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Novel Rogue Waves for a Mixed Coupled Nonlinear Schrödinger Equation on Darboux-Dressing Transformation

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Abstract. Focusing-defocusing mixed coupled nonlinear Schrödinger equation of localised waves in a two-mode nonlinear fiber is investigated. Novel localised wave solutions are constructed by employing the Darboux-dressing transformation. The set of such solutions includes rogue waves on the soliton background. In addition, for the main characteristics of these solutions, we give the graphs to make readers more aware of the characteristics of these solutions. Hopefully our results can be used to help enrich rogue waves phenomena in nonlinear wave field.

AMS subject classifications: 65M10, 78A48
Key words: Mixed coupled nonlinear Schrödinger equation, Darboux-dressing transformation, breather wave, rogue wave.

1. Introduction

Rogue waves are used to describe huge catastrophic waves unexpectedly arising on relatively calm ocean surface [12] and to characterise extreme wave events in optics [20], plasma [16], Bose-Einstein condensate [5], finance [29, 30], and so on. It is the common belief that rogue waves have three main characteristics:

1) The amplitude of the wave is more than twice (or larger) than the average amplitude of the significant wave height [1].

2) They appear from nowhere and disappear without trace [2].

3) The probability distribution function of the amplitude obeys the unusual L-shaped statistics, which means that the frequency of the wave is higher than predicted by the classical Gaussian distribution [1, 20].

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Recently, rogue waves have been encountered in optical fibers, deep water waves and other fields where nonlinear Schrödinger (NLS) equations are employed. In particular, Peregrine [19] constructed a first-order rational solution of the NLS equation by mathematical method, which is a formal description of a single rogue wave. This solution, the peak amplitude of which is three times of the average height, was later named after him. The rogue waves attracted a great attention in recent years — cf. [6, 7, 10, 31, 34, 35], and it is worth noting their close connection to NLS and coupled nonlinear Schrödinger equation(CNLS), Li *et al.* [13] determined reduced and non-reduced vector rogue wave solutions of CNLS using the generalised Darboux transformation (DT), Feng *et al.* [8] employed DT in order to construct multi-breather solutions of NLS on the background of elliptic functions and expressed them via theta functions, Zhang *et al.* [33] used DT in new localised wave solutions; and so on.

On the other hand, the Darboux-dressing transformation has been used in the study of the classical Schrödinger equation [17], integrable vector nonlinear Schrödinger equations [18], the Manakov system [23], the Kundu-nonlinear Schrödinger equation [25], the coupled cubic-quintic nonlinear Schrödinger equations [28] and in other fields [24,26,27,32].

Numerous works are devoted the two-component case (as so called the Manakov system)

$$
\begin{aligned}
&iu_t + \frac{1}{2}u_{xx} + \sigma\left(|u|^2 + |v|^2\right)u = 0, \\
&iv_t + \frac{1}{2}v_{xx} + \sigma\left(|u|^2 + |v|^2\right)v = 0,
\end{aligned} \tag{1.1}
$$

where $u(x,t)$ and $v(x,t)$ are wave envelopes, and x and t are, respectively, transverse and longitudinal coordinates [3, 9]. Every subscripted variable in the Eqs. (1.1) refers to the partial differentiation. If $\sigma = 1$, the equations represent the defocusing case, and if $\sigma \neq 1$ the focusing case.

In this work, we consider breather and rogue waves of the focusing-defocusing mixed coupled nonlinear Schrödinger equation (mCNLSEs)

$$
\begin{aligned}
&iu_t + \frac{1}{2}u_{xx} + \left(|v|^2 - |u|^2\right)u = 0, \\
&iv_t + \frac{1}{2}v_{xx} + \left(|v|^2 - |u|^2\right)v = 0,
\end{aligned} \tag{1.2}
$$

where u and v are respectively related to the focusing and defocusing type nonlinearities [21]. The terms $|u|^2u$, $|v|^2v$ and $|v|^2u$, $|u|^2v$ are self-phase and cross-phase modulations. The Eqs. (1.2) can be regarded as a mixture of the defocusing and focusing conditions of the Manakov system (1.1) and are vigorously studied. For example, Tian *et al.* [21] considered initial-boundary value problems related to the Fokas method, Vijayajayanthi *et al.* [22] studied bright-dark solitons and their collisions in mixed N-coupled nonlinear Schrödinger equations, Kanna *et al.* [11] investigated the soliton collisions with a shape change by intensity redistribution, Ling *et al.* [14] constructed vector rogue wave and bright-dark rogue wave solutions by using the Darboux transformation. However, to the best of the our knowledge, rogue wave and breather wave solutions are still not connected to the Darboux dressing transformation [17].

This article is organised as follows. In Section 2, we follows the considerations [14,21] and determine the Lax pair for the Eqs. (1.2). This Lax pair in used in the construction of the corresponding Darboux transformation and an asymptotic expansion. In Section 3, we construct exact breather wave solutions. In Section 4, we give the higher order rogue wave solutions related to the Darboux-dressing transformation and Taylor series expansion. We visualise certain solutions to discuss interesting nonlinear phenomena. Finally, our conclusions are given in Section 5.

