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Numerical Methods for Semilinear Fractional Diffusion Equations with Time Delay

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Abstract. In this paper, we consider the numerical solutions of the semilinear Riesz space-fractional diffusion equations (RSFDEs) with time delay, which constitute an important class of differential equations of practical significance. We develop a novel implicit alternating direction method that can effectively and efficiently tackle the RSFDEs in both two and three dimensions. The numerical method is proved to be uniquely solvable, stable and convergent with second order accuracy in both space and time. Numerical results are presented to verify the accuracy and efficiency of the proposed numerical scheme.

AMS subject classifications: 65N15, 65N30

Key words: Semilinear Riesz space fractional diffusion equations with time delay, implicit alternating direction method, stability and convergence.

1 Introduction

It is widely known that mathematical models with time delay are of fundamental importance in many scientific and engineering applications, including economics, physics, population ecology and medicine. As a result, the theoretical analysis and numerical computation of many differential equations with time delay have been studied by numerous researchers [1,16,23,24]. Fractional differential equations with delay have received a lot of attentions [2,3,8,40,43,52,63,64] as a result of the development of fractional calculus in

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science and engineering [5, 6, 26, 27, 39, 47–49]. As a typical example, the fractional Bloch equation with delay was proposed to depict the nuclear magnetic resonance [2]. Most of the problems for fractional differential equations and fractional differential equations with delay can not be solved analytically. Thereby, numerical treatment for such type of equations becomes a hot topic in the communities of numerical mathematics. In recent years, the attention on the numerical computations of fractional differential equations has been discussed by many researchers [7, 10–14, 19, 20, 22, 42, 51, 54, 55, 57, 60–62]. For example, in [59] the authors considered a class of variable order fractional advection diffusion equation with a nonlinear reaction term. The two-dimensional RSFDEs with nonlinear reaction term was studied in [21]. Recently, the alternating direction implicit Galerkin-Legendre spectral method was proposed to solve the two-dimensional nonlinear reaction-diffusion equations with the Riesz space-fractional derivatives in [53]. In [58], the authors developed the finite element method to solve the two-dimensional nonlinear Riesz space fractional derivatives Fisher' s equation. In [56], a finite difference scheme was proposed for the two-dimensional diffusion equation with the Riesz space-fractional derivatives.

Very recently, some researchers developed methods on numerical solutions of fractional PDEs with time delay. The finite difference method was developed for solving the semi-linear space-fractional diffusion equations with time delay in [15]. In [45], the authors studied a linearized Crank-Nicolson method for solving the nonlinear fractional diffusing equation with multi-delay. In [41], the authors proposed the invariant subspace approach to solve a class of time-fractional partial differential equations with time delay. However, all of the works mentioned above focus on the one dimensional fractional PDEs with delay. In this paper, we shall develop high order schemes for the semilinear Riesz space-fractional diffusion equations with time delay in both two and three dimensions.

The rest of the paper is organized as follows. In Section 2, we present the numerical methods for the two-dimensional and three-dimensional semilinear RSFDEs with time delay. The stability and convergence of the method are proved in Section 3. Finally, we carry out some numerical experiments to confirm the theoretical results of the proposed method in Section 4.

2 Numerical methods for semilinear RSFDEs with time delay

In this paper, we consider the following two-dimensional and three-dimensional semilinear RSFDEs with time delay:

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = K_x \dfrac{\partial^\alpha u(x,y,t)}{\partial |x|^\alpha} + K_y \dfrac{\partial^\beta u(x,y,t)}{\partial |y|^\beta} + f(x,y,t,u,u(x,y,t-s)), \\ \qquad\qquad 1<\alpha,\beta\le 2, \quad (x,y,t)\in\Omega\times[0,T], \\ u(x,y,t)=0, \qquad (x,y)\in\partial\Omega, \quad t\in[0,T], \\ u(x,y,t)=\varphi(x,y,t), \qquad (x,y,t)\in\Omega\times[-s,0], \end{cases} \tag{2.1}$$

and

$$\begin{cases} \dfrac{\partial u(x,y,z,t)}{\partial t} = K_x \dfrac{\partial^\alpha u(x,y,z,t)}{\partial |x|^\alpha} + K_y \dfrac{\partial^\beta u(x,y,z,t)}{\partial |y|^\beta} + K_z \dfrac{\partial^\gamma u(x,y,z,t)}{\partial |z|^\gamma} \\ \qquad\qquad + g(x,y,z,t,u,u(x,y,z,t-s)), \\ \qquad\qquad 1<\alpha,\beta,\gamma\le 2, \quad (x,y,z,t)\in(0,L_x)\times(0,L_y)\times(0,L_z)\times[0,T], \\ u(0,y,z,t)=u(L_x,y,z,t)=0, \quad u(x,0,z,t)=u(x,L_y,z,t)=0, \\ u(x,y,0,t)=u(x,y,L_z,t)=0, \\ u(x,y,z,t)=\varphi(x,y,z,t), \quad (x,y,z,t)\in(0,L_x)\times(0,L_y)\times(0,L_z)\times[-s,0], \end{cases} \tag{2.2}$$

where $\Omega=(0,L_x)\times(0,L_y)$, and $K_x,K_y,K_z\ge 0$ signify the dispersion coefficients, $s>0$ is a time delay.

Throughout the paper, we assume that

(A1) the solutions of the problem (2.1) and (2.2) have piecewise smooth derivatives with respect to t in the subintervals $(ns,(n+1)s)$, $n=0,1,2,\cdots$,

(A2) the RHS function f in Eq. (2.1) has the first-order continuous partial derivatives with respect to its fourth and fifth arguments. Moreover, the following Lipschitz condition

$$|f(x,y,t,u_1,u_2)-f(x,y,t,\bar{u}_1,\bar{u}_2)|\le Ł_1|u_1-\bar{u}_1|+Ł_2|u_2-\bar{u}_2| \tag{2.3}$$

holds for all u_1, u_2, $\bar{u}_1$, $\bar{u}_2$ over $[0,L_x]\times[0,L_y]\times[0,T]$ with the Lipschitz constants β_1, β_2. Similarly, the RHS function g in Eq. (2.2) has the first-order continuous partial derivatives with respect to its fifth and sixth arguments, and the following Lipschitz condition

$$|g(x,y,z,t,u_1,u_2)-g(x,y,z,t,\bar{u}_1,\bar{u}_2)|\le \bar{L}_1|u_1-\bar{u}_1|+\bar{L}_2|u_2-\bar{u}_2| \tag{2.4}$$

holds for all u_1, u_2, $\bar{u}_1$, $\bar{u}_2$ over $[0,L_x]\times[0,L_y]\times[0,L_z]\times[0,T]$, where L_1, L_1, $\bar{L}_1$, $\bar{L}_2$ are Lipschitz constants.

The Riesz space fractional operators $\frac{\partial^\alpha u}{\partial|x|^\alpha}$, $\frac{\partial^\beta u}{\partial|y|^\beta}$, $\frac{\partial^\gamma u}{\partial|z|^\gamma}$ are defined as (see [20])

$$\begin{aligned} \frac{\partial^\alpha u}{\partial|x|^\alpha} &= -c_\alpha[{}_0D_x^\alpha u + {}_xD_{L_x}^\alpha u], \\ \frac{\partial^\beta u}{\partial|y|^\beta} &= -c_\beta[{}_0D_y^\beta u + {}_yD_{L_y}^\beta u], \\ \frac{\partial^\gamma u}{\partial|z|^\gamma} &= -c_\gamma[{}_0D_z^\gamma u + {}_zD_{L_z}^\gamma u], \end{aligned}$$

where

$$c_\alpha=\frac{1}{2\cos(\frac{\pi\alpha}{2})}, \quad c_\beta=\frac{1}{2\cos(\frac{\pi\beta}{2})}, \quad c_\gamma=\frac{1}{2\cos(\frac{\pi\gamma}{2})},$$

and

$$ {}_0D_x^{\alpha}u(x,y,t)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_0^x(t-\tau)^{1-\alpha}u(\tau,y,t)d\tau, \tag{2.5a} $$

$$ {}_xD_{L_x}^{\alpha}u(x,y,t)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^{L_x}(t-\tau)^{1-\alpha}u(\tau,y,t)d\tau. \tag{2.5b} $$

In a similar manner, we can define the other space Riesz fractional derivatives with respect to y and z, respectively.

