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The Crank-Nicolson/Explicit Scheme for the Natural Convection Equations with Nonsmooth Initial Data

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Abstract. In this article, a Crank-Nicolson/Explicit scheme is designed and analyzed for the time-dependent natural convection problem with nonsmooth initial data. The Galerkin finite element method (FEM) with stable MINI element is used for the velocity and pressure and linear polynomial for the temperature. The time discretization is based on the Crank-Nicolson scheme. In order to simplify the computations, the nonlinear terms are treated by the explicit scheme. The advantages of our numerical scheme can be list as follows: (1) The original problem is split into two linear subproblems, these subproblems can be solved in each time level in parallel and the computational sizes are smaller than the origin one. (2) A constant coefficient linear discrete algebraic system is obtained in each subproblem and the computation becomes easy. The main contributions of this work are the stability and convergence results of numerical solutions with nonsmooth initial data. Finally, some numerical results are presented to verify the established theoretical results and show the performances of the developed numerical scheme.

AMS subject classifications: 65M10, 65N30, 76Q10

Key words: Natural-Convection equations, Crank-Nicolson/Explicit scheme, nonsmooth initial data, error estimates.

1 Introduction

In this paper, we consider the following natural convection equations:

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$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -\kappa\nu^2 jT + \mathbf{f} & \text{in } \Omega\times(0,T^{final}_{time}], \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega\times(0,T^{final}_{time}], \\ T_t - Pr^{-1}\nu\Delta T + \mathbf{u}\cdot\nabla T = g & \text{in } \Omega\times(0,T^{final}_{time}], \\ \mathbf{u} = 0, \quad T = 0 & \text{on } \partial\Omega\times(0,T^{final}_{time}], \\ \mathbf{u}(x,0) = \mathbf{u}_0, \quad T(x,0) = T_0 & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where $\mathbf{u}$, T, p are the velocity, temperature and the pressure, $\mathbf{f}$ and g are the body forces, Ω is a bounded convex polygonal domain, the parameters ν, κ and Pr are the viscosity, Groshoff and Prandtl numbers, $j = (0,1)^T$ is the vector of gravitational acceleration, $T^{final}_{time} > 0$ is the final time.

The natural convection problem is an important system with dissipative nonlinear terms in atmospheric dynamics (see [4, 25]), it not only inherits all difficulties of the Navier-Stokes equations, but also contains strong coupling among variables and nonlinear terms. Hence, finding the numerical solutions becomes a difficult task, and several efficient numerical methods have been developed in recent years, for examples, [5, 8, 27] for the discontinuous methods, [24, 26] for the lattice Boltzmann method, [7] for the stabilized method, [21, 23] for the iterative schemes.

As a classical second order scheme, the Crank-Nicolson scheme has been used to treat various problems. Here we just refer to [20, 22] for the linear problems, [3] for the semilinear parabolic problem and the reference therein as the examples. Generally speaking, the implicit scheme for nonlinear term is unconditionally stable and has optimal error estimates, but we need to treat a nonlinear problem at each step and a lot of computing cost is required. In order to simplify the computations, some variants of the Crank-Nicolson scheme were developed, for examples, the Crank-Nicolson extrapolation scheme [12, 14, 28], the Crank-Nicolson/Newton scheme [9]. The explicit scheme for nonlinear term is another way to treat the nonlinear term, one of the most important advantages is that the discrete algebraic system with a constant coefficient matrix is obtained at each time level. However, a restriction on the time-step was required. For examples we can refer to the Crank-Nicolson/Adams-Bashforth scheme [10, 14, 16, 31] and the references therein.

In this paper, we consider the Crank-Nicolson/Explicit scheme for the natural convection equations with nonsmooth initial data. In this way, the origin problem is split into two linear subproblems, and these subproblems with the constant coefficient matrix can be solved easily in each time level. Compared with [32, 33], the main contributions can be list as follows:

(1) Under some restrictions on time step, almost unconditional stability results of numerical solutions in various norms are established with nonsmooth initial data.

(2) By introducing the weight function and using the negative norm technique, under the same time step conditions, we obtain that the Crank-Nicolson/Explicit

scheme has the same convergence as the Crank-Nicolson extrapolation and Crank-Nicolson/Adams-Bashforth schemes, but the Crank-Nicolson/Explicit scheme implements easier.

(3) The Crank-Nicolson/Explicit scheme splits the origin problem into two linear subproblems and these subproblems can be solved in parallel. Furthermore, the computational size is reduced and the computational cost is saved.

The rest of this article is organized as follows. In Section 2, some basic assumptions and results of the natural convection problem are presented, the spatial semidiscrete numerical scheme and the corresponding stability and convergence results are also provided. Section 3 is devoted to develop the Crank-Niconlson/Explicit scheme for the natural convection equations, the stability results of numerical solutions with nonsmooth initial data are established. Convergence results of numerical solutions are provided in Section 4. Finally, Section 5 provides some numerical results to show the performances of the considered numerical scheme.

2 Preliminaries

2.1 Weak form and some basic results

In this paper, the standard Sobolev spaces and norms are used [1]. For example, the space $L^2(\Omega)^d$, $(d=1,2)$ is associated with the usual L^2-scalar product $(\cdot)$ and L^2-norm $\|\cdot\|_0$. For convenience, we use the notations

$$\begin{aligned}
&X=H_0^1(\Omega)^2, \quad W=H_0^1(\Omega), \quad M=L_0^2(\Omega)=\Big\{\varphi\in L^2(\Omega):\int_\Omega \varphi dx=0\Big\},\\
&Z=L^2(\Omega), \quad V=\{\mathbf{v}\in X:(\nabla\cdot\mathbf{v},q)=0,\ \forall q\in M\},\\
&Y=L^2(\Omega)^2, \quad H=\{\mathbf{v}\in Y,\ \nabla\cdot\mathbf{v}=0,\ \mathbf{v}\cdot n|_{\partial\Omega}=0\}.
\end{aligned}$$

The spaces W and X are equipped with the usual product and norm $\|\nabla\mathbf{u}\|_0^2=(\nabla\mathbf{u},\nabla\mathbf{u})$.