2. Darboux-Dressing Transformation

Because of complete integrability [14,21], the Lax pair of the Eqs. (1.2) appears as the compatibility condition $\Phi_{tx} = \Phi_{xt}$ for the following pair of linear equations:

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \tag{2.1}$$

where $\Phi = (\phi_1, \phi_2, \phi_3)^T$ is a vector eigenfunction in $\mathbb{C}^2$, ϕ_1, ϕ_2 and ϕ_3 are the complex functions of variables x and t, T is the operation of matrix transposition. Besides, U, V are the 3×3 square matrices,

$$U = \begin{pmatrix} i\lambda & -iu^* & iv^* \\ iu & -i\lambda & 0 \\ iv & 0 & -i\lambda \end{pmatrix}, \quad V = \begin{pmatrix} i\lambda^2 + \frac{1}{2}i(u^*u - v^*v) & -i\lambda u^* - \frac{1}{2}u_x^* & i\lambda v^* + \frac{1}{2}v_x^* \\ i\lambda u - \frac{1}{2}u_x & -i\lambda^2 - \frac{1}{2}iu^*u & \frac{1}{2}iuv^* \\ i\lambda v - \frac{1}{2}v_x & -\frac{1}{2}ivu^* & -i\lambda^2 + \frac{1}{2}iv^*v \end{pmatrix},$$

where $\lambda \in \mathbb{C}$ is the spectral parameter and u^* and v^* are the complex conjugate of u and v, respectively.

Based on the study of [14,21], an appropriate Darboux transformation for the Eqs. (1.2) can be constructed as follows.

Theorem 2.1. *The N-folds Darboux transformation has the form*

$$\Phi_{[N]} = \nabla\Phi, \quad \nabla = I_3 - \frac{(\lambda_1 - \lambda_1^*)}{(\lambda - \lambda_1^*)} \frac{\Lambda_{[N-1]}\Lambda_{[N-1]}^*}{\Lambda_{[N-1]}^*\Lambda_{[N-1]}},$$

where

$$\begin{pmatrix} u_{[N]} \\ v_{[N]} \end{pmatrix} = \begin{pmatrix} u_{[N-1]} \\ v_{[N-1]} \end{pmatrix} + \frac{2(\lambda_1^* - \lambda_1)\phi_{[1N-1]}^*}{|\phi_{1[N-1]}|^2 + |\phi_{2[N-1]}|^2 + |\phi_{3[N-1]}|^2} \times \begin{pmatrix} \phi_{2[N-1]} \\ \phi_{3[N-1]} \end{pmatrix},$$

and

$$I_3 = \operatorname{diag}(1,1,1), \quad \Lambda_{[N-1]} = \Phi(x,t,\lambda_1)Z_{[N-1]} = \left(\phi_{1[N-1]}, \phi_{2[N-1]}, \phi_{3[N-1]}\right)^T$$

with

$$Z_{[N-1]} = \left(z_{1[N-1]}, z_{[2N-1]}, z_{3[N-1]}\right)^T$$

being column vector contained free real parameters, Φ is the fundamental solution of the Lax equation (2.1) *with $\lambda = \lambda_1$ depending on the variables x and t, $*$ denotes the conjugate, Φ is a column vector function of λ.*

Proof. The proof is similar to the proofs in [14, 17, 27] and is omitted here. □

We discover that Ref. [14] mainly uses matrix analysis method to solve mCNLS equations and to obtain complete classification of non-singular solutions. In this paper, the breather solution is obtained based on the DT equation and the seed solution, and the rogue wave is obtained by using Taylor expansion. The main differences are:

1. Kuznetsov-Ma soliton and Akhmediev breather are verified.
2. By changing the parameters, the Peregrine structure and the boomeron type bright soliton are obtained.
3. The second-order rogue wave are constructed in this paper.

3. Breather Wave Solutions

To derive the exact breather wave solutions of the Eqs. (1.2), we start with seed solutions of the form

$$u = a_1 \exp(ikx + iwt), \quad v = a_2 \exp(ikx + iwt)$$

with the dispersion relations

$$w = -a_1^2 + a_2^2 - \frac{1}{2}k^2,$$

where a_1 and a_2 are real parameters and k is the wave number. According to [17, 18, 27], the corresponding solutions of the Lax system (2.1) have the form

$$\Phi = \begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \\ \phi_3(x,t) \end{pmatrix} = AFGZ, \tag{3.1}$$

$$F = e^{i\Theta x}, \quad G = e^{i\Lambda t}, \tag{3.2}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(ikx + iwt) & 0 \\ 0 & 0 & \exp(ikx + iwt) \end{pmatrix}, \tag{3.3}$$

and $Z = (z_1, z_2, z_3)^{\mathsf{T}}$ is a free complex vector.

