in\Big[-\dfrac{\pi}{2},0\Big], \end{cases} \tag{5.4}$$

where

$$G(x,t) = 2x(1-x)(2\cos t + x\sin 2t) + 8t^4 \sin t,$$

this equations has a unique true solution

$$u(x,t) = 4x(1-x)\sin x, \quad z(x,t) = x\cos t,$$

it can be calculated that $L_3 = 0.5 < 1$, so Eqs. (5.4) satisfy the conditions of Theorem 3.1, the discrete space variable x, and the space step size $h = 1/N$, where N is a positive integer, using a uniform spatial grid $\{x_i = ih,\ i=0,1,\cdots,N\}$, we get the following semi-discrete equations

$$\begin{cases} \dfrac{\partial u_i(t)}{\partial t} = t^4 \dfrac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{h^2} + 2u_i(t)z_i(t) + 3u_i\Big(t-\dfrac{\pi}{2}\Big)z_i\Big(t-\dfrac{\pi}{2}\Big) \\ \qquad\qquad + G(x_i,t), \quad i=1,2,\cdots,N-1, \quad t\in[0,\pi], \\ z_i(t) = u_i(t)u_i\Big(t-\dfrac{\pi}{2}\Big) + \dfrac{1}{2}z_i\Big(t-\dfrac{\pi}{2}\Big) + x_i\cos t - \dfrac{1}{2}x_i\sin t \\ \qquad\qquad + 8x_i^2(1-x_i)^2\sin 2t, \quad i=1,2,\cdots,N-1, \quad t\in[0,\pi], \\ u_0(t) = u_N(t) = z_0(t) = 0, \quad z_N(t) = \cos t, \quad t\in[0,\pi], \\ u_i(t) = 4x_i(1-x_i)\sin t, \quad z_i(t) = x_i\cos t, \quad i=0,1,\cdots,N, \quad t\in\Big[-\dfrac{\pi}{2},0\Big], \end{cases} \tag{5.5}$$

where $u_i(t-\frac{\pi}{2})$ and $z_i(t-\frac{\pi}{2})$ represent $u(x_i,t-\frac{\pi}{2})$ and $z(x_i,t-\frac{\pi}{2})$ respectively. For each time iteration step from t_n to t_{n+1}, we split the problem (5.5) into two sub-problems, namely non-stiff sub-problem

$$\begin{cases} \dfrac{d\bar{u}_i}{dt} = 2\bar{u}_i(t)\bar{z}_i(t) + 3\bar{u}_i\Big(t-\dfrac{\pi}{2}\Big)\bar{z}_i\Big(t-\dfrac{\pi}{2}\Big) + G(x_i,t), \quad t\in(t_n,t_{n+1}), \\ \bar{u}_i(t_n) = u_{i,n}, \quad \bar{z}_i(t_n) = z_{i,n}, \end{cases} \tag{5.6}$$

and stiff sub-problem

$$\begin{cases} \dfrac{d\tilde{u}_i}{dt} = t^4 \dfrac{\tilde{u}_{i+1}(t)-2\tilde{u}_i(t)+\tilde{u}_{i-1}(t)}{h^2}, \\ \tilde{z}_i(t) = \tilde{u}_i(t)\tilde{u}_i\Big(t-\dfrac{\pi}{2}\Big) + \dfrac{1}{2}\tilde{z}_i\Big(t-\dfrac{\pi}{2}\Big) + x_i\cos t \\ \qquad\qquad -\dfrac{1}{2}x_i\sin t + 8x_i^2(1-x_i)^2\sin 2t, \qquad t\in(t_n,t_{n+1}), \\ \tilde{u}_i(t_n) = \bar{u}_{i,n+1}, \end{cases} \tag{5.7}$$