Set $A=-P\Delta$, where P is the L^2-orthogonal projection of Z onto W or Y onto H. Assume that Ω is such that of the domain of A is given by (see [12,17,30])

$$D(A)=H^2(\Omega)^2\cap V \quad \text{and} \quad Q(A)=H^2(\Omega)\cap W.$$

The following assumption about the prescribed data for problem (1.1) is needed.

Assumption 2.1. (A1) The initial data $\mathbf{u}_0(x)$, $T_0(x)$ and the force $\mathbf{f}$, g satisfy

$$\sup_{0\le t\le T_{time}^{final}}\{\|\mathbf{f}(t)\|_1+\|\mathbf{f}_t(t)\|_0+\|\mathbf{f}_{tt}(t)\|_0+\|g(t)\|_1+\|g_t(t)\|_0+\|g_{tt}(t)\|_0\}\le\widetilde{C},$$

$$\begin{cases}\|\nabla\mathbf{u}_0\|_0+\|\nabla T_0\|_0+\|\mathbf{u}_0\|_{L^\infty}+\|T_0\|_{L^\infty}\le\widetilde{C}, & \mathbf{u}_0\in V\cap L^\infty(\Omega)^2,\quad T_0\in H^1(\Omega)\cap L^\infty(\Omega),\\ \|\nabla\mathbf{u}_0\|_0+\|\nabla T_0\|_0\le\widetilde{C}, & \mathbf{u}_0\in V,\quad T_0\in H^1,\end{cases}$$

where $\widetilde{C}$ is a general positive constant.

Define the bilinear forms $a(\cdot,\cdot)$, $\tilde{a}(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X\times X$, $W\times W$ and $X\times M$ by

$$a(\mathbf{u},\mathbf{v})=\nu(\nabla\mathbf{u},\nabla\mathbf{v}),\quad d(\mathbf{v},q)=(q,\mathrm{div}\mathbf{v}),\quad \tilde{a}(T,\psi)=Pr^{-1}\nu(\nabla T,\nabla\psi),$$

the trilinear forms $b(\mathbf{u},\mathbf{v},\mathbf{w})$ and $\tilde{b}(\mathbf{u},T,\psi)$ on $X\times X\times X$ and $X\times W\times W$ are

$$\begin{aligned}
b(\mathbf{u},\mathbf{v},\mathbf{w})&=((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})+\frac{1}{2}((\mathrm{div}\mathbf{u})\mathbf{v},\mathbf{w})=\frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})-\frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}),\\
\tilde{b}(\mathbf{u},T,\psi)&=((\mathbf{u}\cdot\nabla)T,\psi)+\frac{1}{2}((\mathrm{div}\mathbf{u})T,\psi)=\frac{1}{2}((\mathbf{u}\cdot\nabla)T,\psi)-\frac{1}{2}((\mathbf{u}\cdot\nabla)\psi,T).
\end{aligned}$$

With above notations, for all $(\mathbf{v},q,\psi)\in X\times M\times W$, $0\le t\le T_{time}^{final}$, the weak form of natural convection equations (1.1) aims to find $(\mathbf{u},p,T)\in X\times M\times W$, such that

$$\begin{cases}
(\mathbf{u}_t,\mathbf{v})+a(\mathbf{u},\mathbf{v})-d(\mathbf{v},p)+d(\mathbf{u},q)+b(\mathbf{u},\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v})-\kappa\nu^2(jT,\mathbf{v}),\\
(T_t,\psi)+\tilde{a}(T,\psi)+\tilde{b}(\mathbf{u},T,\psi)=(g,\psi),\\
\mathbf{u}(x,0)=\mathbf{u}_0,\quad T(x,0)=T_0,\quad \mathbf{u}_t|_{\partial\Omega}=0,\quad T_t|_{\partial\Omega}=0.
\end{cases}\tag{2.1}$$

The following results can be found in Chapter 5 of Reference [25].

Theorem 2.1. *Under the assumption of $\partial\Omega\in C^2$ or $\Omega\in\mathbb{R}^2$ is a convex polygon and $T_0\in C^1(0,T_{time}^{final};\Omega)$ with $\boldsymbol{f}=0$, $g=0$, problem* (2.1) *has at least a solution $(\boldsymbol{u},p,T)\in L^2(0,T_{time}^{final};X)\cap H^1(0,T_{time}^{final};V)\times L^2(0,T_{time}^{final};M)\times H^1(0,T_{time}^{final};W)$. In addition, the solution is unique provided that $\nu^{-1}\overline{N}^2\|\nabla T\|_0^2+2\nu N\|\nabla\boldsymbol{u}\|_0\le 2$, and the following prior estimate holds*

$$\|\boldsymbol{u}\|_0^2+\|T\|_0^2+\int_0^{T_{time}^{final}}(\|\nabla\boldsymbol{u}\|_0^2+\|\nabla T\|_0^2)ds\le C\int_0^{T_{time}^{final}}(\|T_{0t}\|_0^2+\|\nabla T_0\|_0^2)ds,$$

where

$$N=\sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in X}\frac{b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})}{\|\nabla\boldsymbol{u}\|_0\|\nabla\boldsymbol{v}\|_0\|\nabla\boldsymbol{w}\|_0},\quad \overline{N}=\sup_{\boldsymbol{u}\in X,T,\psi\in W}\frac{\tilde{b}(\boldsymbol{u},T,\psi)}{\|\nabla\boldsymbol{u}\|_0\|\nabla T\|_0\|\nabla\psi\|_0}.$$

2.2 Spatial discrete Galerkin finite element method

Take a positive parameter $h\to 0$ and define a decomposition $J_h=J_h(\Omega)$ be a family of regular partitioning of triangles K or quadrilaterals K of the domain $\bar{\Omega}$. Based on J_h, we construct the conforming finite element spaces of $X_h\times M_h\times W_h$ of $X\times M\times W$. Furthermore, some assumptions are made about the spaces X_h,M_h and W_h (see [2,6,11,29]).