Substituting Φ (3.2) into the Lax pairs (2.1), leads to the following representations of Θ and Λ:

$$\Theta = \begin{pmatrix} \lambda & -a_1 & a_2 \\ a_1 & -k-\lambda & 0 \\ a_2 & 0 & -k-\lambda \end{pmatrix}, \tag{3.4}$$

$$\Lambda=\begin{pmatrix} \lambda^2+\frac{1}{2}a_1^2-\frac{1}{2}a_2^2 & \frac{1}{2}a_1(-2\lambda+k) & -\frac{1}{2}a_2(-2\lambda+k) \\ -\frac{1}{2}a_1(-2\lambda+k) & -\lambda^2-\frac{1}{2}a_1^2-w & \frac{1}{2}a_1a_2 \\ -\frac{1}{2}a_1(-2\lambda+k) & -\frac{1}{2}a_1a_2 & -\lambda^2+\frac{1}{2}a_2^2-w \end{pmatrix}.$$

It is easily seen that (3.4) satisfies

$$\begin{aligned}&[\Theta,\Lambda]=\Theta\Lambda-\Lambda\Theta=0,\\ &A_x+iA\Theta-UA=0,\\ &A_t+iA\Lambda-VA=0.\end{aligned}$$

According to the Eq. (3.2), the matrix F can be written as

$$F=\exp\left(-\frac{ikx}{2}\right)\begin{pmatrix} f_{11} & -2ia_1\tau\sinh\left(\frac{1}{2}\tau x\right) & 2ia_2\tau\sinh\left(\frac{1}{2}\tau x\right) \\ 2ia_1\tau\sinh\left(\frac{1}{2}\tau x\right) & f_{22} & f_{23} \\ 2ia_2\tau\sinh\left(\frac{1}{2}\tau x\right) & f_{32} & f_{33} \end{pmatrix}, \quad (3.5)$$

where

$$\begin{aligned}
&\tau=\sqrt{4\left(a_1^2-a_2^2\right)-k^2-4k\lambda-4\lambda^2},\\
&f_{11}=i\tau(k+2\lambda)\sinh\left(\frac{1}{2}\tau x\right)+\tau^2\cosh\left(\frac{1}{2}\tau x\right),\\
&f_{22}=-\frac{\tau}{a_1^2-a_2^2}\left((ik+2i\lambda)a_1^2\sinh\left(\frac{1}{2}\tau x\right)-a_1^2\tau\cosh\left(\frac{1}{2}\tau x\right)+a_2^2\tau\exp\left(-\frac{1}{2}ix(k+2\lambda)\right)\right),\\
&f_{23}=\frac{a_1a_2\tau}{a_1^2-a_2^2}\left((ik+2i\lambda)\sinh\left(\frac{1}{2}\tau x\right)-\tau\cosh\left(\frac{1}{2}\tau x\right)+\tau\exp\left(-\frac{1}{2}ix(k+2\lambda)\right)\right),\\
&f_{32}=-\frac{a_1a_2\tau}{a_1^2-a_2^2}\left((ik+2i\lambda)\sinh\left(\frac{1}{2}\tau x\right)-\tau\cosh\left(\frac{1}{2}\tau x\right)+\tau\exp\left(-\frac{1}{2}ix(k+2\lambda)\right)\right),\\
&f_{33}=\frac{\tau}{a_1^2-a_2^2}\left((ik+2i\lambda)a_2^2\sinh\left(\frac{1}{2}\tau x\right)-a_2^2\tau\cosh\left(\frac{1}{2}\tau x\right)+a_1^2\tau\exp\left(-\frac{1}{2}ix(k+2\lambda)\right)\right).
\end{aligned}$$

Similar calculations yield

$$G=\exp\left(-\frac{iwt}{2}\right)\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \quad (3.6)$$

where

$$\begin{aligned}
&\upsilon=\sqrt{\left(a_1^2-a_2^2\right)k^2+4\lambda\left(a_1^2-a_2^2\right)k-w^2+\left(-2a_1^2+2a_2^2-4\lambda^2\right)w-a_2^4+2a_1^2a_2^2-a_2^4-4\lambda^4},\\
&g_{11}=i\left(a_1^2-a_2^2+2\lambda^2+w\right)\upsilon\sinh\left(\frac{1}{2}\upsilon t\right)+\upsilon^2\cosh\left(\frac{1}{2}\upsilon t\right),
\end{aligned}$$

$$g_{12} = ia_1\upsilon(k-2\lambda)\sinh\left(\frac{1}{2}\upsilon t\right), \quad g_{13} = -ia_2\upsilon(k-2\lambda)\sinh\left(\frac{1}{2}\upsilon t\right),$$

$$g_{22} = -\frac{\upsilon}{a_1^2-a_2^2}\Bigg(i\big(a_1^4-a_1^2a_2^2+2a_1^2\lambda^2+a_1^2w\big)\sinh\left(\frac{1}{2}\upsilon t\right)-a_1^2\upsilon\cosh\left(\frac{1}{2}\upsilon t\right)$$
$$+a_2^2\upsilon\exp\left(-\frac{1}{2}it\big(2\lambda^2+w\big)\right)\Bigg),$$