Next, we define $\tau=T/N$, $t_n=n\tau$, $n=0,1,\cdots,N$, $s=m\tau$. Let $h_x=\frac{L_x}{M_x}$, $x_i=ih_x$ for $i=0,1,\cdots,M_x$, and $h_y=\frac{L_y}{M_y}$, $y_j=jh_y$ for $j=0,1,\cdots,M_y$, $h_z=\frac{L_z}{M_z}$, $z_k=kh_z$ for $k=0,1,\cdots,M_z$ be the space step-sizes. Define $u_{i,j}^n$ as the numerical approximation to $u(x_i,y_j,t_n)$ and $u_{i,j,k}^n$ as the numerical approximation to $u(x_i,y_j,z_k,t_n)$. Denote $\bar{\Omega}_h=\{(x_i,y_j)|0\le i\le M_x,\ 0\le j\le M_y\}$, $\Omega_h=\bar{\Omega}_h\cap\Omega$, $\partial\Omega_h=\bar{\Omega}_h\cap\partial\Omega$,

$$ \begin{aligned} &V_h=\{v|v=\{v_{i,j}\},\ 0\le i\le M_x,\ 0\le j\le M_y\},\\ &\mathring{V}_h=\{v|v=\{v_{i,j}\},\ 0\le i\le M_x,\ 0\le j\le M_y,\ v_{i,j}=0,\ \text{if } (x_i,y_j)\in\partial\Omega_h\}. \end{aligned} $$

In addition, we define the following operators

$$ \delta_t v_{i,j}^{n+1/2}=\frac{1}{\tau}(v_{i,j}^{n+1}-v_{i,j}^n),\quad v_{i,j}^{n+1/2}=\frac{1}{2}(v_{i,j}^{n+1}+v_{i,j}^n). $$

In the following, we review some useful lemmas.

Lemma 2.1 ([20]). *Suppose that* $1<\gamma<2$, $v(x)\in C^5[0,L]$. *If* $v(x)=0$, $\forall x\in(-\infty,0]\cup[L,+\infty)$, *then*

$$ {}_0D_x^{\gamma}v(x_i)=\frac{1}{h^{\gamma}}\sum_{k=0}^{i+1}\omega_{\gamma}^{(k)}v(x_{i-k+1})+\mathcal{O}(h^2), \tag{2.6a} $$

$$ {}_xD_L^{\gamma}v(x_i)=\frac{1}{h^{\gamma}}\sum_{k=0}^{m-i+1}\omega_{\gamma}^{(k)}v(x_{i+k-1})+\mathcal{O}(h^2), \tag{2.6b} $$

where

$$ \begin{aligned} &\omega_{\gamma}^{(0)}=\frac{\gamma}{2}g_0^{(\gamma)},\\ &\omega_{\gamma}^{(k)}=\frac{\gamma}{2}g_k^{(\gamma)}+\frac{2-\gamma}{2}g_{k-1}^{(\gamma)},\\ &g_0^{(\gamma)}=1,g_k^{(\gamma)}=(1-\frac{\gamma+1}{k})g_{k-1}^{(\gamma)},\quad k=1,2,\cdots. \end{aligned} $$

Lemma 2.2 ([20]). *Suppose that* $1<\gamma\le2$, *then* $\{g_k^{(\gamma)}\}$ *satisfy*

$$ \begin{cases} g_0^{(\gamma)}=1,\quad g_1^{(\gamma)}=-\gamma,\quad g_2^{(\gamma)}=\dfrac{\gamma(\gamma-1)}{2}>0,\\ 1\ge g_2^{(\gamma)}\ge g_3^{(\gamma)}\ge\cdots\ge0,\quad \displaystyle\sum_{k=0}^{\infty}g_k^{(\gamma)}=0,\quad \sum_{k=0}^{m}g_k^{(\gamma)}<0,\quad m\ge1. \end{cases} \tag{2.7} $$

Lemma 2.3 ([20]). *Suppose that* $1<\gamma\leq 2$, *then* $\{\omega_\gamma^{(k)}\}$ *satisfy*

$$\begin{cases} \omega_\gamma^{(0)}=\dfrac{\gamma}{2}, \quad \omega_\gamma^{(1)}=\dfrac{2-\gamma-\gamma^2}{2}<0, \quad \omega_\gamma^{(2)}=\dfrac{\gamma(\gamma^2+\gamma-4)}{4}, \\ 1\geq\omega_\gamma^{(2)}\geq\omega_\gamma^{(3)}\geq\cdots\geq 0, \quad \displaystyle\sum_{k=0}^{\infty}\omega_\gamma^{(k)}=0, \quad \sum_{k=0}^{m}\omega_\gamma^{(k)}<0, \quad m\geq 2. \end{cases} \tag{2.8}$$

2.1 The two-dimensional case

Now, in order to approximate (2.1), we use

$$\frac{\partial u(x,y,t)}{\partial t}\Big|_{(x_i,y_j,t_{n+\frac{1}{2}})}=\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}+\mathcal{O}(\tau^2)=\delta_t u_{i,j}^{n+1/2}+\mathcal{O}(\tau^2). \tag{2.9}$$

According to the method of [20], we have

$$\begin{aligned} &\frac{\partial^\alpha u(x,y,t)}{\partial|x|^\alpha}\Big|_{(x_i,y_j,t_{n+\frac{1}{2}})} \\ &=-\frac{c_\alpha}{2(h_x)^\alpha}\Big[\Big(\sum_{l=0}^{i+1}\omega_\alpha^{(l)}u(x_{i-l+1},y_j,t_{n+1})+\sum_{l=0}^{M_x-i+1}\omega_\alpha^{(l)}u(x_{i+l-1},y_j,t_{n+1})\Big) \\ &\quad+\Big(\sum_{l=0}^{i+1}\omega_\alpha^{(l)}u(x_{i-l+1},y_j,t_n)+\sum_{l=0}^{M_x-i+1}\omega_\alpha^{(l)}u(x_{i+l-1},y_j,t_n)\Big)\Big]+\mathcal{O}(h_x^2), \end{aligned} \tag{2.10a}$$

$$\begin{aligned} &\frac{\partial^\beta u(x,y,t)}{\partial|y|^\beta}\Big|_{(x_i,y_j,t_{n+\frac{1}{2}})} \\ &=-\frac{c_\beta}{2(h_y)^\beta}\Big[\Big(\sum_{l=0}^{j+1}\omega_\beta^{(l)}u(x_i,y_{j-l+1},t_{n+1})+\sum_{l=0}^{M_y-j+1}\omega_\beta^{(l)}u(x_i,y_{j+l-1},t_{n+1})\Big) \\ &\quad+\Big(\sum_{l=0}^{j+1}\omega_\beta^{(l)}u(x_i,y_{j-l+1},t_n)+\sum_{l=0}^{M_y-j+1}\omega_\beta^{(l)}u(x_i,y_{j+l-1},t_n)\Big)\Big]+\mathcal{O}(h_y^2). \end{aligned} \tag{2.10b}$$

For the nonlinear reaction term, by using the Taylor expansion, we can obtain the following approximation:

$$\begin{aligned} &f(x_i,y_j,t_{n+\frac{1}{2}},u(x_i,y_j,t_{n+\frac{1}{2}}),u(x_i,y_j,t_{n+\frac{1}{2}}-s)) \\ =&f(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u(x_i,y_j,t_n)-\frac{1}{2}u(x_i,y_j,t_{n-1}),u(x_i,y_j,t_{n+\frac{1}{2}}-s))+\mathcal{O}(\tau^2) \\ \sim&f(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^{n}-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m})+\mathcal{O}(\tau^2). \end{aligned} \tag{2.11}$$

Therefore, the numerical method for (2.1) is determined by the following finite difference equation

$$\begin{aligned}\frac{u_{i,j}^n-u_{i,j}^{n-1}}{\tau}=&-\frac{K_x c_\alpha}{2(h_x)^\alpha}\Bigg[\Big(\sum_{l=0}^{i+1}\omega_\alpha^{(l)}u(x_{i-l+1},y_j,t_{n+1})+\sum_{l=0}^{M_x-i+1}\omega_\alpha^{(l)}u(x_{i+l-1},y_j,t_{n+1})\Big)\\&\cdot\Big(\sum_{l=0}^{i+1}\omega_\alpha^{(l)}u(x_{i-l+1},y_j,t_n)+\sum_{l=0}^{M_x-i+1}\omega_\alpha^{(l)}u(x_{i+l-1},y_j,t_n)\Big)\Bigg]\\&-\frac{K_y c_\beta}{2(h_y)^\beta}\Bigg[\Big(\sum_{l=0}^{j+1}\omega_\beta^{(l)}u(x_i,y_{j-l+1},t_{n+1})+\sum_{l=0}^{M_y-j+1}\omega_\beta^{(l)}u(x_i,y_{j+l-1},t_{n+1})\Big)\\&\cdot\Big(\sum_{l=0}^{j+1}\omega_\beta^{(l)}u(x_i,y_{j-l+1},t_n)+\sum_{l=0}^{M_y-j+1}\omega_\beta^{(l)}u(x_i,y_{j+l-1},t_n)\Big)\Bigg]\\&+f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big).\end{aligned}\tag{2.12}$$

Clearly, the numerical method (2.12) is consistent with order $\mathcal{O}(\tau^2+h_x^2+h_y^2)$.