Assumption 2.2. (A2) For each $\mathbf{v}\in D(A)$ and $\psi\in Q(A)$ and $q\in H^1(\Omega)\cap M$, there are the approximations $\pi_h\mathbf{v}\in V_h$, $\mu_h\psi\in W_h$ and $\rho_h q\in M_h$ such that

$$\|\nabla(\mathbf{v}-\pi_h\mathbf{v})\|_0\le ch\|A\mathbf{v}\|_0,\quad \|\nabla(\psi-\mu_h\psi)\|_0\le ch\|A\psi\|_0,\quad \|q-\rho_h q\|_0\le ch\|\nabla q\|_0.$$

For each $\phi_h \in X_h$ or W_h, the following inverse inequalities hold

$$\|\nabla \phi_h\|_0 \leq ch^{-1}\|\phi_h\|_0, \quad \|\phi_h\|_{L^\infty} \leq ch^{-1}\|\phi_h\|_0.$$

Define the discrete analogue of the space V_h as

$$V_h = \{\mathbf{v}_h \in X_h;\ d(\mathbf{v}_h,q_h) = 0,\ \forall q_h \in M_h\},$$

and set $P_h^1 : Y \to V_h$ and $P_h^2 : Z \to W_h$ be the L^2-orthogonal projections defined by

$$\begin{aligned} &(P_h^1\mathbf{v},\mathbf{v}_h) = (\mathbf{v},\mathbf{v}_h), && \mathbf{v} \in Y, \quad \mathbf{v}_h \in V_h,\\ &(P_h^2\psi,\psi_h) = (\psi,\psi_h), && \psi \in Z, \quad \psi_h \in W_h. \end{aligned}$$

Assumption 2.3. (A3) There exists a constant $\beta > 0$ such that

$$\beta\|q_h\|_0 \leq \sup_{0\neq \mathbf{v}_h \in X_h} \frac{|d(\mathbf{v}_h,q_h)|}{\|\mathbf{v}_h\|_1}, \quad \forall q_h \in M_h.$$

The following properties are classical for $i=1,2$ (see [11,19])

$$\|\nabla P_h^i\varphi\|_0 \leq \gamma\|\nabla\varphi\|_0, \quad \|\varphi - P_h^i\varphi\|_0 \leq \gamma h\|\nabla(\varphi - P_h^i\varphi)\|_0, \quad \varphi \in X \text{ or } W, \tag{2.2a}$$

$$\|\varphi - P_h^i\varphi\|_0 + h\|\nabla(\varphi - P_h^i\varphi)\|_0 \leq \gamma h^2\|A\varphi\|_0, \quad \varphi \in D(A) \text{ or } Q(A), \tag{2.2b}$$

for some positive constants γ.

With above notations, for all $(\mathbf{v}_h,q_h,\psi_h) \in X_h \times M_h \times W_h$ and $0 \leq t \leq T_{time}^{final}$, the Galerkin finite element method for problem (2.1) is to seek $(\mathbf{u}_h,p_h,T_h) \in X_h \times M_h \times W_h$ such that

$$\begin{cases} (\mathbf{u}_{ht},\mathbf{v}_h) + a(\mathbf{u}_h,\mathbf{v}_h) - d(\mathbf{v}_h,p_h) + d(\mathbf{u}_h,q_h) + b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h) - \kappa\nu^2(jT_h,\mathbf{v}_h),\\ (T_{ht},\psi_h) + \tilde{a}(T_h,\psi_h) + \tilde{b}(\mathbf{u}_h,T_h,\psi_h) = (g,\psi_h),\\ \mathbf{u}_h(x,0) = \mathbf{u}_{0h} = P_h\mathbf{u}_0, \quad T_h(x,0) = T_{0h} = P_hT_0,\\ \mathbf{u}_{ht}|_{\partial\Omega} = 0, \quad T_{ht}|_{\partial\Omega} = 0. \end{cases} \tag{2.3}$$

With the help of P_h^i, we define the discrete analogue $A_{ih} = -P_h^i\Delta_h$ $(i=1,2)$ and Δ_h by $(-\Delta_h\phi_h,\varphi_h) = (\nabla\phi_h,\nabla\varphi_h)$, $\forall \phi_h,\varphi_h \in X_h$ or W_h with the "discrete" Sobolev norms $\|\phi_h\|_r = \|A_{ih}^{r/2}\phi_h\|_0$ for $r=-1,0,1,2$ and $\phi_h \in X_h$ or W_h. Furthermore, it holds

$$\|\phi_h\|_0 \leq \gamma_0\|\phi_h\|_1, \quad \|\phi_h\|_1 \leq \gamma_0\|A_{ih}\phi_h\|_0. \tag{2.4}$$

The following properties of trilinear terms can be found in [12,13,15,18].