$$g_{23} = \frac{a_1a_2\tau}{a_1^2-a_2^2}\Bigg(i\big(a_1^2-a_2^2+2\lambda^2+w\big)\sinh\left(\frac{1}{2}\upsilon t\right)-\upsilon\cosh\left(\frac{1}{2}\upsilon t\right)$$
$$+\upsilon\exp\left(-\frac{1}{2}it\big(2\lambda^2+w\big)\right)\Bigg),$$

$$g_{32} = -\frac{a_1a_2\tau}{a_1^2-a_2^2}\Bigg(i\big(a_1^2-a_2^2+2\lambda^2+w\big)\sinh\left(\frac{1}{2}\upsilon t\right)-\upsilon\cosh\left(\frac{1}{2}\upsilon t\right)$$
$$+\upsilon\exp\left(-\frac{1}{2}it\big(2\lambda^2+w\big)\right)\Bigg),$$

$$g_{33} = \frac{\upsilon}{a_1^2-a_2^2}\Bigg(i\big(a_1^2a_2^2-a_2^4+2a_2^2\lambda^2+a_2^2w\big)\sinh\left(\frac{1}{2}\upsilon t\right)-a_2^2\upsilon\cosh\left(\frac{1}{2}\upsilon t\right)$$
$$+a_1^2\upsilon\exp\left(-\frac{1}{2}it\big(2\lambda^2+w\big)\right)\Bigg),$$

$$g_{12} = -g_{21}, \quad g_{13} = g_{31}.$$

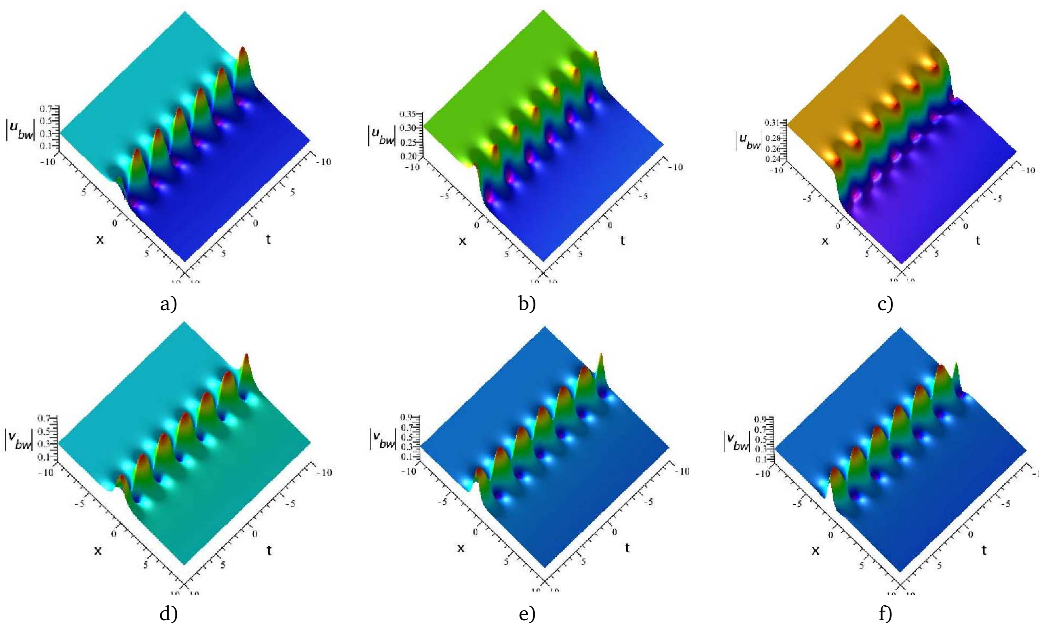
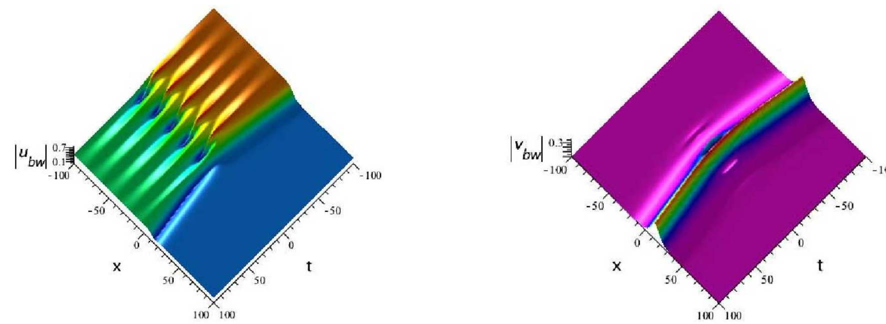


Figure 1: Akhmediev breathers, $a_1 = i/4, a_2 = 1/4, k = 0, \lambda = i, z_1 = 1, z_2 = 1$. (a,d) $z_3 = 1$, (b,e) $z_3 = 10$, (c,f) $z_3 = 10000$.



a) b)

Figure 2: Kuznetsov-Ma solutions, $a_1 = i/4, a_2 = 0, k = 0, \lambda = 1/5i, z_1 = 1, z_2 = 1, z_3 = 1$.

