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On Proximal Relations in Transformation Semigroups Arising from Generalized Shifts

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Abstract. For a finite discrete topological space X with at least two elements, a nonempty set Γ, and a map $\varphi : \Gamma \to \Gamma$, $\sigma_\varphi : X^\Gamma \to X^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (for $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$) is a generalized shift. In this text for $\mathcal{S} = \{\sigma_\psi : \psi \in \Gamma^\Gamma\}$ and $\mathcal{H} = \{\sigma_\psi : \Gamma \overset{\psi}{\to} \Gamma \text{ is bijective}\}$ we study proximal relations of transformation semigroups $(\mathcal{S}, X^\Gamma)$ and $(\mathcal{H}, X^\Gamma)$. Regarding proximal relation we prove:

$$P(\mathcal{S}, X^\Gamma) = \{((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) \in X^\Gamma \times X^\Gamma : \exists \beta \in \Gamma\ (x_\beta = y_\beta)\}$$

and $P(\mathcal{H}, X^\Gamma) \subseteq \{((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) \in X^\Gamma \times X^\Gamma : \{\beta \in \Gamma : x_\beta = y_\beta\} \text{ is infinite}\} \cup \{(x,x) : x \in \mathcal{X}\}$.

Moreover, for infinite Γ, both transformation semigroups $(\mathcal{S}, X^\Gamma)$ and $(\mathcal{H}, X^\Gamma)$ are regionally proximal, i.e., $Q(\mathcal{S}, X^\Gamma) = Q(\mathcal{H}, X^\Gamma) = X^\Gamma \times X^\Gamma$, also for sydetically proximal relation we have $L(\mathcal{H}, X^\Gamma) = \{((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) \in X^\Gamma \times X^\Gamma : \{\gamma \in \Gamma : x_\gamma \neq y_\gamma\} \text{ is finite}\}$.

Key Words: Generalized shift, proximal relation, transformation semigroup.

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1 Preliminaries

By a (left topological) transformation semigroup (S, Z, π) or simply (S, Z) we mean a compact Hausdorff topological space Z (phase space), discrete topological semigroup S

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(phase semigroup) with identity e and continuous map $\pi : S \times Z \to Z$ ($\pi(s,z) = sz, s \in S, z \in Z$) such that for all $z \in Z$ and $s,t \in S$ we have $ez = z$, $(st)z = s(tz)$. If S is a discrete topological group too, then we call the transformation semigroup (S,Z), a *transformation group*. We say $(x,y) \in Z \times Z$ is a *proximal pair* of (S,Z) if there exists a net $\{s_\lambda\}_{\lambda\in\Lambda}$ in S with

$$\lim_{\lambda\in\Lambda} s_\lambda x = \lim_{\lambda\in\Lambda} s_\lambda y.$$

We denote the collection of all proximal pairs of (S,Z) by $P(S,Z)$ and call it *proximal relation* on (S,Z), for more details on proximal relations we refer the interested reader to [4,8].

In the transformation semigroup (S,Z) we call $(x,y) \in Z \times Z$ a regionally proximal pair if there exists a net $\{(s_\lambda, x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ in $S \times Z \times Z$ such that

$$\lim_{\lambda\in\Lambda} x_\lambda = x, \quad \lim_{\lambda\in\Lambda} y_\lambda = y \quad \text{and} \quad \lim_{\lambda\in\Lambda} s_\lambda x_\lambda = \lim_{\lambda\in\Lambda} s_\lambda y_\lambda.$$

We denote the collection of all regionally proximal pairs of (S,Z) by $Q(S,Z)$ and call it regionally proximal relation on (S,Z). Obviously we have $P(S,Z) \subseteq Q(S,Z)$. In the transformation group (T,Z), by [9] we call $L(T,Z) = \{(x,y) \in Z \times Z : \overline{T(x,y)} \subseteq P(T,Z)\}$ the syndetically proximal relation of (T,Z) (for details on the interaction of $L(T,Z)$, $Q(T,Z)$ and $P(T,Z)$ with uniform structure of Z see [5,6,9]).

1.1 A collection of generalized shifts as phase semigroup

For nonempty sets X, Γ and self-map $\varphi : \Gamma \to \Gamma$ define the generalized shift $\sigma_\varphi : X^\Gamma \to X^\Gamma$ by $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ ($(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$). Generalized shifts have been introduced for the first time in [2], in addition dynamical and non-dynamical properties of generalized shifts have been studied in several texts like [3] and [7]. It's well-known that if X has a topological structure, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is continuous (when X^Γ equipped with product topology), in addition If X has at least two elements, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is a homeomorphism if and only if $\varphi : \Gamma \to \Gamma$ is bijective.

Convention. In this text suppose X is a finite discrete topological space with at least two elements, Γ is a nonempty set, $\mathcal{X} := X^\Gamma$, and:

- $\mathcal{S} := \{\sigma_\varphi : \varphi \in \Gamma^\Gamma\}$, is the semigroup of generalized shifts on X^Γ,
- $\mathcal{H} := \{\sigma_\varphi : \varphi \in \Gamma^\Gamma$ and $\varphi : \Gamma \to \Gamma$ is bijective$\}$, is the group of generalized shift homeomorphisms on X^Γ.

