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Definite Condition of the Evolutionary $\vec{p}(x)-$Laplacian Equation

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Abstract. For the nonlinear degenerate parabolic equations, how to find an appropriate boundary value condition to ensure the well-posedness of weak solution has been an interesting and challenging problem. In this paper, we develop the general characteristic function method to study the stability of weak solutions based on a partial boundary value condition.

Key Words: Definite condition, stability, general characteristic function method, weak solution, Laplacian equation.

AMS Subject Classifications: 35B35, 35D30, 35K55

1 Introduction

Consider the evolutionary $\vec{p}(x)-$Laplacian equation [20]

$$
\begin{aligned}
u_t =& \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\,|u_{x_i}|^{p_i(x)-2}\,u_{x_i}\right) + \sum_{i=1}^{N}\frac{\partial b_i(u,x,t)}{\partial x_i} \\
&- b(x,t)|u|^{\sigma(x)-2}u, \quad (x,t)\in\Omega\times(0,T),
\end{aligned} \tag{1.1}
$$

where $a_i(x)$, $p_i(x)$ and $\sigma(x)$ are nonnegative continuous functions with $p_i(x) > 1$ and $\sigma(x) > 1$, $b(x,t)$ and $b_i(s,x,t)$ are Lipschitz functions, and Ω is a smooth bounded domain in $\mathbb{R}^n$ with $\Omega_T = \Omega\times(0,T)$, $T\in(0,\infty)$. A simpler version of Eq. (1.1) is of the form

$$
u_t = \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i(x)-2}u_{x_i}\right), \tag{1.2}
$$

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which is the so-called anisotropic electrorheological fluid equation [1,17]. When $a_i(x) \equiv 1$, the Cauchy-Dirichlet problem and the Cauchy problem for the degenerate and singular quasilinear anisotropic parabolic equations were studied in [13,18,19]. If $a_i(x) \in C^1(\overline{\Omega})$ satisfies

$$a_i(x) > 0, \quad x \in \Omega \quad \text{and} \quad a_i(x) = 0, \quad x \in \partial\Omega, \tag{1.3}$$

the well-posedness of Eq. (1.2) was established in [21]. The degenerate parabolic p-Laplace equation with measurable coefficients was investigated in [6] and the improved integrability of the gradient was naturally formulated in terms of Marcinkiewicz spaces.

Antontsev-Shmarev [3] considered the existence of weak solution of the equation

$$u_t = \operatorname{div}\left(a(x,t)\,|\nabla u|^{p(x)-2}\,\nabla u\right) - b(x,t)|u|^{\sigma(x)-2}u, \quad (x,t) \in \Omega \times (0,T),$$

and investigated the vanishing property of solutions under the suitable assumptions on $b(x,t)$ and the variable exponent $\sigma(x)$ [4]. Chen-Perthame [8] studied the well-posedness and stability of a class of nonlinear hyperbolic-parabolic equations by developing an analytical and effective approach. Recently, we studied the well-posedness of an anisotropic parabolic equation [22]

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\,|u_{x_i}|^{p_i-2}\,u_{x_i}\right) + f(x,t,u,\nabla u), \quad (x,t) \in \Omega \times (0,T).$$

When some diffusion coefficients are degenerate on the boundary $\partial\Omega$ and the others are positive on $\overline{\Omega}$, a new concept–the general characteristic function of the domain Ω, was introduced and applied, and a novel partial boundary value condition was presented to study the stability of weak solutions for anisotropic parabolic equations.

Distinguished from those [21,22] in which $a_i(x)$ is requested to satisfy condition (1.3), in this paper we consider the well-posedness of weak solutions to Eq. (1.1) by only requiring $a_i(x) \geq 0$, $i = 1,2,\cdots,N$ and

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.4a}$$

$$u(x,t) = 0, \qquad (x,t) \in \Sigma_p \times (0,T), \tag{1.4b}$$

where $\Sigma_p \subset \partial\Omega$ is a relatively open subset.

For the associated linear case of Eq. (1.1), i.e., the degenerate linear heat conduction equation of the form

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)u_{x_i}\right) + \sum_{i=1}^{N} b^i(x)u_{x_i} + b(x,t)u + g(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.5}$$

where $a_i(x) = 0$ on the boundary $\partial\Omega$, to ensure the well-posedness and stability of weak solution, according to the Fichera-Oleinik theory [10,16], we need to include a partial boundary condition as (1.4b), in which

$$\Sigma_p = \left\{x \in \partial\Omega : \sum_{i=1}^{N} b^i(x)n_i(x) < 0\right\}, \tag{1.6}$$

where $\vec{n} = \{n_i\}$ is the inner normal vector of Ω.

In order to find the proper partial boundary Σ_p in (1.4b) for the stability of Eq. (1.1), we consider the convection function b_i dependent on the spatial variable x. While the convection term $\sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i}$ in Eq. (1.1) is replaced by a simpler form $\sum_{i=1}^{N} \frac{\partial b_i(u)}{\partial x_i}$, the problem that how to figure out the partial boundary Σ_p in condition (1.4b) becomes more difficult and challenging. We will continue to work on this problem in a subsequent paper.

One of the main features of the present paper is to find the expression of Σ_p explicitly in the boundary value condition (1.4b) and prove the stability of solutions based on the partial boundary value condition (1.4b). The other is that, only under the condition

$$\prod_{i=1}^{N} a_i(x) > 0, \quad x \in \Omega \quad \text{and} \quad \prod_{i=1}^{N} a_i(x) = 0, \quad x \in \partial\Omega, \tag{1.7}$$

we prove the stability of weak solutions without the boundary value condition (1.4b).

In addition, under the condition $a(x) + b(x) > 0$ for $x \in \overline{\Omega}$, the parabolic equation arising in the double phase problem

$$u_t = \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u\right), \tag{1.8}$$

has been widely studied [7,15,20]. It is worth mentioning that, instead of $a(x) + b(x) > 0$ for $x \in \overline{\Omega}$, the methods developed in this paper can be applied to study the well-posedness problem of Eq. (1.8) under the condition $a(x) + b(x) > 0$ only for $x \in \Omega$. In other words, we can consider the case $a(x) + b(x) = 0$ for $x \in \partial\Omega$ or $a(x)b(x) = 0$ for $x \in \partial\Omega$ by means of the general characteristic method.

The rest of the paper is organized as follows. In Section 2, we present the related preliminary results on weak solutions and some technical lemmas, and summarize our main results. In Section 3, we apply the general characteristic function method to explore the stability of weak solutions under the condition

$$\int_{\Omega} [a_i(x)]^{-\frac{1}{p_i(x)-1}} dx < \infty, \quad i = 1, 2, \cdots, N. \tag{1.9}$$

In Section 4, the stability of weak solutions is proved without condition (1.9). In Section 5, the local stability of weak solutions is established if $a_i(x)$ satisfies (1.7).

For convenience of our statement, throughout the whole paper, we use c to represent a constant that may change from line to line.

2 Preliminaries and main results

Let us briefly recall some preliminary results on properties of the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ and variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ [9,12,24].