Substituting (3.5) and (3.6) into (3.1), and according to Theorem 2.1, the corresponding novel breathers are displayed in Figs. 1 and 2.

Solutions in Fig. 1 demonstrate time locality and space periodicity, which is called Akhmediev breathers [4]. Fig. 2 has opposite characteristics, which is called Kuznetsov-Ma soliton [15].

4. Novel Rogue Waves

To construct novel rogue wave solutions of the Eqs. (1.2) of exponential and polynomial functions via Taylor expansions formulas, we choose $\lambda = \pm i\theta(1+\varepsilon)$, $0 < \varepsilon < 1, \theta = \sqrt{-a_1^2 + a_2^2}$. The corresponding breather wave solutions (3.2),(3.3) have the form

$$
\begin{aligned}
&F(\lambda)|_{\lambda=\pm i\theta(1+\varepsilon)} = \Sigma_{N=0}^{\infty} F_N \varepsilon^N, \\
&G(\lambda)|_{\lambda=\pm i\theta(1+\varepsilon)} = \exp\left(\frac{\theta^2 it}{2}\right) \Sigma_{N=0}^{\infty} G_N \varepsilon^N,
\end{aligned}
$$

where F_N, G_N are the N-th coefficient matrices of ε. Analogously,

$$
Z_\varsigma = \Sigma_{\varsigma=0}^{N} (z_{1\varsigma}, z_{2\varsigma}, z_{3\varsigma}) \varepsilon^\varsigma
$$

and

$$
\begin{aligned}
&\Phi(\lambda) = |_{\lambda=\pm i\theta(1+\varepsilon)} = \Sigma_{N=0}^{\infty} \Phi_N \varepsilon^N, \\
&\Phi_N = \mho_0 = \exp\left(\frac{\theta^2 it}{2}\right) A \Sigma_{\varsigma=0}^{N} \Sigma_{\iota}^{N} F_\varsigma G_\iota (z_{1,N-\varsigma-\iota}, z_{2\varsigma,N-\varsigma-\iota}, z_{3\varsigma,N-\varsigma-\iota}).
\end{aligned}
$$

Moreover, novel rogue waves $\Sigma_{\varsigma=0}^{\infty} (z_{1\varsigma}, z_{2\varsigma}, z_{3\varsigma}) \varepsilon^\varsigma$ can be written as

$$
\Sigma_{\varsigma=0}^{\infty} (z_{1\varsigma}, z_{2\varsigma}, z_{3\varsigma}) \varepsilon^\varsigma = \exp(i\Theta X + i\Lambda T) L|_{\lambda=\pm i\theta(1+\varepsilon)},
$$

where

$$
X = \Sigma_{\varsigma=0}^{N} R_N \varepsilon^N, \quad T = \Sigma_{\varsigma=0}^{N} S_N \varepsilon^N, \quad L = (L_1, L_2, L_3)'.
$$

We can use Theorem 2.1 to obtain new higher order rogue waves. For $N=1$, the first-order rogue waves have the form

$$\begin{pmatrix} u_{[r1]} \\ v_{[r1]} \end{pmatrix} = \begin{pmatrix} u_{[0]} \\ v_{[0]} \end{pmatrix} + \frac{2(\lambda_1^* - \lambda_1)\phi_{1[0]}^*}{|\phi_{1[0]}|^2 + |\phi_{2[0]}|^2 + |\phi_{3[0]}|^2} \times \begin{pmatrix} \phi_{2[0]} \\ \phi_{3[0]} \end{pmatrix}, \tag{4.1}$$

where

$$\begin{pmatrix} \varphi_{1[0]} \\ \varphi_{2[0]} \\ \varphi_{3[0]} \end{pmatrix} = \mho_0 = \exp\left(\frac{\theta^2 it}{2}\right) A F_0 G_0 Z_0, \quad Z_0 = \begin{pmatrix} z_{1[0]} \\ z_{2[0]} \\ z_{3[0]} \end{pmatrix},$$

$$F_0 = \begin{pmatrix} 1-\theta x & -ia_1 x & ia_2 x \\ ia_1 x & \frac{-\theta a_1^2 x + \exp(\theta x) a_2^2 - a_1^2}{\theta^2} & \frac{a_1 a_2(\theta x - \exp(\theta x)+1)}{\theta^2} \\ ia_2 x & \frac{a_1 a_2(-\theta x + \exp(\theta x) - 1)}{\theta^2} & \frac{-\theta a_2^2 x + \exp(\theta x) a_1^2 - a_2^2}{\theta^2} \end{pmatrix},$$