Define the following fractional partial difference operators:

$$\delta_x^\alpha u_{i,j}^n=-\frac{K_x c_\alpha}{(h_x)^\alpha}\Bigg[\sum_{l=0}^{i+1}\omega_\alpha^{(l)}u(x_{i-l+1},y_j,t_n)+\sum_{l=0}^{M_x-i+1}\omega_\alpha^{(l)}u(x_{i+l-1},y_j,t_n)\Bigg],\tag{2.13a}$$

$$\delta_y^\beta u_{i,j}^n=-\frac{K_y c_\beta}{(h_y)^\beta}\Bigg[\sum_{l=0}^{j+1}\omega_\beta^{(l)}u(x_i,y_{j-l+1},t_n)+\sum_{l=0}^{M_y-j+1}\omega_\beta^{(l)}u(x_i,y_{j+l-1},t_n)\Bigg].\tag{2.13b}$$

By means of these operator definitions, the numerical method (2.12) can be written as

$$\delta_t u_{i,j}^{n+1/2}=\delta_x^\alpha u_{i,j}^{n+1/2}+\delta_y^\beta u_{i,j}^{n+1/2}+f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big)\tag{2.14}$$

or

$$\begin{aligned}&\Big(1-\frac{\tau}{2}\delta_x^\alpha-\frac{\tau}{2}\delta_y^\beta\Big)u_{i,j}^{n+1}\\=&\Big(1+\frac{\tau}{2}\delta_x^\alpha+\frac{\tau}{2}\delta_y^\beta\Big)u_{i,j}^n+\tau f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big),\\&1\le i\le M_x-1,\quad 1\le j\le M_y-1.\end{aligned}\tag{2.15}$$

The boundary and initial conditions are discretized as

$$\begin{aligned}&u_{0,j}^n=u(0,y_j,t_n)=0,&&u_{M_x,j}^n=u(L_x,y_j,t_n)=0,\\&u_{i,0}^n=u(x_i,0,t_n)=0,&&u_{i,M_y}^n=u(x_i,L_y,t_n)=0,\\&u_{i,j}^k=\varphi(ih_x,jh_y,k\tau),\quad k\le 0.\end{aligned}$$

The operator form (2.15) can be written in the following directional separation product form named as alternating direction implicit methods (ADIM)

$$\begin{aligned}&\Big(1-\frac{\tau}{2}\delta_x^\alpha\Big)\Big(1-\frac{\tau}{2}\delta_y^\beta\Big)u_{i,j}^{n+1}\\&=\Big(1+\frac{\tau}{2}\delta_x^\alpha\Big)\Big(1+\frac{\tau}{2}\delta_y^\beta\Big)u_{i,j}^n+\tau f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big),\\&\quad 1\le i\le M_x-1,\quad 1\le j\le M_y-1,\end{aligned}\tag{2.16}$$

which introduces an additional perturbation term $C[\delta_x^\alpha\delta_y^\beta](u_{i,j}^{n+1}+u_{i,j}^n)$. Clearly, this method is consistent with order $\mathcal{O}(\tau^2+h_x^2+h_y^2)$.

The method (2.16) can now be solved by the following iterative scheme:

- First, for each fixed y_j, solve the problem in the x-direction to obtain an intermediate solution $\bar{u}_{i,j}^n$ in the form

$$\begin{aligned}&\Big(1-\frac{\tau}{2}\delta_x^\alpha\Big)\bar{u}_{i,j}^{n+1}\\&=\Big(1+\frac{\tau}{2}\delta_x^\alpha\Big)\Big(1+\frac{\tau}{2}\delta_y^\beta\Big)u_{i,j}^n+\tau f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big).\end{aligned}\tag{2.17}$$

- Second, for each fixed x_i, solve the problem in the y-direction,

$$\Big(1-\frac{\tau}{2}\delta_y^\beta\Big)u_{i,j}^{n+1}=\bar{u}_{i,j}^{n+1}.\tag{2.18}$$

The boundary and initial conditions are discretized as

$$\begin{aligned}&u_{0,j}^n=u(0,y_j,t_n)=0,\qquad &&u_{M_x,j}^n=u(L_x,y_j,t_n)=0,\\&u_{i,0}^n=u(x_i,0,t_n)=0,\qquad &&u_{i,M_y}^n=u(x_i,L_y,t_n)=0,\\&\bar{u}_{0,j}=0,\qquad &&\bar{u}_{M_x,j}=0,\end{aligned}$$

and

$$u_{i,j}^k=\varphi(ih_x,jh_y,k\tau),\quad k\le 0.$$

Let

$$\begin{aligned}&r_x=\tau\frac{K_xc_\alpha}{2(h_x)^\alpha},\qquad &&r_y=\tau\frac{K_yc_\beta}{2(h_y)^\beta},\\&D_x=(d_{i,j}^x)_{(M_x-1)\times(M_x-1)},\qquad &&D_y=(d_{i,j}^y)_{(M_y-1)\times(M_y-1)},\end{aligned}$$

where

$$d^x_{i,j}=\begin{cases} r_x\omega^{(i-j+1)}_\alpha & \text{for } j<i-1,\\ r_x(\omega^{(0)}_\alpha+\omega^{(2)}_\alpha) & \text{for } j=i-1,\\ 2r_x\omega^{(1)}_\alpha & \text{for } j=i,\\ r_x(\omega^{(0)}_\alpha+\omega^{(2)}_\alpha) & \text{for } j=i+1,\\ r_x\omega^{(j-i+1)}_\alpha & \text{for } j>i+1, \end{cases} \tag{2.19a}$$

$$d^y_{i,j}=\begin{cases} r_y\omega^{(i-j+1)}_\beta & \text{for } j<i-1,\\ r_y(\omega^{(0)}_\beta+\omega^{(2)}_\beta) & \text{for } j=i-1,\\ 2r_y\omega^{(1)}_\beta & \text{for } j=i,\\ r_y(\omega^{(0)}_\beta+\omega^{(2)}_\beta) & \text{for } j=i+1,\\ r_y\omega^{(j-i+1)}_\beta & \text{for } j>i+1. \end{cases} \tag{2.19b}$$

Let

$$(I-D_y)u^n_v=\tilde{u}_v,\quad 1\le v\le M_x-1, \tag{2.20}$$

where $u^n_v=(u^n_{v,1},u^n_{v,2},\cdots,u^n_{v,M_y-1})^T$, $\tilde{u}_v=(\tilde{u}^n_{v,1},\tilde{u}^n_{v,2},\cdots,\tilde{u}^n_{v,M_y-1})^T$. Then (2.17) can be rewritten in the matrix form

$$(I+D_x)\bar{u}^n_w=(I-D_x)\tilde{u}^*_w+\tau f,\quad 1\le w\le M_y-1, \tag{2.21}$$

where $u^*_w=(\tilde{u}_{1,w},\tilde{u}_{2,w},\cdots,\tilde{u}_{M_x-1,w})^T$, $\bar{u}^n_w=(\bar{u}^n_{1,w},\bar{u}^n_{2,w},\cdots,\bar{u}^n_{M_x-1,w})^T$,

$$f=\Big(f\Big(x_1,y_w,t_{n+\frac{1}{2}},\frac{3}{2}u^n_{1,w}-\frac{1}{2}u^{n-1}_{1,w},u^{n+\frac{1}{2}-m}_{1,w}\Big),f\Big(x_2,y_w,t_{n+\frac{1}{2}},\frac{3}{2}u^n_{2,w}-\frac{1}{2}u^{n-1}_{2,w},u^{n+\frac{1}{2}-m}_{2,w}\Big),$$
$$\cdots,f\Big(x_{M_x-1},y_w,t_{n+\frac{1}{2}},\frac{3}{2}u^n_{M_x-1,w}-\frac{1}{2}u^{n-1}_{M_x-1,w},u^{n+\frac{1}{2}-m}_{M_x-1,w}\Big)\Big)^T.$$

Similarly, (2.18) can be written in the matrix form

$$(I+D_y)\hat{u}^{n+1}_v=\hat{u}^*_v,1\le v\le M_x-1, \tag{2.22}$$

where $\hat{u}^{n+1}_v=(u^{n+1}_{v,1},u^{n+1}_{v,2},\cdots,u^{n+1}_{v,M_y-1})^T$, $\hat{u}^*_v=(\bar{u}^n_{v,1},\bar{u}^n_{v,2},\cdots,\bar{u}^n_{v,M_y-1})^T$.