Set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1 \right\}.$$

For any $h \in C_+\left(\overline{\Omega}\right)$, we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, let $L^{p(x)}(\Omega)$ consist of all measurable real-valued functions $u(x)$ which satisfy

$$\int_\Omega |u(x)|^{p(x)} dx < \infty$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Define

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure space of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

From [9,24], we have

Lemma 2.1. *The following three statements are true*

(i) The space $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) ($p(x)$-Hölder's inequality) Let $p(x)$ *and* $q(x) = \frac{p(x)}{p(x)-1}$ *be real functions. Then, the conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$\left| \int_\Omega uv dx \right| \le 2\|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)}.$$

(iii) There holds that

$$\begin{aligned}
&\text{if } \|u\|_{L^{p(x)}(\Omega)} = 1, \quad \text{then } \int_\Omega \|u\|^{p(x)} dx = 1; \\
&\text{if } \|u\|_{L^{p(x)}(\Omega)} > 1, \quad \text{then } \|u\|_{L^{p(x)}(\Omega)}^{p^-} \le \int_\Omega |u|^{p(x)} dx \le \|u\|_{L^{p(x)}(\Omega)}^{p^+}; \\
&\text{if } \|u\|_{L^{p(x)}(\Omega)} < 1, \quad \text{then } \|u\|_{L^{p(x)}(\Omega)}^{p^+} \le \int_\Omega |u|^{p(x)} dx \le \|u\|_{L^{p(x)}(\Omega)}^{p^-}.
\end{aligned}$$

Let $\vec{p}(x)=\{p_i(x)\}$. We define $\vec{P}_+,\vec{P}_-\in\mathbb{R}^N$ by

$$\vec{P}_+=(p_1^+,\cdots,p_N^+),\quad \vec{P}_-=(p_1^-,\cdots,p_N^-),$$

and denote

$$P_+^+=\max\{p_1^+,\cdots,p_N^+\}.$$

For each fixed $t\in[0,T)$, we define the Banach space $V_t(\Omega)$ by

$$\begin{aligned}&V_t(\Omega)=\left\{u(x,t):u(x,t)\in L^2(\Omega)\bigcap W_0^{1,1}(\Omega),\ |\nabla u(x,t)|^{P_+^+}\in L^1(\Omega)\right\},\\&\|u\|_{V_t(\Omega)}=\|u\|_{2,\Omega}+\|\nabla u\|_{P_+^+,\Omega},\end{aligned}$$

and denote by $V_t'(\Omega)$ its dual space. In addition, we denote the Banach space $\mathbf{W}(Q_T)$ by

$$\begin{aligned}&\mathbf{W}(Q_T)=\left\{u:[0,T]\to V_t(\Omega)|u\in L^2(Q_T),\ |\nabla u|^{P_+^+}\in L^1(Q_T),\ u=0\text{ on }\partial\Omega\right\},\\&\|u\|_{\mathbf{W}(Q_T)}=\|\nabla u\|_{P_+^+,Q_T}+\|u\|_{2,Q_T},\end{aligned}$$

and denote by $\mathbf{W}'(Q_T)$ its dual space. According to [5], we know that

$$w\in\mathbf{W}'(Q_T)\iff\begin{cases}w=w_0+\sum\limits_{i=1}^N D_iw_i,\quad w_0\in L^2(Q_T),\quad w_i\in L^{P_+^{+\prime}}(Q_T),\\ \forall\phi\in\mathbf{W}(Q_T),\quad\langle\langle w,\phi\rangle\rangle=\iint_{Q_T}\Big(w_0\phi+\sum\limits_i^N w_iD_i\phi\Big)dxdt,\end{cases}$$

where $\Omega_T=\Omega\times(0,T)$, $T\in(0,\infty)$, and $P_+^{+\prime}=\frac{P_+^+}{P_+^+-1}$.

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)}=\sup\left\{\langle\langle v,\phi\rangle\rangle|\phi\in\mathbf{W}(\mathbf{Q_T}),\ \|\phi\|_{\mathbf{W}(Q_T)}\le 1\right\}.\tag{2.1}$$

Definition 2.1. *A function $u(x,t)$ is said to be a weak solution of Eq.* (1.1) *with the initial value* (1.4a)*, provided that*

$$u\in L^\infty(Q_T),\quad u_t\in\mathbf{W}'(Q_T),\quad a_i(x)u_{x_i}\in L^\infty\left(0,T;L^{p_i(x)}(\Omega)\right),\tag{2.2}$$

and for any function $\varphi\in C_0^1(Q_T)$ there holds

$$\begin{aligned}&\iint_{Q_T}\left[\frac{\partial u}{\partial t}\varphi+\sum_{i=1}^N a_i(x)\,|u_{x_i}|^{p_i(x)-2}\,u_{x_i}\varphi_{x_i}+\sum_{i=1}^N b_i(u,x,t)\varphi_{x_i}\right]dxdt\\&=-\iint_{Q_T}b(x,t)|u|^{\sigma(x)-2}u\varphi(x,t)dxdt.\end{aligned}\tag{2.3}$$

The initial condition (1.4a) is satisfied in the sense of

$$\lim_{t\to 0}\int_\Omega u(x,t)\phi(x)dx=\int_\Omega u_0(x)\phi(x)dx\tag{2.4}$$

for any $\phi(x)\in C_0^\infty(\Omega)$.

Similar to the characteristic function χ of Ω defined by

$$\chi(x)=1,\quad x\in\Omega \quad \text{and} \quad \chi(x)=0,\quad x\in\mathbb{R}^N\setminus\Omega,$$

we give the following definition.

Definition 2.2. *We say that a nonnegative function $\varphi(x)\in C^1(\overline{\Omega})$ is a general characteristic function of Ω, provided that*

$$\varphi(x)=0,\quad x\in\partial\Omega \quad \textit{and} \quad \varphi(x)>0,\quad x\in\Omega. \tag{2.5}$$

This paper is a continuum of our previous work [22], where we proposed a method, currently called the general characteristic function method, to study the stability of weak solutions of the anisotropic parabolic equations. It is notable that the existence of local solutions can be established in an analogous manner as [3, Theorem 4.3], so we will not discuss the existence of weak solutions in this study. In addition, according to [21, Lemma 3.2], if (1.9) is true, then

$$\int_\Omega |\nabla u|dx<\infty,$$

and the trace of u can be defined on the boundary $\partial\Omega$.

Let us summarize our main results on the stability of weak solutions of Eq. (1.1).

Theorem 2.1. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq.* (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively, and with*

$$u(x,t)=v(x,t)=0,\quad (x,t)\in\Sigma_p\times(0,T), \tag{2.6}$$

where

$$\Sigma_p=\left\{x\in\partial\Omega:\sum_{j=1}^N\prod_{k=1,k\neq j}^N a_k(x)a_{jx_i}\neq 0,\ i=1,2,\cdots,N\right\}. \tag{2.7}$$

If $\sigma(x)\geq\sigma^-\geq 2$, $b_i(s,x,t)$ $(i=1,2,\cdots,N)$ is a Lipschitz function, $a_i(x)\in C^1(\overline{\Omega})$ satisfies (1.9) *and*

$$\prod_{k=1}^N a_k(x)=0,\quad x\in\partial\Omega, \tag{2.8a}$$

$$n^{\frac{p_i^+}{p_i^-}}\left(\int_{\Omega\setminus D_n} a_i(x)\left|\sum_{j=1}^N\prod_{k=1,\ k\neq j}^N a_k a_{jx_i}\right|^{p_i(x)}dx\right)^{\frac{1}{p_i^-}}\leq c,\quad i=1,2,\cdots,N, \tag{2.8b}$$

where $D_n=\left\{x\in\Omega:\varphi(x)>\frac{1}{n}\right\}$ for the sufficiently large n, then we have

$$\int_\Omega |u(x,t)-v(x,t)|dx\leq c\int_\Omega |u_0(x)-v_0(x)|dx. \tag{2.9}$$

On the other hand, if $a_i(x) \in C^1(\overline{\Omega})$ does not satisfy (1.9), we can apply the general characteristic function method to obtain the stability result without any boundary value condition.

Theorem 2.2. *Let $u(x,t)$ and $v(x,t)$ be two solutions of Eq.* (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively. Suppose that $\sigma(x) \geq \sigma^- \geq 2$ and $a_i(x) \in C^1(\overline{\Omega})$ is a nonnegative function satisfying* (2.8a) *such that*

$$\int_\Omega a_i(x) \left| \frac{\sum_{k=1}^N \prod_{j=1,\, j\neq k}^N a_j(x) a_{kx_i}}{\prod_{j=1}^N a_j(x)} \right|^{p(x)} dx < \infty, \quad i = 1,2,\cdots,N. \tag{2.10}$$

If there is a positive constant c such that

$$|b_i(u,x,t) - b_i(v,x,t)| \leq c a_i(x) |u - v|, \tag{2.11}$$

then the stability (2.9) *is true without any boundary value condition.*

Condition (2.11) implies that Eq. (1.1) can not be of the hyperbolic characteristic. One of our motivations on condition (2.11) initially comes from the study of a model of strong degenerate parabolic equation arising in mathematical finance, which indicates that condition (2.11) is important and indispensable in the decision theory under the risk. For more details on the model of strong degenerate parabolic equation, one can refer to [2]. Note that the condition $\sigma(x) \geq \sigma^- \geq 2$ ensures that $f(u) = |u|^{\sigma(x)-2}u$ is a Lipschitz function.