$$G_0 = \begin{pmatrix} 1 - it\theta^2 & a_1 t\theta & -a_2 t\theta \\ -a_1 t\theta & \frac{1}{\theta^2}(\chi_1) & \frac{a_1 a_2}{\theta^2}\left(it\theta^2 + 1 - \exp(\frac{\theta^2 it}{2})\right) \\ -a_2 t\theta & \frac{a_1 a_2}{\theta^2}\left(it\theta^2 + 1 - \exp(\frac{\theta^2 it}{2})\right) & \frac{1}{\theta^2}(\chi_2) \end{pmatrix},$$

$$\chi_1 = ia_1^4 t - ia_1^2 a_2^2 t - a_1^2 + a_2^2 \exp\left(\frac{\theta^2 it}{2}\right), \quad \chi_2 = ia_2^4 t - ia_1^2 a_2^2 t + a_2^2 - a_1^2 \exp\left(\frac{\theta^2 it}{2}\right).$$

Fig. 3 shows three different types of first-order rogue waves to illustrate rogue waves (4.1).

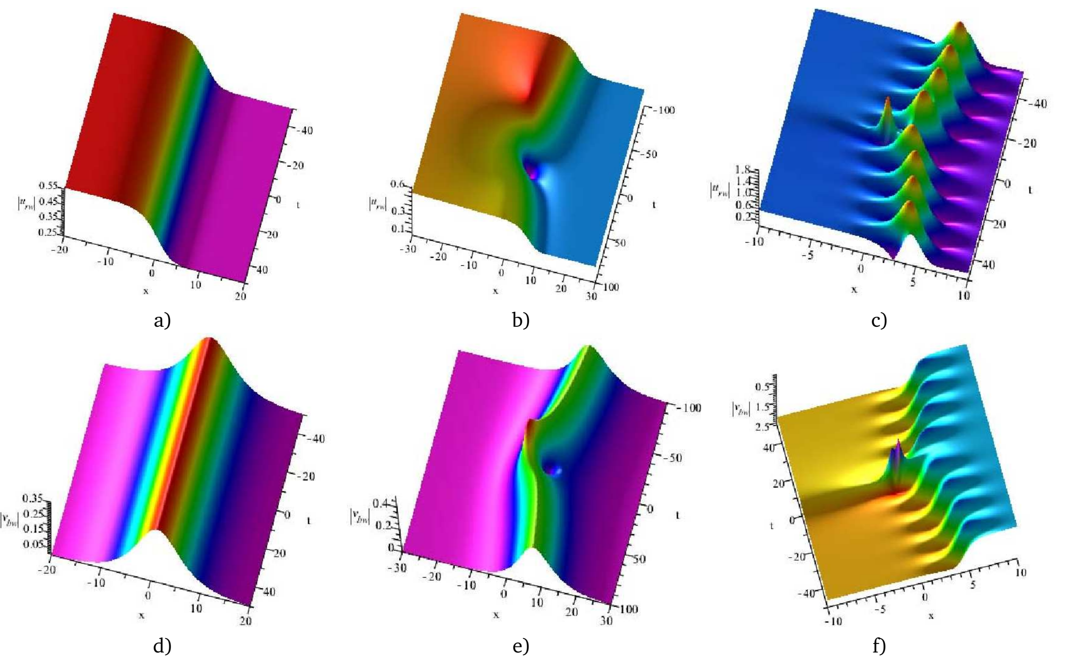


Figure 3: First-order rogue waves from Eqs. (4.1), $a_1 = i/4, a_2 = 0, k = 0, \lambda = 1/5i, z_1 = 1, z_2 = 1$. (a,d) $z_3 = 1$, (b,e) $z_3 = 10$, (c,f) $z_3 = 10000$.

Figs. 3(a,b,d,e) show that the Peregrine structure and boomeron type bright solitons appear with the increase of z_3. According to Figs. 3(b,e), the first-order rogue waves interact with an amplitude-varying soliton in u and one bright soliton and the first-order rogue waves in v. The Peregrine bump coexists and interacts with breather-like solitons, and the breather bends towards to the first-order rogue waves of Eqs. (1.2), cf. Figs. 3(c,f).

Analogously, the second-order rogue waves — i.e. if $N = 2$, have the form

$$\begin{pmatrix} u_{[r2]} \\ v_{[r2]} \end{pmatrix} = \begin{pmatrix} u_{[r1]} \\ v_{[r1]} \end{pmatrix} + \frac{2(\lambda_1^* - \lambda_1)\phi_{1[1]}^*}{|\phi_{1[1]}|^2 + |\phi_{2[1]}|^2 + |\phi_{3[1]}|^2} \times \begin{pmatrix} \phi_{2[1]} \\ \phi_{3[1]} \end{pmatrix}, \tag{4.2}$$

where

$$\begin{pmatrix} \phi_{1[1]} \\ \phi_{2[1]} \\ \phi_{3[1]} \end{pmatrix} = \mho_1 = T_1\Omega_1 + i\theta\Omega_0, \quad T_1 = 2i\theta(I_3 - P_1), \quad P_1 = \frac{\mho_0\mho_0^\dagger}{\mho_0^\dagger\mho_0},$$