Lemma 2.1. *The trilinear forms $b(\cdot,\cdot,\cdot)$ and $\tilde{b}(\cdot,\cdot,\cdot)$ satisfy*

$$\begin{aligned}
&b(\boldsymbol{u}_h,\boldsymbol{v}_h,\mathbf{w}_h)=-b(\boldsymbol{u}_h,\mathbf{w}_h,\boldsymbol{v}_h),\quad |b(\boldsymbol{u}_h,\boldsymbol{v}_h,\mathbf{w}_h)|\le\widehat{C}|\log h|^{1/2}\|\nabla\boldsymbol{u}_h\|_0\|\nabla\boldsymbol{v}_h\|_0\|\mathbf{w}_h\|_0,\\
&\tilde{b}(\boldsymbol{u}_h,T_h,\psi_h)=-\tilde{b}(\boldsymbol{u}_h,\psi_h,T_h),\quad |\tilde{b}(\boldsymbol{u}_h,T_h,\psi_h)|\le\widetilde{C}|\log h|^{1/2}\|\nabla\boldsymbol{u}_h\|_0\|\nabla T_h\|_0\|\psi_h\|_0,\\
&|b(\boldsymbol{u}_h,\boldsymbol{v}_h,\mathbf{w}_h)|+|b(\boldsymbol{v}_h,\boldsymbol{u}_h,\mathbf{w}_h)|+|b(\mathbf{w}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)|\\
&\le\frac{\widehat{C}}{2}\|\boldsymbol{u}_h\|_0^{1/2}\|\boldsymbol{u}_h\|_1^{1/2}\|\boldsymbol{v}_h\|_1\|\mathbf{w}_h\|_0^{1/2}\|\mathbf{w}_h\|_1^{1/2}+\frac{\widehat{C}}{2}\|\boldsymbol{u}_h\|_1\|\boldsymbol{v}_h\|_0^{1/2}\|\boldsymbol{v}_h\|_1^{1/2}\|\mathbf{w}_h\|_0^{1/2}\|\mathbf{w}_h\|_1^{1/2},\\
&|\tilde{b}(\boldsymbol{u}_h,T_h,\psi_h)|+|\tilde{b}(T_h,\boldsymbol{u}_h,\psi_h)|+|\tilde{b}(\psi_h,\boldsymbol{u}_h,T_h)|\le\widetilde{C}\|\boldsymbol{u}_h\|_0^{1/2}\|\boldsymbol{u}_h\|_1^{1/2}\|T_h\|_1\|\psi_h\|_0^{1/2}\|\psi_h\|_1^{1/2},\\
&|b(\boldsymbol{u}_h,\boldsymbol{v}_h,\mathbf{w}_h)|+|b(\boldsymbol{v}_h,\boldsymbol{u}_h,\mathbf{w}_h)|+|b(\mathbf{w}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)|\\
&\le\frac{\widehat{C}}{2}\|A_{1h}\boldsymbol{v}_h\|_0^{1/2}\|\boldsymbol{v}_h\|_1^{1/2}\|\boldsymbol{u}_h\|_0^{1/2}\|\boldsymbol{u}_h\|_1^{1/2}\|\mathbf{w}_h\|_0+\frac{\widehat{C}}{2}\|A_{1h}\boldsymbol{v}_h\|_0^{1/2}\|\boldsymbol{v}_h\|_0^{1/2}\|\boldsymbol{u}_h\|_1\|\mathbf{w}_h\|_0,\\
&|\tilde{b}(\boldsymbol{u}_h,T_h,\psi_h)|+|\tilde{b}(T_h,\boldsymbol{u}_h,\psi_h)|+|\tilde{b}(\psi_h,\boldsymbol{u}_h,T_h)|\\
&\le\frac{\widetilde{C}}{2}\|A_{2h}T_h\|_0^{1/2}\|T_h\|_1^{1/2}\|\boldsymbol{u}_h\|_0^{1/2}\|\boldsymbol{u}_h\|_1^{1/2}\|\psi_h\|_0+\frac{\widetilde{C}}{2}\|A_{2h}T_h\|_0^{1/2}\|T_h\|_0^{1/2}\|\boldsymbol{u}_h\|_1\|\psi_h\|_0,
\end{aligned}$$

for all $\boldsymbol{u}_h,\boldsymbol{v}_h,\in V_h$, $\mathbf{w}_h\in X_h$ and $T_h,\psi_h\in W_h$, where $\widehat{C}>0$ and $\widetilde{C}>0$ are constants.

In order to present the error analysis for time discretization, we recall the following smooth properties of $\mathbf{u}_h$ and T_h, and some errors of $\mathbf{u}-\mathbf{u}_h$, $T-T_h$ and $p-p_h$. Following the techniques used in [13–16], we can obtain the following properties.

Theorem 2.2. *Under the assumptions* (A1)-(A3)*, for all $t\in[0,T_{time}^{final}]$, then the numerical solution $(\boldsymbol{u}_h,T_h)$ of problem* (2.3) *satisfies*

$$\begin{aligned}
&\|\boldsymbol{u}_h(t),T_h(t)\|_0^2+\|\boldsymbol{u}_h(t),T_h(t)\|_1^2+\sigma(t)\|A_h\boldsymbol{u}_h(t),A_hT_h(t)\|_0^2\\
&\quad+\int_0^t\big\{\|\nabla\boldsymbol{u}_h,\nabla T_h\|_0^2+\|A_h\boldsymbol{u}_h,A_hT_h\|_0^2\big\}ds\le C,\quad \sigma^{1+r}(t)\|\boldsymbol{u}_{ht},T_{ht}\|_r^2\le C,\quad r=-1,0,1,2,\\
&\int_0^t\big\{\|\boldsymbol{u}_{hs},T_{hs}\|_0^2+\sigma^r(s)\|\boldsymbol{u}_{hs},T_{hs}\|_r^2\big\}ds\le C,\quad r=1,2,\\
&\int_0^t\big\{\sigma^{2-r}(s)\|A_h^{-r/2}\boldsymbol{u}_{hss},A_h^{-r/2}T_{hss}\|_0^2)\big\}ds\le C,\quad r=0,1,2,\\
&\sigma^3(t)\|\boldsymbol{u}_{htt}(t),T_{htt}(t)\|_0^2+\int_0^t\big\{\sigma^3(s)(\|\boldsymbol{u}_{hss},T_{hss}\|_1^2+\|\boldsymbol{u}_{hsss},T_{hsss}\|_{-1}^2)\big\}ds\le C,
\end{aligned}$$