Theorem 2.3. *Suppose that $p_i(x) \geq p_i^- > 1$ and $\sigma(x) \geq \sigma^- > 1$. Let $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq.* (1.1) *with the different initial values $u_0(x)$ and $v_0(x)$, respectively. If $a_i(x)$ and $b_i(\cdot,x,t)$ satisfy* (2.11), *then there is a constant $\alpha_1 > 1$ such that*

$$\int_\Omega \Big[\prod_{k=1}^N a_k(x) \Big]^{\alpha_1} |u(x,t) - v(x,t)|^2 dx \leq c \int_\Omega \Big[\prod_{k=1}^N a_k(x) \Big]^{\alpha_1} |u(x,0) - v(x,0)|^2 dx. \tag{2.12}$$

From this theorem, we can see that the uniqueness of weak solution of Eq. (1.1) with the initial condition (1.4a) is true, when

$$\prod_{k=1}^N a_k(x) = 0$$

holds on the boundary $\partial\Omega$.

As we know, for solving a given differential equation, it is important to find a suitable definite condition. For example, consider the well-known heat equation

$$u_t = \Delta u, \quad (x,t) \in \Omega \times (0,T),$$

in addition to the initial condition

$$u(x,0)=u_0(x), \quad x\in\Omega,$$

where $u_0(x)$ denotes the initial temperature. One of the following boundary value conditions should be imposed

(i) the Dirichlet condition

$$u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T).$$

(ii) the Neumann condition

$$\frac{\partial u}{\partial n}=0, \quad (x,t)\in\partial\Omega\times(0,T),$$

where n is the outer normal vector of Ω.

(iii) the Robin condition

$$\frac{\partial u}{\partial n}+ku=0, \quad (x,t)\in\partial\Omega\times(0,T),$$

where k is a positive constant.

Theoretically, all these conditions are the so-called definite conditions. But, for a degenerate parabolic equation, the definite conditions generally become more complicated. For example, for the degenerate heat conduction equation

$$u_t=\mathrm{div}(a(x,t)\nabla u),$$

where $a(x,t)\geq 0$, or for the nonlinear heat conduction equation

$$u_t=\mathrm{div}(k(x,t,u)\nabla u), \tag{2.13}$$

where $k(x,t,u)\geq 0$, the above three boundary value conditions (i)-(iii) may be overdetermined. While, for a hyperbolic-parabolic mixed type equation

$$u_t=\mathrm{div}(k(x,t,u)\nabla u)+\mathrm{div}(\vec{b}(u)),$$

in order to obtain the uniqueness of weak solution, apart from one of the above three boundary value conditions, the entropy condition should be imposed accordingly [11, 14]. From the above example, we can see that the boundary value condition (1.4b) plays a crucial role to ensure the well-posedness of weak solution for parabolic equations.

In our previous work [23], we showed that the condition

$$a_i(x)=0, \quad x\in\partial\Omega \quad \text{and} \quad a_i(x)>0, \quad x\in\Omega, \quad i=1,2,\cdots,N, \tag{2.14}$$

can take the place of the boundary value condition (1.4b) for the stability of weak solutions of the evolutionary $\vec{p}(x)-$Laplacian equations. Such a fact is quite easy to understand for the equation

$$u_t=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a_i(x)\,|u_{x_i}|^{p_i(x)-2}\,u_{x_i}\right),\quad (x,t)\in\Omega\times(0,T). \tag{2.15}$$

If $u_1,u_2\in L^1(0,T;W^{1,\vec{p(x)}}(\Omega))$ are two weak solutions of Eq. (2.15) with the initial values $u_{10}(x)$ and $u_{20}(x)$ respectively, since $a_i(x)$ satisfies (2.14), we can deduce that

$$\int_\Omega |u_1(x,t)-u_2(x,t)|^2dx\le\int_\Omega |u_{10}(x)-u_{20}(x)|^2dx. \tag{2.16}$$

In this study, we will extend main results in [23] to Eq. (1.1) under the weaker condition

$$\prod_{i=1}^{N}a_i(x)=0,\quad x\in\partial\Omega\quad\text{and}\quad\prod_{i=1}^{N}a_i(x)>0,\quad x\in\Omega. \tag{2.17}$$

As we can see, even for the simple case like Eq. (2.15), condition (2.17) is not a sufficient condition for (2.16), if no other appreciate condition is imposed.

Roughly speaking, the Laplacian operator Δ represents the difference between the average value of a function in the neighborhood of a point, and its value at that point. Thus, if u stands for the temperature, Δ means whether (and by how much) the material surrounding each point is hotter or colder, on the average, than the material at that point. According to the second law of thermodynamics, heat will flow from hotter bodies to adjacent colder bodies, in proportion to the difference of temperature and of the thermal conductivity of the material between them. When heat flows into (respectively, out of) a material, its temperature increases (respectively, decreases), in proportion to the amount of heat divided by the amount (mass) of material, with a proportionality factor called the specific heat capacity of the material. For (2.13), if $k(x,t,\cdot)|_{x\in\partial\Omega}=0$, we conjecture that the free boundary $\{(x,t)\subset\overline{\Omega}\times[0,T):u(x,t)=0\}$ is in the interior of Q_T. If this is the real case, then the boundary value condition becomes unnecessary and so $\Sigma_p=\emptyset$ sounds reasonable.

3 Stability under partial boundary value condition

Lemma 3.1. *Suppose that $u\in\mathbf{W}(Q_T)$ and $u_t\in\mathbf{W}'(Q_T)$. For any continuous function $h(s)$, let*

$$H(s)=\int_0^s h(s)ds.$$

For a.e. $t_1,t_2\in(0,T)$, there holds

$$\int_{t_1}^{t_2}\int_\Omega h(u)u_tdxdt=\left[\int_\Omega (H(u)(x,t_2)-H(u)(x,t_1))dx\right].$$

This is the generalized version of [5, Corollary 2.1].

For a given general characteristic function $\varphi(x)$, we set

$$\varphi_n(x)=\begin{cases} n\varphi(x), & \varphi(x)<\dfrac{1}{n},\\ 1, & \varphi(x)\geq \dfrac{1}{n}, \end{cases}$$

where n is a large positive integer.

Lemma 3.2. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq.* (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively, under the partial boundary value condition*

$$u(x,t)=v(x,t)=0,\quad (x,t)\in\Sigma_p\times(0,T). \tag{3.1}$$

Suppose that $b_i(s,x,t)$ is a Lipschitz function, $a_i(x)$ satisfies (1.9) *and there are a general characteristic function $\varphi(x)$ and a constant c such that*

$$n^{\frac{p_i^+}{p_i^-}}\left(\int_{\Omega\setminus D_n} a_i(x)\left|\frac{\partial\varphi}{\partial x_i}\right|^{p_i(x)}dx\right)^{\frac{1}{p_i^-}}\leq c,\quad i=1,2,\cdots,N. \tag{3.2}$$

If $\sigma(x)\geq\sigma^-\geq 2$, then we have

$$\int_\Omega |u(x,t)-v(x,t)|dx\leq c\int_\Omega |u_0(x)-v_0(x)|dx,\quad a.e.\ \ t\in[0,T),$$

where $D_n=\left\{x\in\Omega:\varphi(x)>\frac{1}{n}\right\}$ for the sufficiently large n and

$$\Sigma_p=\bigcup_{i=1}^N\{x\in\partial\Omega:\varphi_{x_i}\neq 0\}. \tag{3.3}$$