where the symbols $\bar{u}_{i,n+1}$ and $\{\tilde{u}_{i,n+1},\tilde{z}_{i,n+1}\}$ represent the numerical solutions generated by solving problems (5.6) and (5.7) with methods (2.8) and (2.10) respectively, and let $u_{i,n+1} = \tilde{u}_{i,n+1}$, $z_{i,n+1} = \tilde{z}_{i,n+1}$, where $u_{i,0} = u_i(0)$, $z_{i,0} = z_i(0)$, $i=1,2,\cdots,N-1$. We take $N=100$, that is, space step size $h=1/100$, in order to calculate the approximate solutions u_{ij} and z_{ij} of Eqs. (5.4) at each grid point (x_i,t_j), we use the method (2.7) with time step size $\tau = 1/5m$ $(m=4,8,16,32,64)$ to solve the semi-discrete equations (5.5). A series of effective numerical solutions are obtained, and global errors and convergence orders of CES method are shown in the Table 1. Here,

$$u_{err} = \max_{0\le i\le N,\ 0\le j\le[\frac{\pi}{\tau}]} \|u_{ij} - u(x_i,t_j)\|, \qquad z_{err} = \max_{0\le i\le N,\ 0\le j\le[\frac{\pi}{\tau}]} \|z_{ij} - z(x_i,t_j)\|.$$

Table 1: Global errors and convergence orders of CES method (2.7) for Eq. (5.5) with $h=0.01$.

τ	u_{err}	z_{err}	convergence orders of u	convergence orders of z
1/20	1.180e-02	9.102e-03	-	-
1/40	6.164e-03	4.975e-03	0.937459275	0.871487061
1/80	3.152e-03	2.572e-03	0.967599327	0.951805883
1/160	1.594e-03	1.310e-03	0.983615905	0.973323831
1/320	8.010e-04	6.610e-04	0.992777482	0.986844635

To illustrate the stability of the method (2.7) with time space size $\tau=1/100$, we add the perturbations δ_1, δ_2 to the two equations of the (5.4d) respectively, namely Eq. (5.4d) is rewritten as

$$u(x,t)=4x(1-x)\sin t+\delta_1,\quad z(x,t)=x\cos t+\delta_2,\quad x\in(0,1),\quad t\in\Big[-\frac{\pi}{2},0\Big],$$

the same method can be used to calculate the perturbed numerical solutions $\hat{u}_{ij}$ and $\hat{z}_{ij}$. Take $\delta_1=\delta_2=0.2$, we can get the errors $\|\hat{u}_{ij}-u_{ij}\|$ and $\|\hat{z}_{ij}-z_{ij}\|$, when $i=50$, that is, $x=\frac{1}{2}$, the changes at different time iteration steps are shown in Fig. 2.

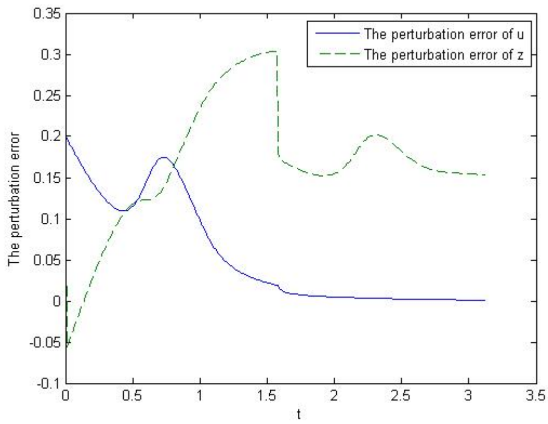

Figure 2: Changes of errors $\|\hat{u}_{ij}-u_{ij}\|$ and $\|\hat{z}_{ij}-z_{ij}\|$ with t when $i=50$.

From the Fig. 2, it can be seen that the error $\|\hat{u}_{ij}-u_{ij}\|$ will approach zero, and $\|\hat{z}_{ij}-z_{ij}\|$ is also controlled within a controllable range with the increase of t, which shows that the method (2.7) is stable.

6 Conclusions

In this paper, a novel CES method based on canonical interpolation operators is proposed to solve the more general nonlinear composite stiff FDAEs, which effectively solves the difficulties caused by algebraic conditions, and stability and convergence of the method is proved. Ultimately, the numerical examples given further verify the theoretical results of CES method.

In the future, we can extend it to higher-order splitting methods, such as the second-order canonical implicit midpoint splitting method or the high-order canonical Runge-Kutta splitting method, and establish corresponding stability and convergence theories for these high-order methods.

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