Equip X^Γ with product (pointwise convergence) topology. Now we may consider $\mathcal{S}$ (resp. $\mathcal{H}$) as a subsemigroup (resp. subgroup) of continuous maps (resp. homeomorphisms) from $\mathcal{X}$ to itself, so $\mathcal{S}$ (resp. $\mathcal{H}$) acts on $\mathcal{X}$ in a natural way.

Our aim in this text is to study $P(T,\mathcal{X})$, $Q(T,\mathcal{X})$, and $L(T,\mathcal{X})$ for $T = \mathcal{H}, \mathcal{S}$. Readers interested in this subject may refer to [1] too.

2 Proximal and regionally proximal relations of $(\mathcal{S},\mathcal{X})$

In this section we prove that

$$P(\mathcal{S},\mathcal{X})=\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\exists\beta\in\Gamma\ (x_\beta=y_\beta)\},$$
$$Q(\mathcal{S},\mathcal{X})=\begin{cases}\mathcal{X}\times\mathcal{X}, & \Gamma \text{ is infinite},\\ P(\mathcal{S},\mathcal{X}), & \Gamma \text{ is finite}.\end{cases}$$

Theorem 2.1. $P(\mathcal{S},\mathcal{X})=\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\exists\beta\in\Gamma\ (x_\beta=y_\beta)\}$.

Proof. First consider $\beta\in\Gamma$ and $(x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}\in\mathcal{X}$ by $x_\beta=y_\beta$. Define $\psi:\Gamma\to\Gamma$ with $\psi(\alpha)=\beta$ for all $\alpha\in\Gamma$. Then

$$\sigma_\psi((x_\alpha)_{\alpha\in\Gamma})=(x_\beta)_{\alpha\in\Gamma}=(y_\beta)_{\alpha\in\Gamma}=\sigma_\psi((y_\alpha)_{\alpha\in\Gamma}),$$
$$((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{S},\mathcal{X}).$$

Conversely, suppose $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{S},\mathcal{X})$. There exists a net $\{\sigma_{\varphi_\lambda}\}_{\lambda\in\Lambda}$ in $\mathcal{S}$ with

$$\lim_{\lambda\in\Lambda}\sigma_{\varphi_\lambda}((x_\alpha)_{\alpha\in\Gamma})=\lim_{\lambda\in\Lambda}\sigma_{\varphi_\lambda}((y_\alpha)_{\alpha\in\Gamma})=:(z_\alpha)_{\alpha\in\Gamma}.$$

Choose arbitrary $\theta\in\Gamma$, then

$$\lim_{\lambda\in\Lambda}x_{\varphi_\lambda(\theta)}=\lim_{\lambda\in\Lambda}y_{\varphi_\lambda(\theta)}=z_\theta$$

in X. Since X is discrete, there exists $\lambda_0\in\Lambda$ such that $x_{\varphi_\lambda(\theta)}=y_{\varphi_\lambda(\theta)}=z_\theta$ for all $\lambda\geq\lambda_0$, in particular for $\beta=\varphi_{\lambda_0(\theta)}$ we have $x_\beta=y_\beta$. □

Lemma 2.1. *For infinite* Γ *we have:* $Q(\mathcal{S},\mathcal{X})=Q(\mathcal{H},\mathcal{X})=\mathcal{X}\times\mathcal{X}$.

Proof. Suppose Γ is infinite, then there exits a bijection $\mu:\Gamma\times\mathbb{Z}\to\Gamma$, in particular $\{\mu(\{\alpha\}\times\mathbb{Z}):\alpha\in\Gamma\}$ is a partition of Γ to its infinite countable subsets. Define bijection $\varphi:\Gamma\to\Gamma$ by $\varphi(\mu(\alpha,n))=\mu(\alpha,n+1)$ for all $\alpha\in\Gamma$ and $n\in\mathbb{Z}$. Consider $p\in X$ and $(x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}\in\mathcal{X}$. For all $n\geq1$ and $\alpha\in\Gamma$ let:

$$x_\alpha^n:=\begin{cases}x_\alpha, & \alpha=\mu(\beta,k)\ \text{ for some }\ \beta\in\Gamma \text{ and } k\leq n,\\ p, & \text{otherwise},\end{cases}$$
$$y_\alpha^n:=\begin{cases}y_\alpha, & \alpha=\mu(\beta,k)\ \text{ for some }\ \beta\in\Gamma \text{ and } k\leq n,\\ p, & \text{otherwise},\end{cases}$$

then:

$$\lim_{n\to+\infty}(x_\alpha^n)_{\alpha\in\Gamma}=(x_\alpha)_{\alpha\in\Gamma},$$
$$\lim_{n\to+\infty}(y_\alpha^n)_{\alpha\in\Gamma}=(y_\alpha)_{\alpha\in\Gamma},$$
$$\lim_{n\to\infty}\sigma_{\varphi^{2n}}((x_\alpha^n)_{\alpha\in\Gamma})=(p_\alpha)_{\alpha\in\Gamma}=\lim_{n\to+\infty}\sigma_{\varphi^{2n}}((y_\alpha^n)_{\alpha\in\Gamma}).$$

By $\sigma_{\varphi^{2n}}\in\mathcal{H}$ for all $n\geq1$ and using the above statements, we have $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in Q(\mathcal{H},\mathcal{X})\subseteq Q(\mathcal{S},\mathcal{X})$. □

Lemma 2.2. *For finite* Γ *and any subsemigroup* $\mathcal{T}$ *of* $\mathcal{S}$ *we have* $Q(\mathcal{T},\mathcal{X})=P(\mathcal{T},\mathcal{X})$.