2.2 The three-dimensional case

Next, we discuss the numerical method for three-dimensional case. We can construct the numerical method for (2.2) as follows:

$$\begin{aligned}\delta_t u^{n+1/2}_{i,j,k}=&\delta^\alpha_x u^{n+1/2}_{i,j,k}+\delta^\beta_y u^{n+1/2}_{i,j,k}+\delta^\gamma_z u^{n+1/2}_{i,j,k}\\ &+g\Big(x_i,y_j,z_k,t_{n+\frac{1}{2}},\frac{3}{2}u^n_{i,j,k}-\frac{1}{2}u^{n-1}_{i,j,k},u^{n+\frac{1}{2}-m}_{i,j,k}\Big),\end{aligned} \tag{2.23}$$

or

$$\begin{aligned}&\Big(1-\frac{\tau}{2}\delta_x^\alpha-\frac{\tau}{2}\delta_y^\beta-\frac{\tau}{2}\delta_z^\gamma\Big)u_{i,j,k}^{n+1}\\&=\Big(1+\frac{\tau}{2}\delta_x^\alpha+\frac{\tau}{2}\delta_y^\beta+\frac{\tau}{2}\delta_z^\gamma\Big)u_{i,j,k}^{n}+\tau g\Big(x_i,y_j,z_k,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j,k}^{n}-\frac{1}{2}u_{i,j,k}^{n-1},u_{i,j,k}^{n+\frac{1}{2}-m}\Big),\\&\qquad 1\le i\le M_x-1,\quad 1\le j\le M_y-1,\quad 1\le k\le M_z-1,\end{aligned}\tag{2.24}$$

where

$$\delta_x^\alpha u_{i,j,k}^n=-\frac{K_x c_\alpha}{(h_x)^\alpha}\Big[\sum_{l=0}^{i+1}\omega_\alpha^{(l)}u_{i-l+1,j,k}^n+\sum_{l=0}^{M_x-i+1}\omega_\alpha^{(l)}u_{i+l-1,j,k}^n\Big],\tag{2.25a}$$

$$\delta_y^\beta u_{i,j,k}^n=-\frac{K_y c_\beta}{(h_y)^\beta}\Big[\sum_{l=0}^{j+1}\omega_\beta^{(l)}u_{i,j-l+1,k}^n+\sum_{l=0}^{M_y-j+1}\omega_\beta^{(l)}u_{i,j+l-1,k}^n\Big],\tag{2.25b}$$

$$\delta_z^\gamma u_{i,j,k}^n=-\frac{K_z c_\gamma}{(h_z)^\gamma}\Big[\sum_{l=0}^{k+1}\omega_\gamma^{(l)}u_{i,j,k-l+1}^n+\sum_{l=0}^{M_z-k+1}\omega_\gamma^{(l)}u_{i,j,k+l-1}^n\Big].\tag{2.25c}$$

Moreover, the operator form (2.24) can be written in the following directional separation product form

$$\begin{aligned}&\Big(1-\frac{\tau}{2}\delta_x^\alpha\Big)\Big(1-\frac{\tau}{2}\delta_y^\beta\Big)\Big(1-\frac{\tau}{2}\delta_z^\gamma\Big)u_{i,j,k}^{n+1}\\&=\Big(1+\frac{\tau}{2}\delta_x^\alpha\Big)\Big(1+\frac{\tau}{2}\delta_y^\beta\Big)\Big(1+\frac{\tau}{2}\delta_z^\gamma\Big)u_{i,j,k}^{n}+\tau g\Big(x_i,y_j,z_k,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j,k}^{n}-\frac{1}{2}u_{i,j,k}^{n-1},u_{i,j,k}^{n+\frac{1}{2}-m}\Big),\end{aligned}\tag{2.26}$$

which introduces an additional perturbation error equal to $\mathcal{O}(\tau^2)$. The additional perturbational error is not large compared to the approximation errors for the other terms in (2.24), and this method is consistent with order $\mathcal{O}(\tau^2+h_x^2+h_y^2+h_z^2)$. The method (2.26) can now be solved by the following iterative scheme:

- First, for each fixed y_j, z_k, solve the problem in the x-direction to obtain an intermediate solution $\bar{u}_{i,j,k}^n$ in the form

$$\begin{aligned}\Big(1-\frac{\tau}{2}\delta_x\Big)\bar{u}_{i,j,k}^n=&\Big(1+\frac{\tau}{2}\delta_x^\alpha\Big)\Big(1+\frac{\tau}{2}\delta_y^\beta\Big)\Big(1+\frac{\tau}{2}\delta_z^\gamma\Big)u_{i,j,k}^n\\&+\tau g\Big(x_i,y_j,z_k,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j,k}^{n}-\frac{1}{2}u_{i,j,k}^{n-1},u_{i,j,k}^{n+\frac{1}{2}-m}\Big).\end{aligned}\tag{2.27}$$

- Second, for each fixed x_i, z_k, solve the problem in the y-direction,

$$(1-\tau\delta_y^\beta)\hat{u}_{i,j,k}^n=\bar{u}_{i,j,k}^n.\tag{2.28}$$

- Finally, for each fixed x_i, y_j, solve the problem in the z-direction,

$$(1-\tau\delta_z^\gamma)u_{i,j,k}^{n+1}=\hat{u}_{i,j,k}^n.\tag{2.29}$$

The boundary and initial conditions are discretized as

$$\begin{aligned}
&u^n_{0,j,k}=u(0,y_j,z_k,t_n)=0, && u^n_{M_x,j,k}=u(L_x,y_j,z_k,t_n)=0,\\
&u^n_{i,0,k}=u(x_i,0,z_k,t_n)=0, && u^n_{i,M_y,k}=u(x_i,L_y,z_k,t_n)=0,\\
&u^n_{i,j,0}=u(x_i,y_j,0,t_n)=0, && u^n_{i,j,L_z}=u(x_i,y_j,L_z,t_n)=0,\\
&u^l_{i,j,k}=\varphi(ih_x,jh_y,kh_z,l\tau), \quad l\le 0,
\end{aligned}$$

and

$$\begin{aligned}
&\bar{u}^n_{0,j,k}=0, && \bar{u}^n_{M_x,j,k}=0,\\
&\hat{u}^n_{i,0,k}=0, && \hat{u}^n_{i,M_y,k}=0.
\end{aligned}$$

3 Theoretical analysis

Without loss of generality, we present the proofs of stability and convergence of the proposed method for solving the two dimensional semilinear Riesz space fractional diffusion equations with time delay in this paper. The three dimensional case can be proved by following similar arguments. For any grid functions $u,v\in\mathring{V}_h$, we define the inner product and norm as

$$(u,v)=h_xh_y\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1}u_{i,j}v_{i,j},\quad \|u\|=\sqrt{(u,u)}.$$

In addition, we introduce the following useful lemmas which play important roles in the subsequent analysis.