Proof. For any given positive integer n, we let

$$g_n(s)=\int_0^s h_n(\tau)d\tau,\quad h_n(s)=2n(1-\mid ns\mid)_+.$$

Clearly, $h_n(s)\in C(\mathbb{R})$ and

$$h_n(s)\geq 0,\quad \mid sh_n(s)\mid\leq 1,\quad \mid g_n(s)\mid\leq 1,\quad \lim_{n\to\infty}g_n(s)=\text{sgn}s,\quad \lim_{n\to\infty}sg_n'(s)=0.$$

By the limit process, we can choose the test function as $\chi_{s,t}\varphi_n g_n(u-v)$, where $[s,t]\subseteq$

$(0,T)$, and $\chi_{s,t}$ is its characteristic function on $[\tau,s]$. Then, we get

$$\begin{aligned}
&\int_s^t\int_\Omega \varphi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\
&\quad+\sum_{i=1}^N\int_s^t\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i})(u-v)_{x_i}g_n'(u-v)\varphi_n(x)dxdt\\
&\quad+\sum_{i=1}^N\int_s^t\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i})g_n(u-v)\varphi_{nx_i}dxdt\\
&\quad+\sum_{i=1}^N\int_s^t\int_\Omega (b_i(u,x,t)-b_i(v,x,t))\cdot(u-v)_{x_i}g_n'(u-v)\varphi_n(x)dxdt\\
&\quad+\sum_{i=1}^N\int_s^t\int_\Omega (b_i(u,x,t)-b_i(v,x,t))\cdot g_n(u-v)\varphi_{nx_i}(x)dxdt\\
=&-\int_s^t\int_\Omega b(x,t)\left(|u|^{\sigma(x)-2}u-|v|^{\sigma(x)-2}v\right)\varphi_n g_n(u-v)dxdt. \qquad (3.4)
\end{aligned}$$

As $n\to\infty$, it follows from Lemma 3.1 that

$$\begin{aligned}
&\lim_{n\to\infty}\int_s^t\int_\Omega \varphi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\
=&\lim_{n\to\infty}\int_s^t\int_\Omega \frac{\partial(\varphi_n(x)G_n(u-v))}{\partial t}dxdt\\
=&\lim_{n\to\infty}\int_\Omega \varphi_n(x)[G_n(u-v)(x,t)-G_n(u-v)(x,s)]dx\\
=&\int_\Omega |u-v|(x,t)dx-\int_\Omega |u-v|(x,s)dx. \qquad (3.5)
\end{aligned}$$

Let

$$D_n=\left\{x\in\Omega:\varphi(x)>\frac{1}{n}\right\}\quad\text{and}\quad q_i(x)=\frac{p_i(x)}{p_i(x)-1}.$$

In view of

$$|\varphi_{nx_i}|=n|\varphi_{x_i}|\quad \text{for}\quad x\in\Omega\setminus D_n,$$

without loss the generality, we assume that

$$\|n[a_i(x)]^{\frac{1}{p_i(x)}}g_n(u-v)\varphi_{x_i}\|_{L^{p_i(x)}(\Omega\setminus D_n)}>1.$$

It follows from condition (3.2) that

$$\begin{aligned}
&\|n[a_i(x)]^{\frac{1}{p_i(x)}}g_n(u-v)\varphi_{x_i}\|_{L^{p_i(x)}(\Omega\setminus D_n)}\\
\le&\|n[a_i(x)]^{\frac{1}{p_i(x)}}\varphi_{x_i}\|_{L^{p_i(x)}(\Omega\setminus D_n)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_{\Omega \setminus D_n} a_i(x) n^{p_i(x)} \left| \frac{\partial \varphi}{\partial x_i} \right|^{p_i(x)} dx \right)^{\frac{1}{p_i^-}} \\
&\leq n^{\frac{p_i^+}{p_i^-}} \left(\int_{\Omega \setminus D_n} a_i(x) \left| \frac{\partial \varphi}{\partial x_i} \right|^{p_i(x)} dx \right)^{\frac{1}{p_i^-}} \leq c.
\end{aligned}
$$

By letting $p(x) = q_i(x)$ in (iii) of Lemma 2.1, we further have

$$
\begin{aligned}
&\left| \int_{\Omega} a_i(x) (|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) \varphi_{n x_i} g_n(u-v) dx \right| \\
= &\left| \int_{\Omega \setminus D_n} a_i(x) (|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) \varphi_{n x_i} g_n(u-v) dx \right| \\
\leq &\left\| [a_i(x)]^{\frac{p_i(x)-1}{p_i(x)}} (|u_{x_i}|^{p_i(x)-1} + |v_{x_i}|^{p_i(x)-1}) \right\|_{L^{q_i(x)}(\Omega \setminus D_n)} \\
&\cdot \left\| n [a_i(x)]^{\frac{1}{p_i(x)}} g_n(u-v) \varphi_{x_i} \right\|_{L^{p_i(x)}(\Omega \setminus D_n)} \\
\leq &c \left[\left(\int_{\Omega \setminus D_n} a_i(x) |u_{x_i}|^{p_i(x)} dx \right)^{\frac{1}{q_i^+}} + \left(\int_{\Omega \setminus D_n} a_i(x) |v_{x_i}|^{p_i(x)} dx \right)^{\frac{1}{q_i^+}} \right] \to 0 \quad \text{as } n \to \infty. \quad (3.6)
\end{aligned}
$$

Considering the convection term, from condition (1.9) we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \left| \int_{\left\{x \in \Omega : |u-v| < \frac{1}{n}\right\}} \varphi_n [b_i(u,x,t) - b_i(v,x,t)] g_n{}'(u-v)(u-v)_{x_i} dx \right| \\
\leq &c \lim_{n\to\infty} \int_{\left\{x \in \Omega : |u-v| < \frac{1}{n}\right\}} \left| \frac{b_i(u,x,t) - b_i(v,x,t)}{u-v} (u-v)_{x_i} \right| dx \\
\leq &c \lim_{n\to\infty} \left\| [a_i(x)]^{-\frac{1}{p_i(x)}} \frac{b_i(u,x,t) - b_i(v,x,t)}{u-v} \right\|_{L^{q_i(x)}(\Omega_n)} \left\| [a_i(x)]^{\frac{1}{p_i(x)}} (u-v)_{x_i} \right\|_{L^{p_i(x)}(\Omega_n)} \\
\leq &c \lim_{n\to\infty} \left\{ \int_{\left\{x \in \Omega : |u-v| < \frac{1}{n}\right\}} [a_i(x)]^{\frac{1}{1-p_i(x)}} \left| \frac{b_i(u,x,t) - b_i(v,x,t)}{u-v} \right|^{\frac{p_i(x)}{p_i(x)-1}} dx \right\}^{\frac{1}{q_{i1}^+}} \\
&\cdot \left\{ \int_{\left\{x \in \Omega : |u-v| < \frac{1}{n}\right\}} a_i(x) |(u-v)_{x_i}|^{p_i(x)} dx \right\}^{\frac{1}{p_{i1}}} = 0, \qquad (3.7)
\end{aligned}
$$

in which p_{i1} is taken to be p_i^- (or p_i^+) if

$$
\left\| [a_i(x)]^{\frac{1}{p_i(x)}} |u_{x_i} - v_{x_i}| \right\|_{L^{p_i(x)}(\Omega_n)} > 1 \quad (\text{or } \leq 1),
$$

respectively, where

$$\Omega_n = \left\{ x \in \Omega : |u(x,t) - v(x,t)| < \frac{1}{n} \right\}.$$

Simultaneously q_{i1} is taken to be q_i^+ (or q_i^-) if

$$\left\| [a_i(x)]^{-\frac{1}{p_i(x)}} \frac{b_i(u,x,t) - b_i(v,x,t)}{u-v} \right\|_{L^{q_i(x)}(\Omega_n)} > 1 \quad (\text{or} \quad \le 1),$$

respectively.