Proof. We must only prove $Q(\mathcal{T},\mathcal{X})\subseteq P(\mathcal{T},\mathcal{X})$. Suppose $(x,y)\in Q(\mathcal{T},\mathcal{X})$, then there exists a net $\{(x_\lambda,y_\lambda,t_\lambda)\}_{\lambda\in\Lambda}$ in $\mathcal{X}\times\mathcal{X}\times\mathcal{T}$ such that

$$\lim_{\lambda\in\Lambda}x_\lambda=x,\quad \lim_{\lambda\in\Lambda}y_\lambda=y,$$
$$\lim_{\lambda\in\Lambda}t_\lambda x_\lambda=\lim_{\lambda\in\Lambda}t_\lambda y_\lambda=:z.$$

Since $\mathcal{X}\times\mathcal{X}\times\mathcal{T}$ is finite, $\{(x_\lambda,y_\lambda,t_\lambda)\}_{\lambda\in\Lambda}$ has a constant subnet like $\{(x_{\lambda_\mu},y_{\lambda_\mu},t_{\lambda_\mu})\}_{\mu\in M}$, so there exists $t\in\mathcal{T}$ such that for all $\mu\in M$ we have $x=x_{\lambda_\mu}$, $y=y_{\lambda_\mu}$ and $t=t_{\lambda_\mu}$, therefore $tx=ty(=z)$ and $(x,y)\in P(\mathcal{T},\mathcal{X})$. $\square$

Theorem 2.2. *We have:*

$$Q(\mathcal{S},\mathcal{X})=\begin{cases}\mathcal{X}\times\mathcal{X}, & \Gamma \text{ is infinite},\\ P(\mathcal{S},\mathcal{X}), & \Gamma \text{ is finite}.\end{cases}$$

Proof. Use Lemmas 2.1 and 2.2. $\square$

3 Proximal and regionally proximal relations of $(\mathcal{H},\mathcal{X})$

Note that for finite Γ, $\mathcal{H}$ is a finite subset of homeomorphisms on $\mathcal{X}$ and $P(\mathcal{H},\mathcal{X})=\{(x,x):x\in\mathcal{X}\}$, also using Lemmas 2.1 and 2.2 we have:

$$Q(\mathcal{H},\mathcal{X})=\begin{cases}\mathcal{X}\times\mathcal{X}, & \Gamma \text{ is infinite},\\ P(\mathcal{H},\mathcal{X})=\{(x,x):x\in\mathcal{X}\}, & \Gamma \text{ is finite}.\end{cases}$$

In this section we show that:

$$\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}):\max(\mathrm{card}(\{\beta\in\Gamma:x_\beta\neq y_\beta\}),\aleph_0)\leq \mathrm{card}(\{\beta\in\Gamma:x_\beta=y_\beta\})\}$$

is a subset of $P(\mathcal{H},\mathcal{X})$, which is a subset of

$$\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\beta\in\Gamma:x_\beta=y_\beta\}\text{ is infinite}\}\cup\{(x,x):x\in\mathcal{X}\}$$

in its turn. In particular, for countable Γ we prove

$$P(\mathcal{H},\mathcal{X})=\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\beta\in\Gamma:x_\beta=y_\beta\}\text{ is infinite}\}\cup\{(x,x):x\in\mathcal{X}\}.$$

Lemma 3.1. *For infinite* Γ, *we have:*

$$P(\mathcal{H},\mathcal{X})\subseteq\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\beta\in\Gamma:x_\beta=y_\beta\}\textit{ is infinite}\}.$$

Proof. Consider $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{H},\mathcal{X})$, then there exists a net $\{\sigma_{\varphi_\lambda}\}_{\lambda\in\Lambda}$ in $\mathcal{H}$ with

$$\lim_{\lambda\in\Lambda}\sigma_{\varphi_\lambda}((x_\alpha)_{\alpha\in\Gamma})=\lim_{\lambda\in\Lambda}\sigma_{\varphi_\lambda}((y_\alpha)_{\alpha\in\Gamma})=:(z_\alpha)_{\alpha\in\Gamma}.$$