Lemma 3.1. *Suppose that* $1<\alpha,\beta\le 2$, D_x, D_y *are defined as* (2.19a) *and* (2.19b) *respectively. Then* D_x, D_y *are strictly diagonally dominant.*

Proof. Firstly, we prove that D_x is strictly diagonally dominant. Since $1<\alpha\le 2$ and $K_x>0$, so $c_\alpha<0$ and $r_x<0$. Then $d^x_{i,i}>0$, $d^x_{i,j}<0$, $(i\neq j)$. From Lemma 2.3, we have

$$\sum_{j=1,j\neq i}^{M_x-1}|d^x_{i,j}|=-r_x\left(\sum_{j=0,j\neq 1}^{i}\omega^{(j)}_\alpha+\sum_{j=0,j\neq 1}^{M_x-i}\omega^{(j)}_\alpha\right)<-r_x(-\omega^{(1)}_\alpha-\omega^{(1)}_\alpha)=2r_x\omega^{(1)}_\alpha=d^x_{i,i}. \tag{3.1}$$

Then, D_x is strictly diagonally dominant. Similarly, we can prove that D_y is strictly diagonally dominant. □

Lemma 3.2. *Suppose that* $1<\alpha,\beta\le 2$, D_x, D_y *are defined as* (2.19a) *and* (2.19b), *respectively. Then* D_x, D_y *are symmetric positive definite.*

Proof. In view of (2.19a) and (2.19b), the symmetry of D_x, D_y is evident. Let λ_x be one eigenvalue of D_x. According to the Gerschgorin's circle theorem [46], we have

$$|\lambda_x - d^x_{i,i}| \leq \sum_{j=1, j\neq i}^{M_x-1} |d^x_{i,j}|.$$

Then

$$d^x_{i,i} - \sum_{j=1, j\neq i}^{M_x-1} |d^x_{i,j}| \leq \lambda_x \leq d^x_{i,i} + \sum_{j=1, j\neq i}^{M_x-1} |d^x_{i,j}|.$$

By using Lemma 3.1, we have $\lambda_x > 0$, thus D_x is positive definite. Similarly, we can prove that D_y is positive definite. □

Lemma 3.3 ([50,56]). *For any $u \in \mathring{V}_h$, there hold $(\delta^\alpha_x u, u) \leq 0$, and $(\delta^\beta_y u, u) \leq 0$.*

Lemma 3.4 (Gronwall's inequality [44]). *Suppose that $\{k_n\}$ and $\{p_n\}$ are nonnegative sequences, and the sequence $\{\phi_n\}$ satisfies*

$$\phi_0 \leq q_0, \quad \phi_n \leq q_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} p_l \phi_l, \quad n \geq 1,$$

where $q_0 \geq 0$. Then it holds that

$$\{\phi_n\} \leq \Big(q_0 + \sum_{l=0}^{n-1} p_l\Big) \exp\Big(\sum_{l=0}^{n-1} k_l\Big), \quad n \geq 1. \tag{3.2}$$

Theorem 3.1. *The difference scheme* (2.15) *is uniquely solvable.*

Proof. Consider the homogeneous form of (2.15) and taking the inner product with $u^{n+1}_{i,j}$, we have

$$(u^{k+1}, u^{k+1}) - \frac{\tau}{2}(\delta^\alpha_x u^{k+1}, u^{k+1}) - \frac{\tau}{2}(\delta^\beta_y u^{k+1}, u^{k+1}) = 0. \tag{3.3}$$

It follows from Lemma 3.3 that

$$(u^{k+1}, u^{k+1}) = \|u^{k+1}\|^2 \leq 0.$$

Thus, $\|u^{k+1}\| = 0$ and $u^{k+1}_{i,j}$ can be solved uniquely. The proof is completed. □

Theorem 3.2. *The difference scheme* (2.17)-(2.18) *is uniquely solvable.*

Proof. According to Lemma 3.2, we know that $I + D_x$, $I + D_y$ are strictly diagonally dominant. Then $I + D_x$, $I + D_y$ are invertible respectively, which means that the difference scheme (2.17)-(2.18) is uniquely solvable. □

Theorem 3.3. *If the solution of the problem* (2.1) *satisfy the conditions* (A1) *and* (A2), *the method* (2.15) *is convergent. Moreover,*

$$\|\epsilon^n\| \leq C(h_x^2+h_y^2+\tau^2), \quad 1 \leq n \leq N,$$

where $\epsilon_{i,j}^n = u(x_i,y_j,t_n)-u_{i,j}^n$, $0 \leq i \leq M_x$, $0 \leq j \leq M_y$ *denotes the corresponding error,* $u_{i,j}^n$, $(0 \leq i \leq M_x,\ 0 \leq j \leq M_y)$ *be the numerical solution of the corresponding difference scheme* (2.15), $C = c_1 T\sqrt{L_x L_y}\exp(8T(L_1+L_2))$, c_1 *is a positive constant.*

Proof. Obviously, we have

$$\begin{aligned}\delta_t \epsilon_{i,j}^{n+1/2} = &\frac{1}{2}\delta_x^\alpha(\epsilon_{i,j}^{n+1}+\epsilon_{i,j}^n)+\frac{1}{2}\delta_y^\beta(\epsilon_{i,j}^{n+1}+\epsilon_{i,j}^n)+f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big)\\ &-f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u(x_i,y_j,t_n)-\frac{1}{2}u(x_i,y_j,t_{n-1}),\frac{1}{2}u(x_i,y_j,t_{n+1-m})\\ &+u(x_i,y_j,t_{n-m})\Big)+R_{i,j}^{n+1/2}. \end{aligned} \tag{3.4}$$

According to (2.12)-(2.15), there exists a positive constant c_1 such that

$$|R_{i,j}^{n+1/2}| \leq c_1(h_x^2+h_y^2+\tau^2), \quad 1 \leq i \leq M_x-1, \quad 1 \leq j \leq M_y-1, \quad 0 \leq i \leq N-1. \tag{3.5}$$

As a result, we can show that

$$\delta_t \epsilon^{n+1/2} = \frac{1}{2}\delta_x^\alpha(\epsilon^{n+1}+\epsilon^n)+\frac{1}{2}\delta_y^\beta(\epsilon^{n+1}+\epsilon^n)+F^{n+1/2}+R^{n+1/2}, \tag{3.6}$$

where

$$\begin{aligned}F^{n+1/2} = \Big\{ &F_{i,j}^{n+1/2} | F_{i,j}^{n+1/2} = f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big)\\ &-f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u(x_i,y_j,t_n)-\frac{1}{2}u(x_i,y_j,t_{n-1}),\frac{1}{2}u(x_i,y_j,t_{n+1-m})+u(x_i,y_j,t_{n-m})\Big),\\ &1 \leq i \leq M_x-1,\ 1 \leq j \leq M_y-1\Big\}, \quad 0 \leq n \leq N-1. \end{aligned}$$

Furthermore, according to the Lipschitz condition (2.3), we can obtain

$$\begin{aligned}&\Big|f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big)\\ &-f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u(x_i,y_j,t_n)-\frac{1}{2}u(x_i,y_j,t_{n-1}),\frac{1}{2}(u(x_i,y_j,t_{n+1-m})+u(x_i,y_j,t_{n-m}))\Big|\\ \leq &L_1\Big|\frac{3}{2}\epsilon_{i,j}^n-\frac{1}{2}\epsilon_{i,j}^{n-1}\Big|+L_2\Big|\frac{3}{2}\epsilon_{i,j}^{n+1-m}-\frac{1}{2}\epsilon_{i,j}^{n-m}\Big|\\ \leq &L_1\Big(\Big|\frac{3}{2}\epsilon_{i,j}^n\Big|+\Big|\frac{1}{2}\epsilon_{i,j}^{n-1}\Big|\Big)+L_2\Big(\Big|\frac{3}{2}\epsilon_{i,j}^{n+1-m}\Big|+\Big|\frac{1}{2}\epsilon_{i,j}^{n-m}\Big|\Big). \end{aligned} \tag{3.7}$$

Making the inner products of (3.6) with $\epsilon^{n+1/2}$, then we have

$$(\delta_t\epsilon^{n+1/2},\epsilon^{n+1/2})=\Big(\frac{1}{2}\delta_x^\alpha(\epsilon^{n+1}+\epsilon^n),\epsilon^{n+1/2}\Big)+\Big(\frac{1}{2}\delta_y^\beta(\epsilon^{n+1}+\epsilon^n),\epsilon^{n+1/2}\Big)$$
$$+(F^{n+1/2},\epsilon^{n+1/2})+(R^{n+1/2},\epsilon^{n+1/2}). \tag{3.8}$$

Noting that

$$\Big(\frac{1}{2}\delta_x^\alpha(\epsilon^{n+1}+\epsilon^n),\epsilon^{n+1/2}\Big)=\frac{1}{4}(\delta_x^\alpha(\epsilon^{n+1}+\epsilon^n),\epsilon^{n+1}+\epsilon^n),$$
$$\Big(\frac{1}{2}\beta_y^\alpha(\epsilon^{n+1}+\epsilon^n),\epsilon^{n+1/2}\Big)=\frac{1}{4}(\delta_y^\beta(\epsilon^{n+1}+\epsilon^n),\epsilon^{n+1}+\epsilon^n),$$

it follows from Lemma 3.3 that

$$\Big(\frac{1}{2}\delta_x^\alpha(\epsilon^{n+1}+\epsilon^n),\epsilon^{n+1/2}\Big)\le 0,\quad \Big(\frac{1}{2}\beta_y^\alpha(\epsilon^{n+1}+\epsilon^n),\epsilon^{n+1/2}\Big)\le 0. \tag{3.9}$$