Why is the limit of (3.7) equal to zero? Here we give a brief explanation for clarity. Denote by

$$\lim_{n\to\infty} \Omega_n = \Omega_0 = \{ x \in \Omega : u(x,t) = v(x,t) \}.$$

If Ω_0 has a positive measure, then

$$\left\{ \int_{\left\{x\in\Omega:|u-v|<\frac{1}{n}\right\}} [a_i(x)]^{\frac{1}{1-p_i(x)}} \left| \frac{b_i(u,x,t) - b_i(v,x,t)}{u-v} \right|^{\frac{p_i(x)}{p_i(x)-1}} dx \right\}^{\frac{1}{q_{i1}^+}} \le c,$$

$$\lim_{n\to\infty} \left\{ \int_{\left\{x\in\Omega:|u-v|<\frac{1}{n}\right\}} a_i(x)|(u-v)_{x_i}|^{p_i(x)} dx \right\}^{\frac{1}{p_{i1}}} = \left\{ \int_{\Omega_0} a_i(x)|(u-v)_{x_i}|^{p_i(x)} dx \right\}^{\frac{1}{p_{i1}}} = 0.$$

If Ω_0 is with a zero measure, then

$$\begin{aligned} &\left\{ \int_{\left\{x\in\Omega:|u-v|<\frac{1}{n}\right\}} a_i(x)|(u-v)_{x_i}|^{p_i(x)} dx \right\}^{\frac{1}{p_{i1}}} \le c, \\ &\lim_{n\to\infty} \left\{ \int_{\left\{x\in\Omega:|u-v|<\frac{1}{n}\right\}} [a_i(x)]^{\frac{1}{1-p_i(x)}} \left| \frac{b_i(u,x,t) - b_i(v,x,t)}{u-v} \right|^{\frac{p_i(x)}{p_i(x)-1}} dx \right\}^{\frac{1}{q_{i1}^+}} \\ &\le c \left\{ \int_{\Omega_0} [a_i(x)]^{\frac{1}{1-p_i(x)}} dx \right\}^{\frac{1}{q_{i1}^+}} = 0. \end{aligned}$$

For either case, we can see that the limit of (3.7) is zero.

Meanwhile, since on the part of the boundary there have

$$\begin{aligned} &\Sigma_p = \bigcup_{i=1}^{N} \{ x \in \partial\Omega : \varphi_{x_i} \neq 0 \}, \\ &u(x,t) = v(x,t) = 0, \quad x \in \Sigma_p, \end{aligned}$$

we find

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega |b_i(u,x,t)-b_i(v,x,t)||g_n(u-v)\varphi_{nx_i}|dx\\=&c\lim_{n\to\infty} n\int_{\Omega\setminus D_n}|b_i(u,x,t)-b_i(v,x,t)||\varphi_{x_i}|dx\\=&\int_{\partial\Omega}|b_i(u,x,t)-b_i(v,x,t)|\varphi_{x_i}d\Sigma=0.\end{aligned}\tag{3.8}$$

If $\sigma(x)\geq 2$, then

$$\begin{aligned}&\left|-\int_s^t\int_\Omega b(x,t)\left(|u|^{\sigma(x)-2}u-|v|^{\sigma(x)-2}v\right)\varphi_n g_n(u-v)dxdt\right|\\\leq& c\int_s^t\int_\Omega |u(x,\tau)-v(x,\tau)|\,dxd\tau.\end{aligned}\tag{3.9}$$

As $n\to\infty$ in (3.4), we arrive at

$$\begin{aligned}&\int_\Omega |u(x,t)-v(x,t)|\,dx\\\leq&\int_\Omega |u(x,s)-v(x,s)|\,dx+c\int_s^t\int_\Omega |u(x,\tau)-v(x,\tau)|dxd\tau\quad \text{a.e. } t\in[0,T).\end{aligned}$$

Letting $s\to 0$, we obtain

$$\int_\Omega |u(x,t)-v(x,t)|\,dx\leq c\int_\Omega |u_0(x)-v_0(x)|\,dx\quad \text{a.e. } t\in[0,T).$$

Thus, we complete the proof. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. From the above analyses, we can see that the proof of Theorem 2.1 can be processed in a straightforward way by choosing the test function

$$\varphi(x)=\prod_{j=1}^N a_j(x),\tag{3.10}$$

as we discussed in the proof of Lemma 3.2. Since $a_i(x)$ satisfies (2.8a), by (3.10) and (2.8b), it is easy to verify that condition (3.2) holds. By using (3.10), condition (2.7) described in Theorem 2.1 becomes the same as the partial boundary value condition (3.3). Thus, Theorem 2.1 follows from Lemma 3.2 immediately. □

Theorem 3.1. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq.* (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively, under the partial boundary value condition* (3.1). *If $\sigma(x)\geq\sigma^-\geq 2$, $a_i(x)=a(x)$ $(i=1,2,\cdots,N)$, $a(x)\in C^1(\overline{\Omega})$ satisfies* (1.9) *and*

$$a(x)=0,\quad x\in\partial\Omega,\tag{3.11}$$

and $b_i(s,x,t)$ is a Lipschitz function, then we have

$$\int_\Omega |u(x,t)-v(x,t)|dx \le c\int_\Omega |u_0(x)-v_0(x)|dx, \quad a.e. \ t\in[0,T),$$

where

$$\Sigma_p=\bigcup_{i=1}^N\{x\in\partial\Omega : a_{x_i}\neq 0\}. \tag{3.12}$$

Proof. We now choose the test function $\varphi(x)=a(x)^\beta$ with $\beta\ge P_+^+-1$ in Lemma 3.2. Then the partial boundary value condition (3.12) is the same as (3.3). Meanwhile, in the proof of Lemma 3.2, we notice that condition (3.2) plays a pivotal role to ensure inequality (3.6). When $\varphi(x)=a(x)^\beta$ with $\beta\ge P_+^+-1$, we can deduce

$$\begin{aligned}
&\|n[a_i(x)]^{\frac{1}{p_i(x)}}g_n(u-v)\varphi_{x_i}\|_{L^{p_i(x)}(\Omega\backslash D_n)}\\
\le&\|n[a(x)]^{\frac{1}{p_i(x)}}\varphi_{x_i}\|_{L^{p_i(x)}(\Omega\backslash D_n)}\\
\le&\left(\int_{\Omega\backslash D_n}a(x)n^{p_i(x)}\left|a^\beta\right|^{\frac{1}{\beta}(1+p_i(x)(\beta-1))}dx\right)^{\frac{1}{p_i^-}}\\
\le&c\left(\int_{\Omega\backslash D_n}n^{p_i(x)-\frac{1}{\beta}-p_i(x)+\frac{p_i(x)}{\beta}}dx\right)^{\frac{1}{p_i^-}}\\
\le&c\Big(n^{\frac{p_i^+-1}{\beta}-1}\Big)^{\frac{1}{p_i^-}}\le c.
\end{aligned}$$

So, inequality (3.6) holds too. The rest of the proof is closely similar to that of Lemma 3.2. To avoid needless repetition, we omit it. □

4 Stability without partial boundary value condition

Lemma 4.1. *Let $u(x,t)$ and $v(x,t)$ be two solutions of Eq.* (1.1) *with the initial values $u_0(x)$, $v_0(x)$ respectively. If $\sigma(x)\ge\sigma^-\ge 2$ and $\varphi(x)\in C^1(\overline{\Omega})$ is a general characteristic function of Ω such that*

$$\int_\Omega a_i(x)\left|\frac{\varphi_{x_i}}{\varphi}\right|^{p_i(x)}dx<\infty, \quad i=1,2,\cdots,N, \tag{4.1}$$

and there are nonnegative functions $g_i(x)$, $i=1,2,\cdots,N$, such that

$$|b_i(u,x,t)-b_i(v,x,t)|\le cg_i(x)|u-v|, \tag{4.2a}$$

$$\int_\Omega\left|\frac{g_i(x)\varphi_{x_i}}{\varphi}\right|dx<\infty, \quad \int_\Omega g_i(x)^{q_i(x)}[a(x)]^{-\frac{1}{p_i(x)-1}}dx<\infty, \quad i=1,2,\cdots,N, \tag{4.2b}$$

then we have

$$\int_\Omega|u(x,t)-v(x,t)|dx\le c\int_\Omega|u(x,0)-v(x,0)|dx, \quad a.e. \ t\in[0,T). \tag{4.3}$$