Choose distinct $\theta_1,\cdots,\theta_n\in\Gamma$. For all $i\in\{1,\cdots,n\}$ we have

$$\lim_{\lambda\in\Lambda}x_{\varphi_\lambda(\theta_i)}=\lim_{\lambda\in\Lambda}y_{\varphi_\lambda(\theta_i)}=z_{\theta_i}\quad\text{in}\quad X,$$

so there exists $\lambda_1,\cdots,\lambda_n\in\Lambda$ with $x_{\varphi_\lambda(\theta_i)}=y_{\varphi_\lambda(\theta_i)}=z_{\theta_i}$ for all $\lambda\geq\lambda_i$. There exists $\mu\in\Lambda$ with $\mu\geq\lambda_1,\cdots,\lambda_n$, thus $x_{\varphi_\mu(\theta_i)}=y_{\varphi_\mu(\theta_i)}$ for $i=1,\cdots,n$. Since $\varphi_\mu:\Gamma\to\Gamma$ is bijective and $\theta_1,\cdots,\theta_n$ are pairwise distinct, $\{\varphi_\mu(\theta_1),\cdots,\varphi_\mu(\theta_n)\}$ has exactly n elements and $\{\varphi_\mu(\theta_1),\cdots,\varphi_\mu(\theta_n)\}\subseteq\{\beta\in\Gamma:x_\beta=y_\beta\}$. Hence $\{\beta\in\Gamma:x_\beta=y_\beta\}$ has at least n elements (for all $n\geq 1$) and it is infinite. □

Theorem 3.1. *We have:*

$$\begin{aligned}P(\mathcal{H},\mathcal{X})\subseteq&\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\beta\in\Gamma:x_\beta=y_\beta\}\text{ is infinite}\}\\&\cup\{(x,x):x\in\mathcal{X}\}.\end{aligned}$$

Proof. Use Lemma 3.1 and the fact that for finite Γ, $\mathcal{H}$ is a finite subset of homeomorphisms on $\mathcal{X}$. So for finite Γ we have $P(\mathcal{H},\mathcal{X})=\{(w,w):w\in\mathcal{X}\}$. □

Lemma 3.2. *For infinite countable* Γ*, we have*

$$P(\mathcal{H},\mathcal{X})=\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\beta\in\Gamma:x_\beta=y_\beta\}\ \textit{is infinite}\}.$$

Proof. Using Lemma 3.1 we must only prove:

$$P(\mathcal{H},\mathcal{X})\supseteq\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\beta\in\Gamma:x_\beta=y_\beta\}\ \text{is infinite}\}.$$

Consider $(x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}\in\mathcal{X}$ with infinite set $\{\beta\in\Gamma:x_\beta=y_\beta\}=\{\beta_1,\beta_2,\cdots\}$ and distinct β_is. Also suppose $\Gamma=\{\alpha_1,\alpha_2,\cdots\}$ with distinct α_is. For all $n\geq 1$ there exists bijection $\varphi_n:\Gamma\to\Gamma$ with $\varphi_n(\alpha_i)=\beta_i$ for $i\in\{1,\cdots,n\}$. Let $\alpha\in\Gamma$, there exists $i\geq 1$ with $\alpha=\alpha_i$. Since for all $n\geq i$ we have

$$x_{\varphi_n(\alpha)}=x_{\varphi_n(\alpha_i)}=x_{\beta_i}=y_{\beta_i}=y_{\varphi_n(\alpha_i)}=y_{\varphi_n(\alpha)},$$

we have

$$\lim_{n\to\infty}x_{\varphi_n(\alpha)}=\lim_{n\to\infty}y_{\varphi_n(\alpha)}.$$

Therefore

$$\begin{aligned}&\lim_{n\to\infty}\sigma_{\varphi_n}((x_\alpha)_{\alpha\in\Gamma})=\lim_{n\to\infty}(x_{\varphi_n(\alpha)})_{\alpha\in\Gamma}=\lim_{n\to\infty}(y_{\varphi_n(\alpha)})_{\alpha\in\Gamma}=\lim_{n\to\infty}\sigma_{\varphi_n}((y_\alpha)_{\alpha\in\Gamma}),\\&((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{H},\mathcal{X}).\end{aligned}$$

Thus, we complete the proof. □

Theorem 3.2. *For countable* Γ,

$$\begin{aligned}P(\mathcal{H},\mathcal{X}) =&\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\beta\in\Gamma:x_\beta=y_\beta\}\ \textit{is infinite}\}\\&\cup\{(x,x):x\in\mathcal{X}\}.\end{aligned}$$

Proof. First note that for finite Γ, $\mathcal{H}$ is finite and $P(\mathcal{H},\mathcal{X})=\{(x,x):x\in\mathcal{X}\}$. Now use Lemma 3.2. □

Lemma 3.3. *For infinite* Γ, *we have:*

$$\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}):\mathrm{card}(\{\beta\in\Gamma:x_\beta\neq y_\beta\})\leq\ \mathrm{card}(\{\beta\in\Gamma:x_\beta=y_\beta\})\}\subseteq P(\mathcal{H},\mathcal{X}).$$

In particular,

$$\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}):\{\beta\in\Gamma:x_\beta\neq y_\beta\}\ \textit{is finite}\}\subseteq P(\mathcal{H},\mathcal{X}).$$