where

$$\begin{aligned}
&\sigma(t)=\min\{1,t\},\quad \|\boldsymbol{u}_h(t),T_h(t)\|_r=\|\boldsymbol{u}_h(t)\|_r^2+\|T_h(t)\|_r^2\quad \textit{with}\quad r=-1,0,1,\\
&\|A_h\boldsymbol{u}_h(t),A_hT_h(t)\|_0=\|A_h\boldsymbol{u}_h(t)\|_0^2+\|A_hT_h(t)\|_0^2.
\end{aligned}$$

Theorem 2.3. *Under the assumptions* (A1)-(A3)*, for all $0<t\le T_{time}^{final}$, it holds*

$$\sigma^{1/2}(t)\|\boldsymbol{u}-\boldsymbol{u}_h,T-T_h\|_0\le Ch^2,\quad \sigma^{1/2}(t)\|\boldsymbol{u}-\boldsymbol{u}_h,T-T_h\|_1+\sigma(t)\|p-p_h\|_0\le Ch.$$

Lemma 2.2 ([12]). *Let C' and Δt, a_k, b_k, d_k be non-negative numbers for integers $k \geq 0$. If*

$$a_n + \Delta t \sum_{k=r}^{n} b_k \leq \Delta t \sum_{k=r-1}^{n-1} d_k a_k + C', \quad \forall n \geq r-1.$$

Then

$$a_n + \Delta t \sum_{k=r}^{n} b_k \leq C' \exp\Big(\Delta t \sum_{k=r-1}^{n-1} d_k\Big), \quad \forall n \geq r-1.$$

3 The Crank-Nicolson/Explicit scheme for the natural convection equations

In this section, we denote the time step $\Delta t = \frac{T_{time}^{final}}{N}$ with N is an integer and set $t_n = n\Delta t$. Let

$$\mathbf{u}_h^0 = \mathbf{u}_{0h} = P_h \mathbf{u}_0, \quad T_h^0 = T_{0h} = P_h T_0.$$

Firstly, we find $(\mathbf{u}_h^1, p_h^1, T_h^1) \in X_h \times M_h \times W_h$ by the Euler-backward scheme with the explicit scheme to treat the nonlinear terms

$$\left\{ \begin{array}{l} (d_t \mathbf{u}_h^1, \mathbf{v}_h) + a(\mathbf{u}_h^1, \mathbf{v}_h) - d(\mathbf{v}_h, p_h^1) + d(\mathbf{u}_h^1, q_h) + b(\mathbf{u}_h^0, \mathbf{u}_h^0, \mathbf{v}_h) \\ \quad = (\mathbf{f}(t_1), \mathbf{v}_h) - \kappa \nu^2 (j T_h^0, \mathbf{v}_h), \\ (d_t T_h^1, \psi_h) + \tilde{a}(T_h^1, \psi_h) + \tilde{b}(\mathbf{u}_h^0, T_h^0, \psi_h) = (g(t_1), \psi_h), \end{array} \right. \tag{3.1}$$

for all $(\mathbf{v}_h, q_h, \psi_h) \in X_h \times M_h \times W_h$, where $d_t \phi_h^n = \frac{\phi_h^n - \phi_h^{n-1}}{\Delta t}$, ϕ can take $\mathbf{u}$ and T.

Based on the numerical solution $(\mathbf{u}_h^1, p_h^1, T_h^1)$, for all $(\mathbf{v}_h, q_h, \psi_h) \in X_h \times M_h \times W_h$, find $(\mathbf{u}_h^n, p_h^n, T_h^n) \in X_h \times M_h \times W_h$ with $n = 2, \cdots, N$ by

$$\left\{ \begin{array}{l} (d_t \mathbf{u}_h^n, \mathbf{v}_h) + a(\bar{\mathbf{u}}_h^n, \mathbf{v}_h) - d(\mathbf{v}_h, p_h^n) + d(\bar{\mathbf{u}}_h^n, q_h) + b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{v}_h) \\ \quad = (\bar{\mathbf{f}}(t_n), \mathbf{v}_h) - \kappa \nu^2 (j \overline{T}_h^n, \mathbf{v}_h), \\ (d_t T_h^n, \psi_h) + \tilde{a}(\overline{T}_h^n, \psi_h) + \tilde{b}(\mathbf{u}_h^{n-1}, T_h^{n-1}, \psi_h) = (\overline{g}(t_n), \psi_h). \end{array} \right. \tag{3.2}$$

Here and below, we use the following notations frequently

$$\overline{\phi}_h^n = \frac{\phi_h^n + \phi_h^{n-1}}{2}, \quad \overline{\phi}_h(t_n) = \frac{\phi_h(t_n) + \phi_h(t_{n-1})}{2}, \quad \text{where } \phi \text{ can take } \mathbf{u} \text{ or } T.$$