Proof. By a process of limit, the test function φ can be chosen as $\varphi = \chi_{s,t} g_n(\varphi(u-v))$, where $[s,t] \subseteq (0,T)$, and $\chi_{s,t}$ is its characteristic function on $[\tau,s]$. Then we have

$$\begin{aligned}
&\int_s^t \int_\Omega g_n(\varphi(u-v)) \frac{\partial(u-v)}{\partial t} dxdt \\
&+\sum_{i=1}^N \int_s^t \int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i})(u-v)_{x_i} g_n{}'(\varphi(u-v))\varphi(x) dxdt \\
&+\sum_{i=1}^N \int_s^t \int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i})(u-v) g_n{}'(\varphi(u-v))\varphi_{x_i} dxdt \\
&+\sum_{i=1}^N \int_s^t \int_\Omega (b_i(u,x,t) - b_i(v,x,t)) \cdot (u-v)_{x_i} g_n{}'(\varphi(u-v))\varphi(x) dxdt \\
&+\sum_{i=1}^N \int_s^t \int_\Omega (b_i(u,x,t) - b_i(v,x,t)) \cdot (u-v) g_n{}'(\varphi(u-v))\varphi_{x_i}(x) dxdt \\
=&-\int_s^t \int_\Omega b(x,t) \left(|u|^{\sigma(x)-2}u - |v|^{\sigma(x)-2}v\right) g_n(\varphi(u-v)) dxdt. \qquad (4.4)
\end{aligned}$$

As $n \to \infty$, it follows from Lemma 3.1 that

$$\begin{aligned}
&\lim_{n\to\infty} \int_s^t \int_\Omega g_n(\varphi(u-v)) \frac{\partial(u-v)}{\partial t} dxdt \\
=&\lim_{n\to\infty} \int_s^t \int_\Omega \frac{\partial G_n(\varphi(u-v))}{\partial t} dxdt \\
=&\lim_{n\to\infty} \int_\Omega \varphi_n(x)[G_n(\varphi(u-v))(x,s) - G_n(\varphi(u-v))(x,\tau)] dx \\
=&\int_\Omega |u-v|(x,t) dx - \int_\Omega |u-v|(x,s) dx. \qquad (4.5)
\end{aligned}$$

We further have

$$\begin{aligned}
&\left| \int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) \varphi_{x_i} g_n{}'(\varphi(u-v))(u-v) dx \right| \\
=&\left| \int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i}) \varphi_{x_i} g_n{}'(\varphi(u-v))(u-v) dx \right| \\
\leq&\left\| [a_i(x)]^{\frac{p_i(x)-1}{p_i(x)}} (|u_{x_i}|^{p_i(x)-1} + |v_{x_i}|^{p_i(x)-1}) \right\|_{L^{q_i(x)}(\Omega)} \\
&\cdot \left\| n[a_i(x)]^{\frac{1}{p_i(x)}} g_n{}'(\varphi(u-v))(u-v)\varphi \frac{\varphi_{x_i}}{\varphi} \right\|_{L^{p_i(x)}(\Omega)} \to 0 \quad \text{as } n \to \infty. \qquad (4.6)
\end{aligned}$$

Specifically, if $\{x\in\Omega: u-v=0\}$ has zero measure, then

$$\begin{aligned}&\lim_{n\to 0}\int_{\{\Omega:\varphi|u-v|<\frac{1}{n}\}} a_i(x)(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)})dx\\ =&\int_{\{\Omega:|u-v|=0\}} a_i(x)(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)})dx=0.\end{aligned}$$

In view of (4.1) and the fact $|\varphi(u-v)g_n'(\varphi(u-v))|\le c$, we get

$$\int_{\{\Omega:\varphi|u-v|<\frac{1}{n}\}}\left|[a_i(x)]^{\frac{1}{p_i(x)}}\varphi(u-v)g_n'(\varphi(u-v))\frac{\varphi_{x_i}}{\varphi}\right|^{p_i(x)}dx\le c, \tag{4.7a}$$

$$\lim_{n\to 0}\left|\int_\Omega a_i(x)(u-v)g_n'(\varphi(u-v))\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)\varphi_{x_i}dx\right|=0. \tag{4.7b}$$

If $\{x\in\Omega: u-v=0\}$ has a positive measure, it follows from (4.1) and the Lebesgue dominated convergence theorem that

$$\lim_{n\to 0}\int_{\{\Omega:\varphi|u-v|<\frac{1}{n}\}}\left|a_i^{\frac{1}{p_i(x)}}\varphi(u-v)g_n'(\varphi(u-v))\frac{\varphi_{x_i}}{\varphi}\right|^{p_i(x)}dx=0.$$

Since

$$\int_{\{\Omega:\varphi|u-v|<\frac{1}{n}\}} a_i(x)(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)})dx\le\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)})dx\le c,$$

we see that (4.7b) is also true. We further derive

$$\lim_{n\to\infty}\left|\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i})\varphi_{x_i}(u-v)g_n'(\varphi(u-v))dx\right|=0. \tag{4.8}$$

Considering the convection term, from (4.2b) we have

$$\begin{aligned}&\left|\int_s^t\int_\Omega\varphi(x)[b_i(x,t,u)-b_i(x,t,v)]g_n'(\varphi(u-v))(u-v)_{x_i}dxdt\right|\\ \le& c\int_s^t\int_\Omega\left|g_i(x)\varphi(x)(u-v)g_n'(\varphi(u-v))\right|\left|(u-v)_{x_i}\right|dxdt\\ \le& c\int_s^t\left(\int_{\Omega\setminus\Omega_n}a_i(x)\left(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)}\right)dx\right)^{\frac{1}{p_{i1}}}\\ &\cdot\left(\int_{\Omega\setminus\Omega_n}\left|g_i(x)a_i(x)^{\frac{-1}{p_i(x)}}\varphi(x)(u-v)g_n'(\varphi(u-v))\right|^{\frac{p_i(x)}{p_i(x)-1}}dx\right)^{\frac{1}{q_i^+}}dt\\ \to& 0\quad\text{as}\ \ n\to\infty,\end{aligned} \tag{4.9}$$

where

$$\Omega_n=\left\{x\in\Omega:|u(x,t)-v(x,t)|<\frac{1}{n}\right\},$$

and p_{i1} is taken to be p_i^- (or p_i^+) if

$$\left\|[a_i(x)]^{\frac{1}{p_i(x)}}|u_{x_i}|^{p_i(x)-1}\right\|_{L^{p_i(x)}(\Omega_n)}>1 \quad (\text{or}\quad \le 1),$$

respectively. Using

$$\int_0^T\int_\Omega\left|\frac{g_i(x)\varphi_{x_i}}{\varphi}\right|dxdt\le c,$$

we get

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_s^t\int_\Omega[b_i(x,t,u)-b_i(x,t,v)]g_n'(\varphi(u-v))\varphi_{x_i}(u-v)dxdt\right|\\ =&\lim_{n\to\infty}c\int_s^t\int_\Omega|\varphi(u-v)g_n'(\varphi(u-v))|\left|\frac{g_i(x)\varphi_{x_i}}{\varphi}\right|dxdt=0.\end{aligned}\tag{4.10}$$

Since $\sigma(x)\ge 2$, it gives

$$\begin{aligned}&\lim_{n\to\infty}\left|-\int_s^t\int_\Omega b(x,t)\left(|u|^{\sigma(x)-2}u-|v|^{\sigma(x)-2}v\right)g_n(\varphi(u-v))dxdt\right|\\ \le& c\int_s^t\int_\Omega|u(x,t)-v(x,t)|dxdt.\end{aligned}\tag{4.11}$$

Let $n\to\infty$ in (4.4). Combining (4.5)-(4.6) and (4.8)-(4.11), we have

$$\begin{aligned}&\int_\Omega|u(x,t)-v(x,t)|\,dx\\ \le&\int_\Omega|u_0(x)-v_0(x)|\,dx+c\int_s^t\int_\Omega b(x,t)|u(x,t)-v(x,t)|dxdt.\end{aligned}$$

By virtue of Gronwall's inequality, we arrive at the desired result. □

Proof of Theorem 2.2. The proof of Theorem 2.2 can be processed by choosing the test function

$$\varphi(x)=\prod_{j=1}^N a_j(x),$$

as we did in the proof of Lemma 4.1. Since $a_i(x)$ satisfies (2.10), we can see that condition (4.1) is true. By virtue of (2.10) and (2.11), we can verify that conditions (4.2a)-(4.2b) are also true. Consequently, Theorem 2.2 follows from Lemma 4.1 immediately. □

From Theorem 2.2, we can derive the following corollary.