Proof. Suppose Γ is infinite. For $(x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}\in\mathcal{X}$, let:

$$A:=\{\alpha\in\Gamma:x_\alpha=y_\alpha\},\quad B:=\{\alpha\in\Gamma:x_\alpha\neq y_\alpha\}$$

with $\mathrm{card}(B)\leq\mathrm{card}(A)$. There exists a one to one map $\lambda:B\to A$. By $\mathrm{card}(\Gamma)=\mathrm{card}(A)+\mathrm{card}(B)$ and $\mathrm{card}(B)\leq\mathrm{card}(A)$, A is infinite. Since A is infinite, we have $\mathrm{card}(A)=\mathrm{card}(A)\aleph_0$ so there exists a bijection $\varphi:A\times\mathbb{N}\to A$. For all $\theta\in A$ let $K_\theta=\varphi(\{\theta\}\times\mathbb{N})\cup\lambda^{-1}(\theta)$. Thus K_θs are disjoint infinite countable subsets of Γ, as a matter of fact $\{K_\theta:\theta\in A\}$ is a partition of Γ to some of its infinite countable subsets. For all $\theta\in A$, $\{\alpha\in K_\theta:x_\alpha=y_\alpha\}=\varphi(\{\theta\}\times\mathbb{N})$ is infinite and K_θ is infinite countable. By Lemma 3.2 there exists a sequence $\{\psi_n^\theta\}$ of permutations on K_θ such that

$$\lim_{n\to\infty}\sigma_{\psi_n^\theta}(x_\alpha)_{\alpha\in K_\theta}=\lim_{n\to\infty}\sigma_{\psi_n^\theta}(y_\alpha)_{\alpha\in K_\theta}.$$

For all $n\geq 1$ let

$$\psi_n=\bigcup_{\theta\in A}\psi_n^\theta,$$

then $\psi_n:\Gamma\to\Gamma$ is bijective and

$$\lim_{n\to\infty}\sigma_{\psi_n}(x_\alpha)_{\alpha\in\Gamma}=\lim_{n\to\infty}\sigma_{\psi_n}(y_\alpha)_{\alpha\in\Gamma},$$

which completes the proof. □

Theorem 3.3. *The collection* $\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}):\max(\mathrm{card}(\{\beta\in\Gamma:x_\beta\neq y_\beta\}),\aleph_0)\leq\mathrm{card}(\{\beta\in\Gamma:x_\beta=y_\beta\})\}$ *is a subset of* $P(\mathcal{H},\mathcal{X})$.

Proof. If Γ is finite, then

$$\begin{aligned}&\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}):\max(\mathrm{card}(\{\beta\in\Gamma:x_\beta\neq y_\beta\}),\aleph_0)\\\leq&\mathrm{card}(\{\beta\in\Gamma:x_\beta=y_\beta\})\}=\varnothing.\end{aligned}$$

Use Lemma 3.3 to complete the proof. □

4 Syndetically proximal relations of $(\mathcal{H},\mathcal{X})$

In this section we prove:

$$L(\mathcal{H},\mathcal{X})=\begin{cases}\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\gamma\in\Gamma:x_\gamma\neq y_\gamma\}\text{ is finite}\}, & \Gamma\text{ is infinite},\\ \{(x,x):x\in\mathcal{X}\}, & \Gamma\text{ is finite}.\end{cases}$$

Lemma 4.1. *For $(x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma},(u_\alpha)_{\alpha\in\Gamma}\in\mathcal{X}$, and $p,q\in X$ let:*

$$z_\alpha:=\begin{cases}q, & x_\alpha\neq y_\alpha,\\ u_\alpha, & x_\alpha=y_\alpha,\end{cases}\quad\text{and}\quad w_\alpha:=\begin{cases}p, & x_\alpha\neq y_\alpha,\\ u_\alpha, & x_\alpha=y_\alpha.\end{cases}$$

We have:

1). if $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{H},\mathcal{X})$, then $((z_\alpha)_{\alpha\in\Gamma},(w_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{H},\mathcal{X})$,

2). if $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in L(\mathcal{H},\mathcal{X})$, then $((z_\alpha)_{\alpha\in\Gamma},(w_\alpha)_{\alpha\in\Gamma})\in L(\mathcal{H},\mathcal{X})$.

Proof. 1) Suppose $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{H},\mathcal{X})$, then there exists a net $\{\sigma_{\varphi_\lambda}\}_{\lambda\in\Lambda}$ in $\mathcal{H}$ such that

$$\lim_{\lambda\in\Lambda}\sigma_{\varphi_\lambda}((x_\alpha)_{\alpha\in\Gamma})=\lim_{\lambda\in\Lambda}\sigma_{\varphi_\lambda}((y_\alpha)_{\alpha\in\Gamma}).$$

Thus

$$\lim_{\lambda\in\Lambda}((x_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma})=\lim_{\lambda\in\Lambda}((y_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma}),$$

i.e., for all $\alpha\in\Gamma$ there exists $\kappa_\alpha\in\Lambda$ such that:

$$\forall\lambda\geq\kappa_\alpha\ (x_{\varphi_\lambda(\alpha)}=y_{\varphi_\lambda(\alpha)}).$$