From Eqs. (3.1) and (3.2), we obtain

$$\begin{cases} (\mathbf{u}_h^1,\mathbf{v}_h)+\Delta t\Big[a(\mathbf{u}_h^1,\mathbf{v}_h)-d(\mathbf{v}_h,p_h^1)\Big] \\ \quad =(\mathbf{u}_h^0,\mathbf{v}_h)+\Delta t\Big[(\mathbf{f}(t_1),\mathbf{v}_h)-\kappa\nu^2(jT_h^0,\mathbf{v}_h)-b(\mathbf{u}_h^0,\mathbf{u}_h^0,\mathbf{v}_h)\Big], \\ d(\mathbf{u}_h^1,q_h)=0, \\ (T_h^1,\psi_h)+\Delta t\tilde{a}(T_h^1,\psi_h)=(T_h^0,\psi_h)+\Delta t\Big[(g(t_1),\psi_h)-\tilde{b}(\mathbf{u}_h^0,T_h^0,\psi_h)\Big], \end{cases} \tag{3.3a}$$

$$\begin{cases} (\mathbf{u}_h^n,\mathbf{v}_h)+\Delta t\Big[\frac{1}{2}a(\mathbf{u}_h^n,\mathbf{v}_h)-d(\mathbf{v}_h,p_h^n)\Big]=(\mathbf{u}_h^{n-1},\mathbf{v}_h) \\ \quad +\Delta t\Big[(\bar{\mathbf{f}}(t_n),\mathbf{v}_h)-\frac{1}{2}a(\mathbf{u}_h^{n-1},\mathbf{v}_h)-\kappa\nu^2(j\overline{T}_h^n,\mathbf{v}_h)-b(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},\mathbf{v}_h)\Big], \\ d(\overline{\mathbf{u}}_h^n,q_h)=0, \\ (T_h^n,\psi_h)+\Delta t\frac{1}{2}\tilde{a}(T_h^n,\psi_h) \\ =(T_h^{n-1},\psi_h)+\Delta t\Big[(\overline{g}(t_n),\psi_h)-\frac{1}{2}\tilde{a}(T_h^{n-1},\psi_h)-\tilde{b}(\mathbf{u}_h^{n-1},T_h^{n-1},\psi_h)\Big]. \end{cases} \tag{3.3b}$$

For the given $(u_h^0,T_h^0)\in X_h\times W_h$, set

$$\overline{A}(T_h^1,\psi_h)=(T_h^1,\psi_h)+\Delta t\tilde{a}(T_h^1,\psi_h).$$

Taking $\psi_h=T_h^1$ in $\overline{A}(\cdot,\cdot)$, one finds that

$$\overline{A}(T_h^1,T_h^1)=\|T_h^1\|_0^2+\Delta tPr^{-1}\nu\|\nabla T_h^1\|_0^2\geq\alpha\|T_h^1\|_1^2,$$

where $\alpha=\min\{1,\Delta tPr^{-1}\nu\}$ and $\overline{A}(\cdot,\cdot)$ is coercive. Hence, the third equation of problem (3.3a) has a unique solution $T_h^1\in W_h$.

Furthermore, for the given $(\mathbf{u}_h^0,T_h^0)\in X_h\times W_h$, denote

$$\overline{B}(\mathbf{u}_h^1,\mathbf{v}_h)=(\mathbf{u}_h^1,\mathbf{v}_h)+\Delta ta(\mathbf{u}_h^1,\mathbf{v}_h).$$

Then the first equation of problem (3.3a) can be rewritten as

$$\overline{B}(\mathbf{u}_h^1,\mathbf{v}_h)-\Delta td(\mathbf{v}_h,p_h^1)=(\mathbf{u}_h^0,\mathbf{v}_h)+\Delta t\Big[(\mathbf{f}(t_1),\mathbf{v}_h)-\kappa\nu^2(jT_h^0,\mathbf{v}_h)-b(\mathbf{u}_h^0,\mathbf{u}_h^0,\mathbf{v}_h)\Big].$$

Choosing $\mathbf{v}_h=\mathbf{u}_h^1$ in $\overline{B}(\cdot,\cdot)$, we obtain

$$\overline{B}(\mathbf{u}_h^1,\mathbf{u}_h^1)=\|\mathbf{u}_h^1\|_0^2+\Delta t\nu\|\nabla\mathbf{u}_h^1\|_0^2\geq\overline{\alpha}\|\mathbf{u}_h\|_1^2,$$

where $\overline{\alpha}=\min\{1,\nu\Delta t\}$ and $\overline{B}(\cdot,\cdot)$ is also coercive. While $d(\cdot,\cdot)$ satisfies the discrete inf$-$sup condition (A3), then the first and second equations in problem (3.3a) admit a unique solution $(\mathbf{u}_h^1,p_h^1)\in X_h\times M_h$. As a consequence, problem (3.1) has a unique solution $(\mathbf{u}_h^1,p_h^1,T_h^1)\in X_h\times M_h\times W_h$.

By the same analysis, we can establish the existence and uniqueness of numerical solutions in scheme (3.3b). It means that problem (3.2) admits a unique solution $(\mathbf{u}_h^n,p_h^n,T_h^n)\in X_h\times M_h\times W_h$.