Corollary 4.1. *Let $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq.* (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively. If $\sigma(x) \geq \sigma^- \geq 2$, $a_i(x) = a(x)$, $i = 1,\cdots,N$, $a(x)$ satisfies* (3.11) *and*

$$\int_\Omega a(x)^{-(p_i(x)-1)}dx < \infty, \quad i = 1,\cdots,N,$$

and $b_i(s,x,t)$ is a Lipschitz function and satisfies (2.11)*, then there holds*

$$\int_\Omega |u(x,t) - v(x,t)|dx \leq c\int_\Omega |u_0(x) - v_0(x)|dx \quad a.e.\ t \in [0,T).$$

5 Local stability

Lemma 5.1. *Suppose that $p_i(x) \geq p_i^- > 1$, $\sigma(x) \geq \sigma^- > 1$, and $u(x,t)$ and $v(x,t)$ be two weak solutions of Eq.* (1.1) *with the different initial values $u_0(x)$ and $v_0(x)$, respectively. If $a_i(x)$ and $b_i(\cdot,x,t)$ satisfy*

$$|b_i(u,x,t) - b_i(v,x,t)| \leq cg_i(x)|u - v|, \qquad i = 1,\cdots,N, \tag{5.1a}$$

$$\left|a_i(x)^{-1}g_i(x)\right| \leq c, \qquad i = 1,\cdots,N, \tag{5.1b}$$

and there is a general characteristic function $\varphi(x)$ such that

$$a_i(x)\varphi^{p_i(x)(\alpha_1-1)} \leq c\varphi(x)^{\alpha_1}, \quad i = 1,\cdots,N, \tag{5.2}$$

then we have

$$\int_\Omega \varphi(x)^{\alpha_1}|u(x,t) - v(x,t)|^2dx \leq c\int_\Omega \varphi(x)^{\alpha_1}|u(x,0) - v(x,0)|^2dx, \tag{5.3}$$

where $\alpha_1 > 1$ is a constant.

Proof. Let $\varphi(x)$ be a general characteristic function of Ω, and denote $D_\lambda = \{x \in \Omega : \varphi(x) > \lambda\}$ as before. Set

$$\xi_\lambda = [\varphi(x) - \lambda]_+^{\alpha_1}.$$

For any fixed $\tau, s \in [0,T]$, we may choose $\chi_{[\tau,s]}(u - v)\xi_\lambda$ as a test function, where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau,s]$. Then we have

$$\begin{aligned}
&\int_s^t\int_\Omega (u-v)\xi_\lambda(x)\frac{\partial(u-v)}{\partial t}dxdt\\
&+\sum_{i=1}^N\int_s^t\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i})(u-v)_{x_i}\xi_\lambda(x)dxdt\\
&+\sum_{i=1}^N\int_s^t\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i})(u-v)\xi_{\lambda x_i}dxdt\\
&+\sum_{i=1}^N\int_s^t\int_\Omega (b_i(u,x,t)-b_i(v,x,t))\cdot(u-v)_{x_i}\xi_\lambda(x)dxdt\\
&+\sum_{i=1}^N\int_s^t\int_\Omega (b_i(u,x,t)-b_i(v,x,t))\cdot(u-v)\xi_{\lambda x_i}(x)dxdt\\
=&-\int_s^t\int_\Omega b(x,t)\left(|u|^{\sigma(x)-2}u-|v|^{\sigma(x)-2}v\right)(u-v)\xi_\lambda(x)dxdt. \qquad (5.4)
\end{aligned}$$

Estimating the first term on the left of (5.4) yields

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_s^t\int_\Omega (u-v)\xi_\lambda(x)^{\alpha_1}\frac{\partial(u-v)}{\partial t}dxdt=\frac{1}{2}\int_s^t\int_\Omega \varphi(x)^{\alpha_1}\frac{\partial(u-v)^2}{\partial t}dxdt\\
=&\frac{1}{2}\left[\int_\Omega \varphi(x)^{\alpha_1}|u(x,t)-v(x,t)|^2dx-\int_\Omega \varphi(x)^{\alpha_1}|u(x,s)-v(x,s)|^2dx\right]. \qquad (5.5)
\end{aligned}$$

Estimating the second term on the left of (5.4) for any $i\in\{1,2,\cdots,N\}$, we get

$$\int_s^t\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i})(u-v)_{x_i}\xi_\lambda(x)dxdt\geq 0 \qquad (5.6)$$

and

$$\begin{aligned}
&\int_s^t\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i})(u-v)\xi_{\lambda x_i}dxdt\\
\leq&\left(\int_s^t\int_\Omega a_i(x)\left(|\nabla u|^{p_i(x)}+|\nabla v|^{p_i(x)}\right)dxdt\right)^{\frac{1}{q_{i1}}}\\
&\cdot\left(\int_\tau^s\int_{D_\lambda} a_i(x)\,|\nabla\xi_\lambda|^{p_i(x)}\,|u-v|^{p_i(x)}dxdt\right)^{\frac{1}{p_{i1}}}\\
\leq&\left(\int_s^t\int_\Omega a_i(x)\left(|\nabla u|^{p_i(x)}+|\nabla v|^{p_i(x)}\right)dxdt\right)^{\frac{1}{q_{i1}}}\\
&\cdot\left(\int_\tau^s\int_{D_\lambda} a_i(x)\,|\nabla\varphi|^{(\alpha_1-1)p_i(x)}\,|\nabla\varphi|^{p_i(x)}|u-v|^{p_i(x)}dxdt\right)^{\frac{1}{p_{i1}}}\\
\leq&c\left(\int_\tau^s\int_{D_\lambda} a_i(x)\,|\varphi|^{(\alpha_1-1)p_i(x)}\,|u-v|^{p_i(x)}dxdt\right)^{\frac{1}{p_{i1}}}, \qquad (5.7)
\end{aligned}$$

where q_{i1} is taken to be q_i^+ (or q_i^-) if

$$\left\| [a_i(x)]^{\frac{p_i(x)-1}{p_i(x)}} \left(|u_{x_i}|^{p_i(x)-1} + |v_{x_i}|^{p_i(x)-1} \right) \right\|_{L^{q_i(x)}(\Omega)} > 1 \quad (\text{or} \ \leq 1),$$

respectively, and simultaneously p_{i1} is taken to be p_i^+ (or p_i^-) if

$$\left\| \left| [a_i(x)]^{\frac{1}{p_i(x)}} |\nabla \xi \right| |u-v| \right\|_{L^{p_i(x)}(D_\lambda)} > 1 \quad (\text{or} \ \leq 1),$$

respectively. Using $|\varphi_{x_i}| \leq |\nabla \varphi| \leq c$ leads to

$$\begin{aligned}
&\lim_{\lambda\to 0} \left| \int_s^t \int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i})(u-v)\xi_{\lambda x_i} dxdt \right| \\
\leq& \lim_{\lambda\to 0} c \left(\int_\tau^s \int_{\Omega_\lambda} a_i(x)(\varphi-\lambda)_+^{p_i(x)(\alpha_1-1)} |u-v|^{p_i(x)} dxdt \right)^{\frac{1}{p_{i1}}} \\
&\cdot \left(\int_\tau^s \int_{\Omega_\lambda} a_i(x)(|u_{x_i}|^{p_i(x)} + |v_{x_i}|^{p_i(x)}) dxdt \right)^{\frac{1}{q_{i1}}} \\
\leq& c \left(\int_\tau^s \int_\Omega a_i(x) \varphi^{p_i(x)(\alpha_1-1)} |u-v|^{p_i(x)} dxdt \right)^{\frac{1}{p_{i1}}} \\
\leq& c \left(\int_\tau^s \int_\Omega a_i(x) \varphi^{p_i(x)(\alpha_1-1)} |u-v|^{p_i(x)} dxdt \right)^{\frac{1}{p_{i1}}}.
\end{aligned} \tag{5.8}$$