Hence, for all $\lambda\geq\kappa_\alpha$ we have $z_{\varphi_\lambda(\alpha)}=u_{\varphi_\lambda(\alpha)}=w_{\varphi_\lambda(\alpha)}$. On the other hand the net $\{(u_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma}\}_{\lambda\in\Lambda}$ has a convergent subnet like $\{(u_{\varphi_{\lambda_\theta}(\alpha)})_{\alpha\in\Gamma}\}_{\theta\in T}$ to a point of $\mathcal{X}$, say $(v_\alpha)_{\alpha\in\Gamma}$, since $\mathcal{X}$ is compact. For all $\alpha\in\Gamma$ there exists $\theta_\alpha\in T$ such that $\lambda_{\theta_\alpha}\geq\kappa_\alpha$, and moreover

$$\forall\theta\geq\theta_\alpha\ (u_{\varphi_{\lambda_\theta}(\alpha)}=v_\alpha).$$

Note that for all $\theta\geq\theta_\alpha$ we have $\lambda_\theta\geq\kappa_\alpha$, leads us to:

$$\forall\theta\geq\theta_\alpha\ (z_{\varphi_{\lambda_\theta}(\alpha)}=v_\alpha=w_{\varphi_{\lambda_\theta}(\alpha)}).$$

Hence

$$\lim_{\theta\in T}\sigma_{\varphi_{\lambda_\theta}}((z_\alpha)_{\alpha\in\Gamma})=\lim_{\theta\in T}\sigma_{\varphi_{\lambda_\theta}}((w_\alpha)_{\alpha\in\Gamma})\quad\text{and}\quad((z_\alpha)_{\alpha\in\Gamma},(w_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{H},\mathcal{X}).$$

2) Now suppose $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in L(\mathcal{H},\mathcal{X})$ and $((s_\alpha)_{\alpha\in\Gamma},(t_\alpha)_{\alpha\in\Gamma})$ is an element of $\overline{\mathcal{H}((z_\alpha)_{\alpha\in\Gamma},(w_\alpha)_{\alpha\in\Gamma})}$. There exists a net $\{\sigma_{\varphi_\lambda}\}_{\lambda\in\Lambda}$ in $\mathcal{H}$, with

$$((s_\alpha)_{\alpha\in\Gamma},(t_\alpha)_{\alpha\in\Gamma})=\lim_{\lambda\in\Lambda}\sigma_{\varphi_\lambda}((z_\alpha)_{\alpha\in\Gamma},(w_\alpha)_{\alpha\in\Gamma})=\lim_{\lambda\in\Lambda}((z_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma},(w_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma}).$$

On the other hand the net $\{((x_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma},(y_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma})\}_{\lambda\in\Lambda}$ has a convergent subnet in compact space $\mathcal{X}\times\mathcal{X}$, without loss of generality we may suppose $\{((x_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma},(y_{\varphi_\lambda(\alpha)})_{\alpha\in\Gamma})\}_{\lambda\in\Lambda}$ itself converges to a point of $\mathcal{X}\times\mathcal{X}$ like $((m_\alpha)_{\alpha\in\Gamma},(n_\alpha)_{\alpha\in\Gamma})$. Hence

$$((m_\alpha)_{\alpha\in\Gamma},(n_\alpha)_{\alpha\in\Gamma})\in\overline{\mathcal{H}((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})}\subseteq P(\mathcal{H},\mathcal{X}).$$

Now for $\alpha\in\Gamma$ there exists $\kappa\in\Lambda$ such that:

$$\forall\lambda\geq\kappa\ ((m_\alpha,n_\alpha)=(x_{\varphi_\lambda(\alpha)},y_{\varphi_\lambda(\alpha)})).$$

Hence we have:

$$\begin{aligned} m_\alpha\neq n_\alpha &\Rightarrow (\forall\lambda\geq\kappa\ (x_{\varphi_\lambda(\alpha)}\neq y_{\varphi_\lambda(\alpha)}))\\ &\Rightarrow (\forall\lambda\geq\kappa\ (z_{\varphi_\lambda(\alpha)}=q\wedge w_{\varphi_\lambda(\alpha)}=p))\\ &\Rightarrow \lim_{\lambda\in\Lambda} z_{\varphi_\lambda(\alpha)}=q\wedge\lim_{\lambda\in\Lambda} w_{\varphi_\lambda(\alpha)}=p\\ &\Rightarrow (s_\alpha,t_\alpha)=(q,p)\end{aligned}$$

and

$$\begin{aligned} m_\alpha=n_\alpha &\Rightarrow (\forall\lambda\geq\kappa\ (x_{\varphi_\lambda(\alpha)}=y_{\varphi_\lambda(\alpha)}))\\ &\Rightarrow (\forall\lambda\geq\kappa\ (z_{\varphi_\lambda(\alpha)}=w_{\varphi_\lambda(\alpha)}))\\ &\Rightarrow s_\alpha=\lim_{\lambda\in\Lambda} z_{\varphi_\lambda(\alpha)}=\lim_{\lambda\in\Lambda} w_{\varphi_\lambda(\alpha)}=t_\alpha\\ &\Rightarrow s_\alpha=t_\alpha.\end{aligned}$$