Moreover, we know that

$$(\delta_t\epsilon^{n+1/2},\epsilon^{n+1/2})=\frac{1}{2\tau}(\epsilon^{n+1}-\epsilon^n,\epsilon^{n+1}+\epsilon^n)=\frac{1}{2\tau}(\|\epsilon^{n+1}\|^2-\|\epsilon^n\|^2). \tag{3.10}$$

By means of (3.7) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(F^{n+1/2},\epsilon^{n+1/2})&\le\|F^{n+1/2}\|\cdot\|\epsilon^{n+1/2}\|\\&\le\frac{1}{2}\|L_1\Big(\Big|\frac{3}{2}\epsilon_{i,j}^n\Big|+\Big|\frac{1}{2}\epsilon_{i,j}^{n-1}\Big|\Big)+L_2\Big(\Big|\frac{3}{2}\epsilon_{i,j}^{n+1-m}\Big|+\Big|\frac{1}{2}\epsilon_{i,j}^{n-m}\Big|\Big)\Big\|\cdot\|\epsilon^{n+1}+\epsilon^n\|\\&\le(L_1+L_2)(\|\epsilon^{n+1}\|+\|\epsilon^n\|)\cdot(\|\epsilon^n\|+\|\epsilon^{n-1}\|+\|\epsilon^{n+1-m}\|+\|\epsilon^{n-m}\|).\end{aligned} \tag{3.11}$$

For the third term on the right side of (3.8), we can obtain

$$(R^{n+1/2},\epsilon^{n+1/2})\le\|R^{n+1/2}\|\cdot\|\epsilon^{n+1/2}\|\le\frac{1}{2}\|R^{n+1/2}\|\cdot(\|\epsilon^{n+1}\|+\|\epsilon^n\|). \tag{3.12}$$

Substituting (3.9)-(3.12) into (3.8), we obtain that

$$\frac{\|\epsilon^{n+1}\|^2-\|\epsilon^n\|^2}{2\tau}\le(L_1+L_2)(\|\epsilon^{n+1}\|+\|\epsilon^n\|)\cdot(\|\epsilon^n\|+\|\epsilon^{n-1}\|+\|\epsilon^{n+1-m}\|+\|\epsilon^{n-m}\|)$$
$$+\frac{1}{2}\|R^{n+1/2}\|\cdot(\|\epsilon^{n+1}\|+\|\epsilon^n\|), \tag{3.13}$$

namely,

$$\|\epsilon^{n+1}\|\le\|\epsilon^n\|+2\tau(L_1+L_2)(\|\epsilon^n\|+\|\epsilon^{n-1}\|+\|\epsilon^{n+1-m}\|+\|\epsilon^{n-m}\|)+\tau\|R^{n+1/2}\|. \tag{3.14}$$

Replacing n by k and summing over k from 1 to n, and noticing $\epsilon^k=0$ for $-m\le k\le 0$, we have

$$\begin{aligned}\|\epsilon^{n+1}\|\le&\|\epsilon^0\|+8\tau(L_1+L_2)\sum_{k=1}^{n}\|\epsilon^k\|+\tau\sum_{k=1}^{n}\|R^{k+1/2}\|\\=&8\tau(L_1+L_2)\sum_{k=1}^{n}\|\epsilon^k\|+\tau\sum_{k=1}^{n}\|R^{k+1/2}\|.\end{aligned}\tag{3.15}$$

In view of (3.5), we obtain

$$\begin{aligned}\|\epsilon^{n+1}\|\le&8\tau(L_1+L_2)\sum_{k=1}^{n}\|\epsilon^k\|+\tau\sum_{k=1}^{n}c_1\sqrt{L_xL_y}(h_x^2+h_y^2+\tau^2)\\\le&\|\epsilon^0\|+8\tau(L_1+L_2)\sum_{k=1}^{n}\|\epsilon^k\|+c_1T\sqrt{L_xL_y}(h_x^2+h_y^2+\tau^2).\end{aligned}\tag{3.16}$$

Using the Gronwall Lemma 3.4, we have

$$\begin{aligned}\|\epsilon^{n+1}\|\le&c_1T\sqrt{L_xL_y}\exp(8n\tau(L_1+L_2))(h_x^2+h_y^2+\tau^2)\\\le&c_1T\sqrt{L_xL_y}\exp(8T(L_1+L_2))(h_x^2+h_y^2+\tau^2),\quad 0\le n\le N-1.\end{aligned}\tag{3.17}$$

Consequently,

$$\|\epsilon^n\|\le C(h_x^2+h_y^2+\tau^2),\quad 1\le n\le N,\tag{3.18}$$

where $C=c_1T\sqrt{L_xL_y}\exp(8T(L_1+L_2))$, c_1 is a positive constant satisfies (3.5). The proof is completed. □

Next, we will analyze the stability of the scheme (2.15). Let $U_{i,j}^n$ be the solution of

$$\begin{aligned}&\Big(1-\frac{\tau}{2}\delta_x^\alpha-\frac{\tau}{2}\delta_y^\beta\Big)U_{i,j}^{n+1}\\=&\Big(1+\frac{\tau}{2}\delta_x^\alpha+\frac{\tau}{2}\delta_y^\beta\Big)U_{i,j}^n+\tau f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}U_{i,j}^n-\frac{1}{2}U_{i,j}^{n-1},U_{i,j}^{n+\frac{1}{2}-m}\Big),\\&1\le i\le M_x-1,\quad 1\le j\le M_y-1.\end{aligned}\tag{3.19}$$

The boundary and initial conditions are discretized as

$$\begin{aligned}&U_{0,j}^n=u(0,y_j,t_n)=0,&&U_{M_x,j}^n=u(L_x,y_j,t_n)=0,\\&U_{i,0}^n=u(x_i,0,t_n)=0,&&U_{i,M_y}^n=u(x_i,L_y,t_n)=0,\\&U_{i,j}^k=\varphi(ih_x,jh_y,k\tau)a+\psi_{i,j}^k,\quad k\le 0.\end{aligned}$$

Then, we can obtain the following stability result.

Theorem 3.4. *The difference scheme* (2.15) *is unconditionally stable.*

Proof. Let $u_{i,j}^n$, $(0\le i\le M_x, 0\le j\le M_y)$ be the numerical solution of (2.15). And $\varepsilon_{i,j}^n=U_{i,j}^n-u_{i,j}^n$, $0\le i\le M_x$, $0\le j\le M_y$ denotes the corresponding error. Then, we have

$$\delta_t\epsilon_{i,j}^{n+1/2}=\frac{1}{2}\delta_x^\alpha(\varepsilon_{i,j}^{n+1}+\varepsilon_{i,j}^n)+\frac{1}{2}\delta_y^\beta(\varepsilon_{i,j}^{n+1}+\varepsilon_{i,j}^n)+f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}U_{i,j}^n-\frac{1}{2}U_{i,j}^{n-1},U_{i,j}^{n+\frac{1}{2}-m}\Big)$$
$$-f\Big(x_i,y_j,t_{n+\frac{1}{2}},\frac{3}{2}u_{i,j}^n-\frac{1}{2}u_{i,j}^{n-1},u_{i,j}^{n+\frac{1}{2}-m}\Big), \tag{3.20a}$$
$$\varepsilon_{i,j}^n=\psi_{i,j}^n,\quad 0\le i\le M_x,\quad 0\le j\le M_y,\quad -m\le n\le 0, \tag{3.20b}$$
$$\varepsilon_{i,j}^n=0,\quad (x_i,y_j)\in\partial\Omega,\quad 1\le n\le N. \tag{3.20c}$$

Similar to the proof of Theorem 3.3, we have

$$\|\varepsilon^{n+1}\|\le\|\varepsilon^0\|+8\tau(L_1+L_2)\sum_{k=1}^{n}\|\varepsilon^k\|+4\tau L_2\sum_{k=-m}^{0}\|\varepsilon^k\|. \tag{3.21}$$

It follows from Lemma 3.4 that

$$\begin{aligned}\|\varepsilon^{n+1}\|&\le\Big(\|\varepsilon^0\|+4\tau L_2\sum_{k=-m}^{0}\|\varepsilon^k\|\Big)\exp(8n\tau(L_1+L_2))\\&\le\Big(\|\varepsilon^0\|+4\tau L_2\sum_{k=-m}^{0}\|\varepsilon^k\|\Big)\exp(8T(L_1+L_2))\\&\le(1+4sL_2)\sqrt{L_xL_y}\exp(8T(L_1+L_2))\max_{\substack{1\le i\le M_x-1,1\le j\le M_y-1,\\-m\le k\le 0}}|\psi_{i,j}^k|.\end{aligned} \tag{3.22}$$

The proof is completed. □

Remark 3.1. Similar to the proofs of Theorem 3.3 and Theorem 3.4, it can be proved that the following results hold

- If the solution of the problem (2.2) satisfy the conditions (A1) and (A2), the method (2.24) is convergent. And the convergence order is $\mathcal{O}(h_x^2+h_y^2+h_z^2+\tau^2)$.
- The difference scheme (2.24) is unconditionally stable.