$$\Omega_1 = \exp\left(\frac{\theta^2 it}{2}\right)A(F_0G_1 + F_1G_0)Z_0 + F_0G_0Z_1, \quad Z_0 = \begin{pmatrix} z_{1[0]} \\ z_{2[0]} \\ z_{3[0]} \end{pmatrix}, \quad Z_1 = \begin{pmatrix} z_{1[1]} \\ z_{2[1]} \\ z_{3[1]} \end{pmatrix},$$

$$F_1 = \begin{pmatrix} \frac{x}{3}(-\theta^3x^2 + 3\theta^2x - 3\theta) & -\frac{ia_1x^3}{\theta^2} & \frac{ia_2x^3}{\theta^2} \\ \frac{ia_1x^3}{\theta^2} & f_1[22] & f_1[23] \\ \frac{ia_2x^3}{\theta^2} & f_1[32] & f_1[33] \end{pmatrix},$$

$$G_1 = \begin{pmatrix} -\frac{t}{3}(-\theta^6t^2 + 3\theta^4t + 6i\theta^2) & \frac{a_1t}{3\theta}(-\theta^6t^2 + 3\theta^2) & \frac{-a_2t}{3\theta}(-\theta^6t^2 + 3\theta^2) \\ \frac{-a_1t}{3\theta}(-\theta^6t^2 + 3\theta^2) & g_1[22] & g_1[23] \\ \frac{-a_2t}{3\theta}(-\theta^6t^2 + 3\theta^2) & g_1[32] & g_1[33] \end{pmatrix}$$

with

$$\begin{aligned}
f_1[22] &= \frac{x}{3\theta^2}\left(-\theta^3a_1^2x^2 - 3a_1^2\theta^2x + 3\theta\exp(x\theta)a_2^2 - 3\theta a_1^2\right), \\
f_1[23] &= \frac{-a_1a_2x}{3\theta^2}\left(-\theta^3x^2 - 3x\theta^2 + 3\theta\exp(x\theta) - 3\theta\right), \\
f_1[32] &= \frac{a_1a_2x}{3\theta^2}\left(-\theta^3x^2 - 3x\theta^2 + 3\theta\exp(x\theta) - 3\theta\right), \\
f_1[33] &= \frac{-x}{3\theta^2}\left(-\theta^3a_2^2x^2 - 3a_2^2\theta^2x + 3\theta\exp(x\theta)a_1^2 - 3\theta a_2^2\right), \\
g_1[22] &= \frac{t}{3}\left(ia_1^2\theta^4t^2 + 3a_1^2\theta^2t - 6ia_1^2 + 6ia_2^2\exp\left(\frac{it}{2}\theta^2\right)\right), \\
g_1[23] &= \frac{-a_1a_2t}{3}\left(i\theta^4t^2 + 3\theta^2t - 6i + 6i\exp\left(\frac{it}{2}\theta^2\right)\right), \\
g_1[32] &= \frac{-a_1a_2t}{3}\left(i\theta^4t^2 + 3\theta^2t - 6i + 6i\exp\left(\frac{it}{2}\theta^2\right)\right), \\
g_1[33] &= \frac{t}{3}\left(ia_2^2\theta^4t^2 + 3a_2^2\theta^2t - 6ia_2^2 + 6ia_1^2\exp\left(\frac{it}{2}\theta^2\right)\right).
\end{aligned}$$