Hence for $(v_\alpha)_{\alpha\in\Gamma}:=(s_\alpha)_{\alpha\in\Gamma}$, we have:

$$s_\alpha=\begin{cases} q, & m_\alpha\neq n_\alpha,\\ v_\alpha, & m_\alpha=n_\alpha,\end{cases}\quad\text{and}\quad t_\alpha=\begin{cases} p, & m_\alpha\neq n_\alpha,\\ v_\alpha, & m_\alpha=n_\alpha.\end{cases}\tag{4.1}$$

Using 1), $((m_\alpha)_{\alpha\in\Gamma},(n_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{H},\mathcal{X})$ and (4.1) we have $((s_\alpha)_{\alpha\in\Gamma},(t_\alpha)_{\alpha\in\Gamma})\in P(\mathcal{H},\mathcal{X})$, which completes the proof. □

Lemma 4.2. *We have:*

$$L(\mathcal{H},\mathcal{X})\subseteq\{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})\in\mathcal{X}\times\mathcal{X}:\{\gamma\in\Gamma:x_\gamma\neq y_\gamma\}\text{ is finite}\}.$$

Proof. Consider $(x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}\in\mathcal{X}$ such that $B:=\{\alpha\in\Gamma:x_\alpha\neq y_\alpha\}$ is infinite. Choose distinct $p,q\in X$ and let:

$$z_\alpha:=\begin{cases} q, & \alpha\in B,\\ p, & \alpha\notin B.\end{cases}$$

By Lemma 4.1, if $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}) \in \overline{L(\mathcal{H},\mathcal{X})}$, then $((z_\alpha)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma}) \in L(\mathcal{H},\mathcal{X})$. We show $((q)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma}) \in \overline{\mathcal{H}((z_\alpha)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma})}$. Suppose U is an open neighbourhood of $((q)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma})$, then there exists distinct $\alpha_1,\cdots,\alpha_n \in \Gamma$ such that for:

$$V_\alpha = \begin{cases} \{q\}, & \alpha = \alpha_1,\cdots,\alpha_n, \\ X, & \alpha \neq \alpha_1,\cdots,\alpha_n, \end{cases} \quad \text{and} \quad W_\alpha = \{p\}, \quad (\forall \alpha \in \Gamma),$$

we have

$$\prod_{\alpha\in\Gamma} V_\alpha \times \prod_{\alpha\in\Gamma} W_\alpha \subseteq U.$$

Since B is infinite, we could choose distinct $\beta_1,\cdots,\beta_n \in B$ such that $\{\alpha_1,\cdots,\alpha_n\} \cap \{\beta_1,\cdots,\beta_n\} = \varnothing$. Define $\psi : \Gamma \to \Gamma$ by

$$\psi(\alpha) := \begin{cases} \alpha_i, & \alpha = \beta_i, \quad i = 1,\cdots,n, \\ \beta_i, & \alpha = \alpha_i, \quad i = 1,\cdots,n, \\ \alpha, & \text{otherwise}, \end{cases}$$

then $\psi : \Gamma \to \Gamma$ is bijective, $\sigma_\psi \in \mathcal{H}$ and

$$\sigma_\psi((z_\alpha)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma}) = (\sigma_\psi((z_\alpha)_{\alpha\in\Gamma}),\sigma_\psi((p)_{\alpha\in\Gamma})) = ((z_{\psi(\alpha)})_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma}) \in U.$$

Hence $((q)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma}) \in \overline{\mathcal{H}((z_\alpha)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma})}$. Since $((q)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma}) \notin P(\mathcal{H},\mathcal{X})$, we have $((z_\alpha)_{\alpha\in\Gamma},(p)_{\alpha\in\Gamma}) \notin L(\mathcal{H},\mathcal{X})$, which leads to $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}) \notin L(\mathcal{H},\mathcal{X})$ and completes the proof. □

The proof of the following lemma is similar to that of Lemma 3.1.

Lemma 4.3. *For* $((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}) \in \mathcal{X}\times\mathcal{X}$ *if* $\{\alpha \in \Gamma : x_\alpha \neq y_\alpha\}$ *is finite and* $((z_\alpha)_{\alpha\in\Gamma},(w_\alpha)_{\alpha\in\Gamma}) \in \overline{\mathcal{H}((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma})}$, *then* $\{\alpha \in \Gamma : z_\alpha \neq w_\alpha\}$ *is finite satisfying* $\mathrm{card}(\{\alpha\in\Gamma : z_\alpha \neq w_\alpha\}) \leq \mathrm{card}(\{\alpha\in\Gamma : x_\alpha \neq y_\alpha\})$.