4 Numerical examples

In order to verify and demonstrate our theoretical results, we present some numerical examples in this section. Define the error and convergence orders as

$$E(h_x,h_y,\tau)=\max_{\substack{1\le i\le M_x,1\le j\le M_y,\\1\le n\le N}}|u_{i,j}^n-u(x_i,y_j,t_n)|,$$

$$E(h_x,h_y,h_z,\tau)=\max_{\substack{1\le i\le M_x,1\le j\le M_y,\\ 1\le k\le M_z,1\le n\le N}}|u^n_{i,j,k}-u(x_i,y_j,z_k,t_n)|,$$

$$Order=\log_2\Big(\frac{E(2h_x,2h_y,2\tau)}{E(h_x,h_y,\tau)}\Big)\quad\text{or}\quad Order=\log_2\Big(\frac{E(2h_x,2h_y,2h_z,2\tau)}{E(h_x,h_y,h_z,\tau)}\Big).$$

Example 4.1. Consider the following two-dimensional Riesz space fractional diffusion equations with delay

$$\begin{cases}\dfrac{\partial u(x,y,t)}{\partial t}=K_x\dfrac{\partial^\alpha u(x,y,t)}{\partial|x|^\alpha}+K_y\dfrac{\partial^\beta u(x,y,t)}{\partial|y|^\beta}\\ \qquad\qquad +f(x,y,t,u(x,y,t),u(x,y,t-s)), & t\in[0,T],\\ u(x,y,t)=x^2(x-1)^2y^2(y-1)^2e^{-t}, & t\in[-s,0],\quad 0\le x\le 1,\quad 0\le y\le 1,\\ u(0,y,t)=u(1,y,t)=0, & u(x,0,t)=u(x,1,t)=0,\quad 0\le t\le T,\end{cases}\tag{4.1}$$

where $T=2$, $s=0.5$, $K_x=K_y=1$, $1<\alpha,\beta\le 2$,

$$\begin{aligned}&f(x,y,t,u(x,y,t),u(x,y,t-s))\\ =&(u(x,y,t)u(x,y,t-s))^2-u(x,y,t)-(xy)^8(1-x)^8(1-y)^8e^{-4t+2s}\\ &+\frac{K_xe^{-t}y^2(y-1)^2}{2\cos(\frac{\alpha\pi}{2})}\Big\{\frac{24}{\Gamma(5-\alpha)}[x^{4-\alpha}+(1-x)^{4-\alpha}]\\ &-\frac{12}{\Gamma(4-\alpha)}[x^{3-\alpha}+(1-x)^{3-\alpha}]+\frac{2}{\Gamma(3-\alpha)}[x^{2-\alpha}+(1-x)^{2-\alpha}]\Big\}\\ &+\frac{K_ye^{-t}x^2(x-1)^2}{2\cos(\frac{\beta\pi}{2})}\Big\{\frac{24}{\Gamma(5-\beta)}[y^{4-\beta}+(1-y)^{4-\beta}]-\frac{12}{\Gamma(4-\beta)}[y^{3-\beta}+(1-y)^{3-\beta}]\\ &+\frac{2}{\Gamma(3-\beta)}[y^{2-\beta}+(1-y)^{2-\beta}]\Big\},\end{aligned}$$

and the exact solution is

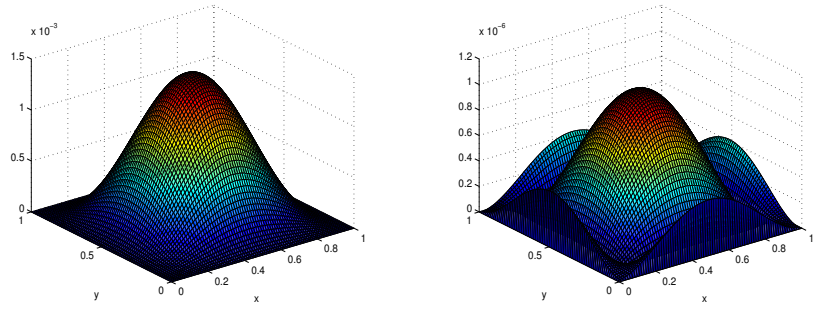
$$u(x,y,t)=x^2(1-x)^2y^2(1-y)^2e^{-t}.$$

One can directly verify that the RHS term satisfies the Lipschitz condition (2.3). When $\alpha=1.6$, $\beta=1.6$ and $\alpha=1.5$, $\beta=1.8$, the errors and the computing orders of (2.17)-(2.18) are shown in Table 1. The numerical results show that the error is very small, and the convergence order is close to 2. Fig. 1 shows that numerical solutions and absolute errors of (4.1) at $t=1$ for $\alpha=\beta=1.6$ with $h_x=h_y=\tau=1/80$. The corresponding results of (4.1) at $t=1$ with $\alpha=1.6$, $\beta=1.8$ are shown in Fig. 2. From the above tables and figures, we can see that these numerical results are consistent with the our theoretical results in Section 3.

Example 4.2. Consider the following Riesz space fractional diffusion equations with de-

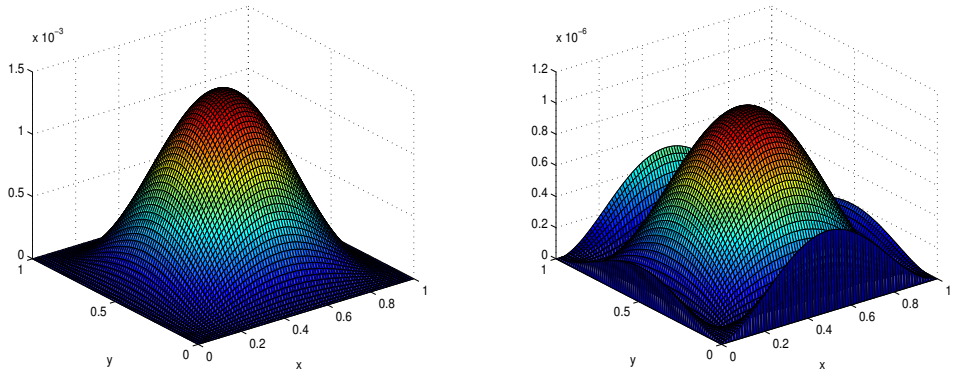
Table 1: The errors and convergence orders of the method (2.17)-(2.18).

$\alpha=1.6,\ \beta=1.6$			$\alpha=1.5,\ \beta=1.8$		
$\tau=h_x=h_y$	$E(h_x,h_y,\tau)$	Order	$\tau=h_x=h_y$	$E(h_x,h_y,\tau)$	Order
$\frac{1}{20}$	3.4911E-05		$\frac{1}{20}$	3.5631E-05	
$\frac{1}{40}$	8.5541E-06	2.0290	$\frac{1}{40}$	8.7483E-06	2.0261
$\frac{1}{80}$	2.0944E-06	2.0321	$\frac{1}{80}$	2.1493E-06	2.0251
$\frac{1}{160}$	5.1331E-07	2.0286	$\frac{1}{160}$	5.2789E-07	2.0256



(a) (b)

Figure 1: Numerical solutions and absolute errors of (4.1) by using the method (2.17)-(2.18) at $t=1$ with $\alpha=1.6$, $\beta=1.6$, $h_x=h_y=\tau=1/80$, (a) numerical solutions, (b) absolute errors.