Denote $\Omega_{i1} = \{x \in \Omega : p_i(x) \geq 2\}$ and $\Omega_{i2} = \{x \in \Omega : 1 < p_i(x) < 2\}$. Since $u, v \in L^\infty$, it follows from (5.2) that

$$a_i(x)\varphi^{p_i(x)(\alpha_1-1)} \leq c\varphi(x)^{\alpha_1}$$

and

$$\begin{aligned}
&\int_\tau^s \int_{\Omega_{i1}} a_i(x)\varphi^{p_i(x)(\alpha_1-1)} |u-v|^{p_i(x)} dxdt \\
\leq& c \int_\tau^s \int_{\Omega_{i1}} a_i(x)\varphi^{p_i(x)(\alpha_1-1)} |u-v|^2 dxdt \\
\leq& c \int_\tau^s \int_\Omega \varphi^{\alpha_1} |u-v|^2 dxdt.
\end{aligned} \tag{5.9}$$

Using Hölder's inequality, we have

$$\begin{aligned}
&\int_\tau^s \int_{\Omega_{i2}} a_i(x)\varphi^{p_i(x)(\alpha_1-1)} |u-v|^{p_i(x)} dxdt \\
\leq& \int_\tau^s \left(\int_{\Omega_{i2}} a_i(x)\varphi^{p_i(x)(\alpha_1-1)} |u-v|^2 dx \right)^{\frac{1}{p_{i2}}} \left(\int_{\Omega_{i2}} a_i(x)\varphi^{p_i(x)(\alpha_1-1)} dx \right)^{\frac{1}{q_{i2}}} dt \\
\leq& c \left(\int_\tau^s \int_\Omega \varphi^{\alpha_1} |u-v|^2 dxdt \right)^{\frac{1}{p_{i2}}},
\end{aligned} \tag{5.10}$$

where p_{i2} is taken to be $(\frac{2}{p_i(x)})^+$ or $(\frac{2}{p_i(x)})^-)$ if

$$\left\|\left[a_i(x)\varphi^{p_i(x)(\alpha_1-1)}\right]^{\frac{p_i(x)}{2}}|u-v|^{p_i(x)}\right\|_{L^{\frac{2}{p_i(x)}}(\Omega_{i2})}>1\quad(\text{or}\quad\leq 1),$$

respectively, and simultaneously q_{i2} is taken be $(\frac{2}{2-p_i(x)})^+$ or $((\frac{2}{2-p_i(x)})^-)$ if

$$\left\|\left[a_i(x)\varphi^{p_i(x)(\alpha_1-1)}\right]^{1-\frac{p_i(x)}{2}}\right\|_{L^{\frac{2}{2-p_i(x)}}(\Omega_{i2})}>1\quad(\text{or}\quad\leq 1),$$

respectively. From (5.7)-(5.10), we get

$$\begin{aligned}&\lim_{\lambda\to 0}\left|\int_s^t\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i})(u-v)\xi_{\lambda x_i}dxdt\right|\\ \leq& c\left(\int_\tau^s\int_\Omega\varphi^{\alpha_1}|u-v|^2dxdt\right)^l\end{aligned}\tag{5.11}$$

for $l\leq 1$. To estimate the third term on the left of (5.4), we use

$$|b_i(u,x,t)-b_i(v,x,t)|\leq cg_i(x)|u-v|\quad\text{and}\quad\left|a_i(x)^{-1}g_i(x)\right|\leq c$$

to derive that

$$\begin{aligned}&\lim_{\lambda\to 0}\left|\int_s^t\int_\Omega(b_i(u,x,t)-b_i(v,x,t))(u-v)_{x_i}\xi_\lambda(x)dxdt\right|\\ =&\left|\int_s^t\int_\Omega(b_i(u,x,t)-b_i(v,x,t))(u-v)_{x_i}\varphi(x)^{\alpha_1}dxdt\right|\\ \leq&\int_s^t\left(\int_\Omega\left|\varphi(x)^{\alpha_1}a_i(x)^{-1}(b_i(u,x,t)-b_i(v,x,t))\right|^{q_i(x)}dx\right)^{\frac{1}{q_{i1}}}\\ &\quad\cdot\left(\int_\Omega a_i(x)(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)})dx\right)^{\frac{1}{p_{i1}}}dt\\ \leq& c\int_s^t\left(\int_\Omega\left|\varphi(x)^{\alpha_1}a_i(x)^{-1}(b_i(u,x,t)-b_i(v,x,t))\right|^{q_i(x)}dx\right)^{\frac{1}{q_{i1}}}dt\\ \leq& c\left(\int_s^t\int_\Omega\left|\varphi(x)^{\alpha_1}a_i(x)^{-1}g_i(x)(u-v)|\right|^{q_i(x)}dxdt\right)^{\frac{1}{q_{i1}}}\\ \leq& c\left(\int_s^t\int_\Omega\varphi(x)^{\alpha_1}|u-v|^2dxdt\right)^{\frac{1}{q_{i1}}}.\end{aligned}\tag{5.12}$$

Meanwhile, it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
&\lim_{\lambda\to 0}\left|\int_s^t\int_\Omega (b_i(u,x,t)-b_i(v,x,t))(u-v)\xi_{\lambda x_i}(x)dxdt\right| \\
\leq &\int_s^t\int_\Omega |b_i(u,x,t)-b_i(v,x,t)||u-v|\alpha_1\varphi(x)^{\alpha_1-1}|\varphi_{x_i}|dxdt \\
\leq &c\int_s^t\int_\Omega |u-v|g_i(x)\varphi(x)^{\alpha_1-1}|\varphi_{x_i}|dxdt \\
\leq &\left(\int_s^t\int_\Omega \varphi(x)^{\alpha_1}|u-v|^2dxdt\right)^{\frac{1}{2}}\left(\int_s^t\int_\Omega \left|g_i(x)\varphi(x)^{\frac{\alpha_1}{2}-1}\right|^2dxdt\right)^{\frac{1}{2}} \\
\leq &c\left(\int_s^t\int_\Omega \varphi(x)^{\alpha_1}|u-v|^2dxdt\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.13}$$

To estimate the last term on the left of (5.4), in view of $\sigma(x)\geq\sigma^->1$, we have

$$\begin{aligned}
&\left|\int_s^t\int_\Omega b(x,t)\left(|u|^{\sigma(x)-2}u-|v|^{\sigma(x)-2}v\right)(u-v)\varphi(x)^{\alpha_1}dxdt\right| \\
\leq &c\left(\int_0^s\int_\Omega \varphi(x)^{\alpha_1}|u(x,t)-v(x,t)|^2dxdt\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.14}$$

By (5.5)-(5.14), letting $\lambda\to 0$ in (5.2) leads to

$$\begin{aligned}
&\int_\Omega \varphi(x)^{\alpha_1}|u(x,t)-v(x,t)|^2dx \\
\leq &\int_\Omega \varphi(x)^{\alpha_1}|u(x,s)-v(x,s)|^2dx+c\left(\int_0^s\int_\Omega \varphi(x)^{\alpha_1}|u(x,t)-v(x,t)|^2dxdt\right)^l,
\end{aligned} \tag{5.15}$$

where $l\leq 1$. Using the generalized Gronwall's inequality, we obtain

$$\int_\Omega \varphi(x)^{\alpha_1}|u(x,t)-v(x,t)|^2dx\leq c\int_\Omega \varphi(x)^{\alpha_1}|u(x,s)-v(x,s)|^2dx.$$

Hence, we arrive at (5.3) immediately by letting $s\to 0$. □

Proof of Theorem 2.3. We choose the characteristic function

$$\varphi(x)=\Big[\prod_{k=1}^N a_k(x)\Big],$$

according to condition (2.11), Theorem 2.3 follows from Lemma 5.1 immediately. □

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