Proof. For $n \geq 1$, if there exists distinct $\alpha_1,\cdots,\alpha_n \in \Gamma$ with $z_{\alpha_i} \neq w_{\alpha_i}$ for $i = 1,\cdots,n$, then let:

$$U_\alpha := \begin{cases} \{z_\alpha\}, & \alpha = \alpha_1,\cdots,\alpha_n, \\ X, & \alpha \neq \alpha_1,\cdots,\alpha_n, \end{cases} \quad \text{and} \quad V_\alpha := \begin{cases} \{w_\alpha\}, & \alpha = \alpha_1,\cdots,\alpha_n, \\ X, & \alpha \neq \alpha_1,\cdots,\alpha_n. \end{cases}$$

Thus

$$U := \prod_{\alpha\in\Gamma} U_\alpha \times \prod_{\alpha\in\Gamma} V_\alpha$$

is an open neighbourhood of $((z_\alpha)_{\alpha\in\Gamma},(w_\alpha)_{\alpha\in\Gamma})$, and there exists bijection $\varphi : \Gamma \to \Gamma$ with

$$(\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi((y_\alpha)_{\alpha\in\Gamma})) = ((x_{\varphi(\alpha)})_{\alpha\in\Gamma},(y_{\varphi(\alpha)})_{\alpha\in\Gamma}) \in U.$$

Hence $x_{\varphi(\alpha_i)} = z_{\alpha_i}$ and $y_{\varphi(\alpha_i)} = w_{\alpha_i}$ for all $i = 1,\cdots,n$. Therefore $x_{\varphi(\alpha_i)} \neq y_{\varphi(\alpha_i)}$ for all $i = 1,\cdots,n$, which leads to $\{\varphi(\alpha_1),\cdots,\varphi(\alpha_n)\} \subseteq \{\alpha \in \Gamma : x_\alpha \neq y_\alpha\}$, so $n = \mathrm{card}(\{\varphi(\alpha_1),\cdots,\varphi(\alpha_n)\}) \leq \mathrm{card}(\{\alpha\in\Gamma : x_\alpha \neq y_\alpha\})$ (note that φ is one to one), which leads to the desired result. □

Lemma 4.4. *For infinite* Γ *we have:*

$$L(\mathcal{H},\mathcal{X}) \supseteq \{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}) \in \mathcal{X}\times\mathcal{X} : \{\gamma\in\Gamma : x_\gamma \neq y_\gamma\} \text{ is finite}\}.$$

Proof. Use Lemmas 4.3 and 3.3. $\square$

Theorem 4.1. *We have:*

$$L(\mathcal{H},\mathcal{X}) = \begin{cases} \{((x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}) \in \mathcal{X}\times\mathcal{X} : \{\gamma\in\Gamma : x_\gamma \neq y_\gamma\} \text{ is finite}\}, & \Gamma \text{ is infinite},\\ \{(x,x) : x\in\mathcal{X}\}, & \Gamma \text{ is finite}. \end{cases}$$

Proof. For infinite Γ use Lemmas 4.2 and 4.4, also for finite Γ note that $P(\mathcal{H},\mathcal{X}) = \{(x,x) : x\in\mathcal{X}\}$. $\square$

5 More details

In transformation semigroup (S,W) we say a nonempty subset D of W is invariant if $SD := \{sw : s\in S, w\in D\} \subseteq W$. For closed invariant subset D of W we may consider action of S on D in a natural way. For closed invariant subset D of W one may verify easily,

$$P(S,D)\subseteq P(S,W), \quad Q(S,D)\subseteq Q(S,W), \quad \text{and} \quad L(S,D)\subseteq L(S,W).$$

Suppose Z is a compact Hausdorff topological space with at least two elements, by Tychonoff's theorem Z^Γ is also compact Hausdorff. Again for $\varphi:\Gamma\to\Gamma$ one may consider $\sigma_\varphi : Z^\Gamma \to Z^\Gamma$ $(\sigma_\varphi((z_\alpha)_{\alpha\in\Gamma}) = (z_{\varphi(\alpha)})_{\alpha\in\Gamma})$, also $\mathcal{S} := \{\sigma_\varphi : Z^\Gamma \to Z^\Gamma | \varphi\in\Gamma^\Gamma\}$, and $\mathcal{H} := \{\sigma_\varphi : Z^\Gamma\to Z^\Gamma | \varphi\in\Gamma^\Gamma \text{ and } \varphi:\Gamma\to\Gamma \text{ is bijective }\}$. Then for each finite nonenpty subset A of Z, A^Γ is a closed invariant subset of $(\mathcal{S},Z^\Gamma)$ (resp. $(\mathcal{H},Z^\Gamma)$) and A is a discrete (and finite) subset of Z. But using previous sections we know about $P(T,A^\Gamma)$, $Q(T,A^\Gamma)$, and $L(T,A^\Gamma)$ for $T=\mathcal{H},\mathcal{S}$. Hence for $T=\mathcal{H},\mathcal{S}$ by:

$$\begin{aligned} &\bigcup\{P(T,A^\Gamma) : A \text{ is a finite subset of } Z\} \subseteq P(T,Z^\Gamma),\\ &\bigcup\{Q(T,A^\Gamma) : A \text{ is a finite subset of } Z\} \subseteq Q(T,Z^\Gamma),\\ &\bigcup\{L(T,A^\Gamma) : A \text{ is a finite subset of } Z\} \subseteq L(T,Z^\Gamma), \end{aligned}$$

we will have more data about $P(T,Z^\Gamma)$, $Q(T,Z^\Gamma)$, $L(T,Z^\Gamma)$.

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