(a) (b)

Figure 2: Numerical solutions and absolute errors of (4.1) by using the method (2.17)-(2.18) at $t=1$ with $\alpha=1.5$, $\beta=1.8$, $h_x=h_y=\tau=1/80$, (a) numerical solutions, (b) absolute errors.

lay

$$\begin{cases} \dfrac{\partial u(x,y,z,t)}{\partial t}=K_x\dfrac{\partial^\alpha u(x,y,z,t)}{\partial|x|^\alpha}+K_y\dfrac{\partial^\beta u(x,y,z,t)}{\partial|y|^\beta}+K_z\dfrac{\partial^\gamma u(x,y,z,t)}{\partial|z|^\gamma} \\ \qquad\qquad +f(x,y,z,t,u(x,y,z,t),u(x,y,z,t-s)),\quad t\in[0,T], \\ u(x,y,z,t)=x^2(x-1)^2y^2(y-1)^2z^2(z-1)^2e^{-t},\quad t\in[-s,0], \\ \quad 0\le x\le 1,\quad 0\le y\le 1,\quad 0\le z\le 1, \\ u(0,y,z,t)=u(1,y,z,t)=0,\quad u(x,0,z,t)=u(x,1,z,t)=0, \\ u(x,y,0,t)=u(x,y,1,t)=0,\quad 0\le t\le T, \end{cases} \tag{4.2}$$

Table 2: The errors and convergence orders of the method (2.27)-(2.29).

$\alpha=1.2$, $\beta=1.2$, $\gamma=1.2$		
$\tau=h_x=h_y=h_z$	$E(h_x,h_y,h_z,\tau)$	Order
$\frac{1}{10}$	6.3469E-06	
$\frac{1}{20}$	1.5913E-06	1.9958
$\frac{1}{40}$	3.9605E-07	2.0065
$\frac{1}{80}$	9.8278E-07	2.0107
$\alpha=1.7$, $\beta=1.8$, $\gamma=1.9$		
$\tau=h_x=h_y=h_z$	$E(h_x,h_y,h_z,\tau)$	Order
$\frac{1}{10}$	3.4262E-06	
$\frac{1}{20}$	1.1281E-06	1.6027
$\frac{1}{40}$	3.0127E-07	1.9048
$\frac{1}{80}$	7.4678E-08	2.0123

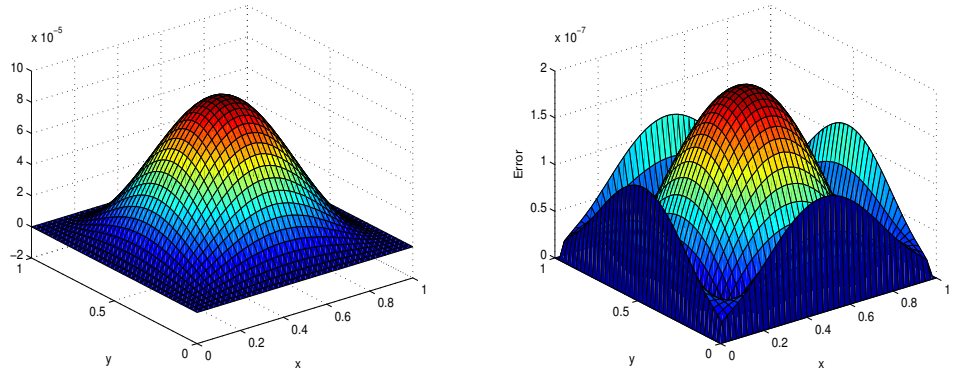
where $T=2$, $s=0.5$, $K_x=K_y=K_z=1$, $1<\alpha,\beta,\gamma\leq 2$,

$$
\begin{aligned}
&f(x,y,z,t,u(x,y,z,t),u(x,y,z,t-s))\\
=&u(x,y,z,t)u(x,y,z,t-s)-u(x,y,z,t)-(xyz)^4(1-x)^4(1-y)^4(1-z)^4e^{-2t+s}\\
&+\frac{K_xe^{-t}y^2(y-1)^2z^2(z-1)^2}{2\cos(\frac{\alpha\pi}{2})}\Big\{\frac{24}{\Gamma(5-\alpha)}[x^{4-\alpha}+(1-x)^{4-\alpha}]\\
&-\frac{12}{\Gamma(4-\alpha)}[x^{3-\alpha}+(1-x)^{3-\alpha}]+\frac{2}{\Gamma(3-\alpha)}[x^{2-\alpha}+(1-x)^{2-\alpha}]\Big\}\\
&+\frac{K_ye^{-t}x^2(x-1)^2z^2(z-1)^2}{2\cos(\frac{\beta\pi}{2})}\Big\{\frac{24}{\Gamma(5-\beta)}[y^{4-\beta}+(1-y)^{4-\beta}]\\
&-\frac{12}{\Gamma(4-\beta)}[y^{3-\beta}+(1-y)^{3-\beta}]+\frac{2}{\Gamma(3-\beta)}[y^{2-\beta}+(1-y)^{2-\beta}]\Big\}\\
&+\frac{K_ze^{-t}x^2(x-1)^2y^2(y-1)^2}{2\cos(\frac{\gamma\pi}{2})}\Big\{\frac{24}{\Gamma(5-\gamma)}[z^{4-\gamma}+(1-z)^{4-\gamma}]\\
&-\frac{12}{\Gamma(4-\gamma)}[z^{3-\gamma}+(1-z)^{3-\gamma}]+\frac{2}{\Gamma(3-\gamma)}[z^{2-\gamma}+(1-z)^{2-\gamma}]\Big\},
\end{aligned}
$$

and the exact solution is

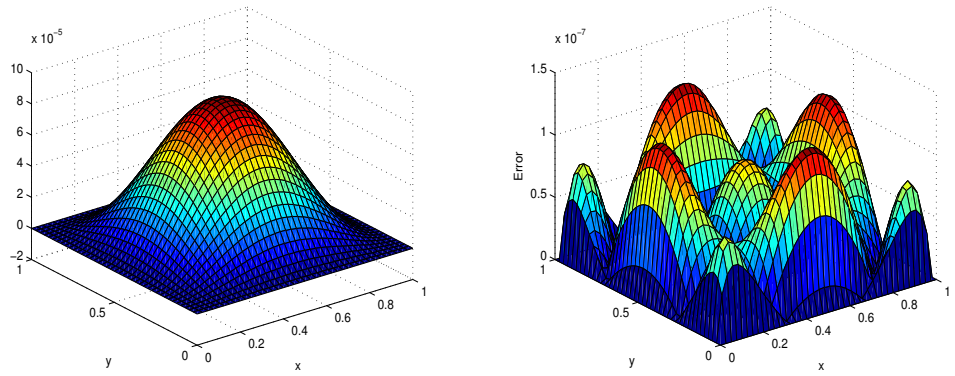
$$u(x,y,z,t)=x^2(1-x)^2y^2(1-y)^2z^2(1-z)^2e^{-t}.$$

Similarly, one can verify that the RHS term satisfies the Lipschitz condition (2.4). We list the errors and convergence orders of (2.27)-(2.29) in Table 2 with respect to $h_x=h_y=h_z=\tau$. The presented results clearly indicate that the method (2.27)-(2.29) is almost second order accuracy. Fig. 3 shows that the numerical solutions and absolute errors of (4.2) at $t=1$, $z=0.5$ for $\alpha=\beta=\gamma=1.2$. The corresponding numerical results for (2.27)-(2.29)



(a) (b)

Figure 3: Numerical solutions and absolute errors of (4.2) by using the method (2.27)-(2.29) at $t=1$, $z=0.5$ with $\alpha=\beta=\gamma=1.2$, $h_x=h_y=h_z=\tau=1/40$, (a) numerical solutions, (b) absolute errors.



(a) (b)

Figure 4: Numerical solutions and absolute errors of (4.2) by using the method (2.27)-(2.29) at $t=1$, $z=0.5$ with $\alpha=1.7$, $\beta=1.8$, $\gamma=1.9$, $h_x=h_y=h_z=\tau=1/40$, (a) numerical solutions, (b) absolute errors.

with $\alpha=1.7$, $\beta=1.8$, $\gamma=1.9$ are shown in Fig. 4. As we all known, it is very difficult to solve the three-dimensional fractional differential equations. From the above tables and figures, we can see that the proposed methods is efficient and accurate to solve the three-dimensional semilinear Riesz space fractional diffusion equations with time delay.

5 Conclusions

In this paper, second order implicit alternating direction methods have been constructed for solving a class of two/three-dimensional semilinear Riesz space fractional diffusion

equations with time delay subject to homogeneous Dirichlet boundary conditions. We established the sharp stability and convergence estimates for the proposed numerical methods. The benchmarking numerical examples verify the effectiveness and efficiency of our numerical scheme. For the future study, we plan to apply the newly derived numerical methods to some inverse problems in the fractional setting, say in particular the Schiffer problem, which is a longstanding problem in the inverse scattering theory [4,9,17,18,25,28–38], but was recently solved in the fractional setting associated with the fractional Helmholtz equation [5].

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