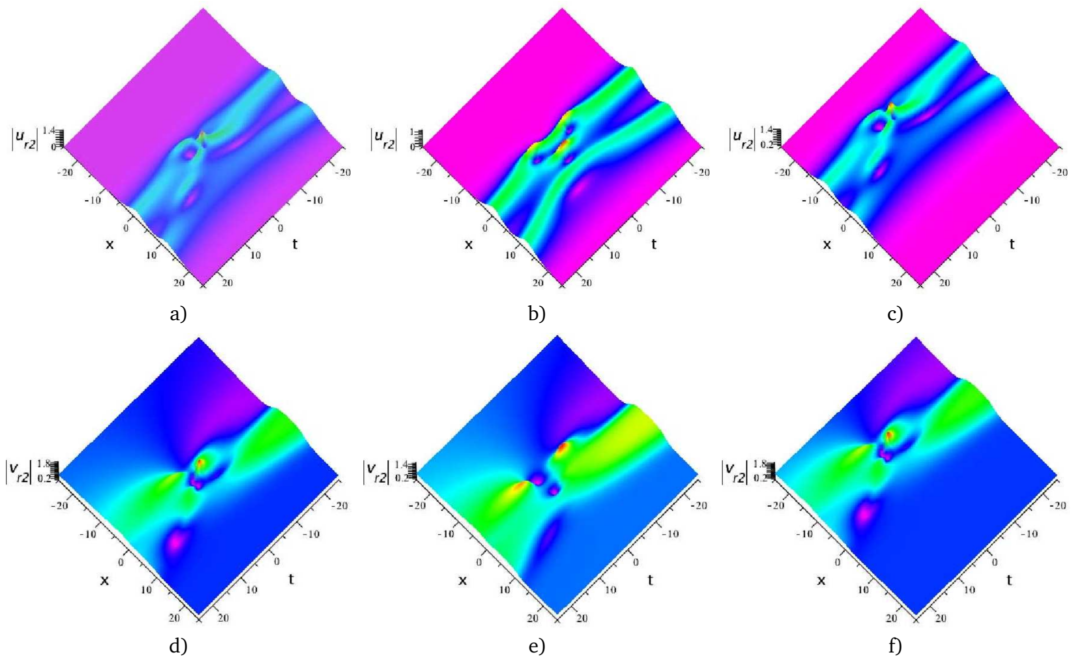


Figure 4: Second-order rogue waves from Eqs. (4.2), $a_1 = 0, a_2 = 1/2, k = 0, r_0 = r_1 = 0, l_1 = l_2 = l_3 = 1$. (a, d) $s_0 = s_1 = 0$, (b, e) $s_0 = 0, s_1 = 10$, (c, f) $s_0 = 10, s_1 = 0$.

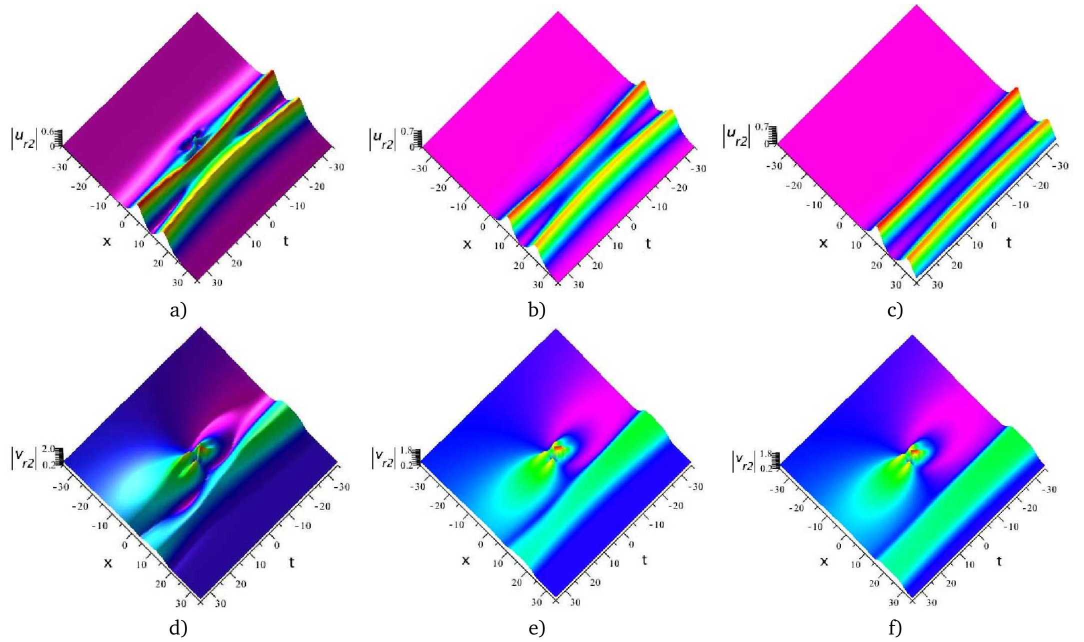


Figure 5: Second-order rogue waves obtained from Eqs. (4.2), $a_1 = 0, a_2 = 1/2, k = 0, r_0 = r_1 = s_0 = s_1 = 0, l_1 = l_2 = 1$. (a, d) $l_3 = 5$, (b, e) $l_3 = 50$, (c, f) $l_3 = 1000$.

Fig. 4 shows that the shape of the second-order rogue wave of u_{r2} changes when s_1 grows. The solution contains a single peak and splits into a wave with three peaks. The solution is called the three sisters or the rogue wave triplet. It can be noted that the change of s_2 has no effect on u_{r2} but only on v_{r2}. The top line in Fig. 5 representing u_{r2}, displays the interaction between second-order rogue waves and bright-soliton waves, whereas the next line demonstrates the interaction between second-order rogue waves and dark-soliton waves of v_{r2}.

The N-order rogue waves can be constructed by the same method, However, the corresponding calculations are tedious, the expressions cumbersome, and are omitted here.

5. Conclusions and Discussions

We study the breather and rogue waves related to the Eqs. (1.2) via the Darboux-dressing transformation. Based on the Lax pair, suitable periodic seed solution and the Taylor series expansion, the novel solutions (novel breather solution and novel rogue waves) are constructed. $a_1, a_2, k, \lambda, l_i, r_i, s_i,\ i = 1,2,3$ are all free parameters, which play an important role in controlling the dynamic properties. Visualisation of these solutions can help to further understand the characteristics of the Eqs. (1.2). We also hope that in the future, such solutions can be observed in experiments.

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