Theorem 3.1. *Suppose that the assumptions* (A1)-(A3) *hold and the time step* Δt *satisfies*

$$\begin{cases} c'2\Delta t\le 1, & \boldsymbol{u}_0\in V\cap L^\infty(\Omega)^2, \quad T_0\in W\cap L^\infty(\Omega),\\ c'|\log h|\Delta t\le 1, & \boldsymbol{u}_0\in V, \quad T_0\in W,\end{cases} \tag{3.4}$$

where $c'>0$ *is a constant, the numerical solution* $(\boldsymbol{u}_h^m,T_h^m)$ $(m=1,2)$ *of schemes* (3.1)-(3.2) *satisfies*

$$\|\boldsymbol{u}_h^m,T_h^m\|_0^2+\min\{\nu,Pr^{-1}\nu\}\sum_{n=1}^m\|\boldsymbol{u}_h^n,T_h^n\|_1^2\Delta t\le C_0', \tag{3.5a}$$

$$\min\{\nu,Pr^{-1}\nu\}\|\boldsymbol{u}_h^m,T_h^m\|_1^2+\min\{\nu^2,Pr^{-2}\nu^2\}\|A_h\boldsymbol{u}_h^n,A_hT_h^n\|_0^2+\|p_h^n\|_0^2\le C_1', \tag{3.5b}$$

$$\min\{\nu^2,Pr^{-2}\nu^2\}\|A_h\boldsymbol{u}_h^m,A_hT_h^m\|_0^2\Delta t+\min\{\nu,Pr^{-1}\nu\}\Delta t^2\sum_{n=1}^m\|d_t\boldsymbol{u}_h^n,d_tT_h^n\|_1^2\le C_2', \tag{3.5c}$$

with some positive constants

$$C_0'\ge\|\boldsymbol{u}_h^0,T_h^0\|_0^2,\quad C_1'\ge\min\{\nu,Pr^{-1}\nu\}\|\boldsymbol{u}_h^0,T_h^0\|_1^2\quad\textit{and}\quad C_2'\ge C_1',$$

all these constants depend on the data Pr^{-1}, ν, κ, Ω, T_{time}^{final}, $\boldsymbol{u}_0$, T_0, f *and* g.

Proof. For $r=0,1$, taking $\mathbf{v}_h=2A_h^r\mathbf{u}_h^1\Delta t\in V_h$, $\psi_h=2A_h^rT_h^1\Delta t\in W_h$, $q_h=0$ in (3.1) and $\mathbf{v}_h=2A_h^r\mathbf{u}_h^2\Delta t\in V_h$, $\psi_h=2A_h^rT_h^2\Delta t\in W_h$, $q_h=0$ in (3.2) with $n=2$, respectively, we get

$$\begin{cases} (d_t\mathbf{u}_h^1,2A_h^r\mathbf{u}_h^1\Delta t)+a(\mathbf{u}_h^1,2A_h^r\mathbf{u}_h^1\Delta t)+b(\mathbf{u}_h^0,\mathbf{u}_h^0,2A_h^r\mathbf{u}_h^1\Delta t)\\ \quad=(\mathbf{f}(t_1),2A_h^r\mathbf{u}_h^1\Delta t)-\kappa\nu^2(jT_h^0,2A_h^r\mathbf{u}_h^1\Delta t),\\ (d_tT_h^1,2A_h^rT_h^1\Delta t)+\tilde a(T_h^1,2A_h^rT_h^1\Delta t)+\tilde b(\mathbf{u}_h^0,T_h^0,2A_h^rT_h^1\Delta t)=(g(t_1),2A_h^rT_h^1\Delta t),\end{cases}$$
$$\begin{cases} (d_t\mathbf{u}_h^2,2A_h^r\mathbf{u}_h^2\Delta t)+a(\overline{\mathbf{u}}_h^2,2A_h^r\mathbf{u}_h^2\Delta t)+b(\mathbf{u}_h^1,\mathbf{u}_h^1,2A_h^r\mathbf{u}_h^2\Delta t)\\ \quad=(\bar{\mathbf{f}}(t_2),2A_h^r\mathbf{u}_h^2\Delta t)-\kappa\nu^2(j\overline{T}_h^2,2A_h^r\mathbf{u}_h^2\Delta t),\\ (d_tT_h^2,2A_h^rT_h^2\Delta t)+\tilde a(\overline{T}_h^2,2A_h^rT_h^2\Delta t)+\tilde b(\mathbf{u}_h^1,T_h^1,2A_h^rT_h^2\Delta t)=(\overline{g}(t_2),2A_h^rT_h^2\Delta t).\end{cases}$$

Thanks to Lemma 2.2 and the Cauchy inequality, we have

$$\begin{cases} \|\mathbf{u}_h^1\|_r^2-\|\mathbf{u}_h^0\|_r^2+\|d_t\mathbf{u}_h^1\|_r^2\Delta t^2+2\nu\|\mathbf{u}_h^1\|_{r+1}^2\Delta t\\ \le\nu\|\mathbf{u}_h^1\|_{r+1}^2\Delta t+2\nu^{-1}\gamma_0^{2(1-r)}\|\mathbf{f}(t_1)\|_0^2\Delta t+2k^2\nu^3\gamma_0^{2(1-r)}\|T_h^0\|_0^2\Delta t\\ \quad-2b(\mathbf{u}_h^0,\mathbf{u}_h^0,A_h^r\mathbf{u}_h^1)\Delta t,\\ \|T_h^1\|_r^2-\|T_h^0\|_r^2+\|d_tT_h^1\|_r^2\Delta t^2+2Pr^{-1}\nu\|T_h^1\|_{r+1}^2\Delta t\\ \le\dfrac{1}{2}Pr^{-1}\nu\|T_h^1\|_{r+1}^2\Delta t+2Pr\nu^{-1}\gamma_0^{2(1-r)}\|g(t_1)\|_0^2\Delta t-2\tilde b(\mathbf{u}_h^0,T_h^0,A_h^rT_h^1)\Delta t,\end{cases} \tag{3.6a}$$

$$
\left\{
\begin{aligned}
&\|\mathbf{u}_h^2\|_r^2-\|\mathbf{u}_h^1\|_r^2+\|d_t\mathbf{u}_h^2\|_r^2\Delta t^2+\frac{\nu}{2}(\|\mathbf{u}_h^2\|_{r+1}^2-\|\mathbf{u}_h^1\|_{r+1}^2)\Delta t\\
&\le\frac{\nu}{8}\|\mathbf{u}_h^2\|_{r+1}^2\Delta t+16\nu^{-1}\gamma_0^{2(1-r)}\|\bar{\mathbf{f}}(t_2)\|_0^2\Delta t+16k^2\nu^3\gamma_0^{2(1-r)}\|\overline{T}_h^2\|_0^2\Delta t\\
&\qquad-2b(\mathbf{u}_h^1,\mathbf{u}_h^1,A_h^r\mathbf{u}_h^2)\Delta t\\
&\le\frac{\nu}{8}\|\mathbf{u}_h^2\|_{r+1}^2\Delta t+16\nu^{-1}\gamma_0^{2(1-r)}\|\bar{\mathbf{f}}(t_2)\|_0^2\Delta t+16k^2\nu^3\gamma_0^{2(1-r)}\|\overline{T}_h^2\|_0^2\Delta t\\
&\qquad-2b(\mathbf{u}_h^1,\mathbf{u}_h^1,A_h^r\mathbf{u}_h^2)\Delta t+2|b(\mathbf{u}_h^0,\mathbf{u}_h^0,A_h^r\mathbf{u}_h^2)|\Delta t,\\
&\|T_h^2\|_r^2-\|T_h^1\|_r^2+\|d_tT_h^2\|_r^2\Delta t^2+\frac{Pr^{-1}\nu}{2}(\|T_h^2\|_{r+1}^2-\|T_h^1\|_{r+1}^2)\Delta t\\
&\le\frac{Pr^{-1}\nu}{16}\|T_h^2\|_{r+1}^2\Delta t+\frac{16\gamma_0^{2(1-r)}}{Pr^{-1}\nu}\|\overline{g}(t_2)\|_0^2\Delta t-2\tilde{b}(\mathbf{u}_h^1,T_h^1,A_h^rT_h^2)\Delta t\\
&\le\frac{Pr^{-1}\nu}{16}\|T_h^2\|_{r+1}^2\Delta t+\frac{16\gamma_0^{2(1-r)}}{Pr^{-1}\nu}\|\overline{g}(t_2)\|_0^2\Delta t\\
&\qquad-2\tilde{b}(\mathbf{u}_h^1,T_h^1,A_h^rT_h^2)\Delta t+2|\tilde{b}(\mathbf{u}_h^0,T_h^0,A_h^rT_h^2)|\Delta t.
\end{aligned}
\right.
\tag{3.6b}
$$

For $r=0,1$ and $i=1,2$, using Lemma 2.2, (2.2a) and (2.4), we obtain

$$
\begin{aligned}
&2|b(\mathbf{u}_h^0,\mathbf{u}_h^0,A_h^r\mathbf{u}_h^i)|\Delta t\le\frac{\nu}{8}\|\mathbf{u}_h^i\|_{r+1}^2\Delta t+\frac{2\widehat{C}\gamma_0^{2(1-r)}}{\nu}|\log h|^r\|\mathbf{u}_h^0\|_1^4\Delta t,\quad \mathbf{u}_0\in V,\\
&2|b(\mathbf{u}_h^0,\mathbf{u}_h^0,A_h^r\mathbf{u}_h^i)|\Delta t\\
&\le 2|(((\mathbf{u}_h^0-\mathbf{u}_0)\cdot\nabla)\mathbf{u}_h^0,A_h^r\mathbf{u}_h^i)|\Delta t+|(\mathrm{div}\mathbf{u}_h^0(\mathbf{u}_h^0-\mathbf{u}_0),A_h^r\mathbf{u}_h^i)|\Delta t\\
&\qquad+2|((\mathbf{u}_0\cdot\nabla)\mathbf{u}_h^0,A_h^r\mathbf{u}_h^i)|\Delta t+|(\mathrm{div}\mathbf{u}_h^0\cdot\mathbf{u}_0,A_h^r\mathbf{u}_h^i)|\Delta t\\
&\le c\|\mathbf{u}_h^0-\mathbf{u}_0\|_0\|\mathbf{u}_h^0\|_1\|A_h^r\mathbf{u}_h^i\|_{L^\infty}\Delta t+c\|\mathbf{u}_0\|_{L^\infty}\|\mathbf{u}_h^0\|_1\|A_h^r\mathbf{u}_h^i\|_0\Delta t\\
&\le\frac{\nu}{8}\|\mathbf{u}_h^i\|_{r+1}^2\Delta t+c(\|\nabla\mathbf{u}_0\|_0^2+\|\mathbf{u}_0\|_{L^\infty}^2)\|\mathbf{u}_h^0\|_1^2\Delta t,\quad \mathbf{u}_0\in V\cap L^\infty(\Omega)^2,\\
&2|b(\mathbf{u}_h^1,\mathbf{u}_h^1,A_h^r\mathbf{u}_h^2)|\Delta t\le\frac{\nu}{8}\|\mathbf{u}_h^2\|_{r+1}^2\Delta t+\frac{\nu}{8}\|A_h\mathbf{u}_h^1\|_0^2\Delta t+c\|\mathbf{u}_h^1\|_0^2\|\mathbf{u}_h^1\|_1^4\Delta t,\\
&2|\tilde{b}(\mathbf{u}_h^0,T_h^0,A_h^rT_h^i)|\Delta t\\
&\le\widetilde{C}\gamma_0^{1-r}|\log h|^{r/2}\|\mathbf{u}_h^0\|_1\|T_h^0\|_1\|T_h^i\|_{r+1}\Delta t\\
&\le\frac{Pr^{-1}\nu}{8}\|T_h^i\|_{r+1}^2\Delta t+\frac{2\widetilde{C}\gamma_0^{2(1-r)}}{Pr^{-1}\nu}|\log h|^r\|\mathbf{u}_h^0\|_1^2\|T_h^0\|_1^2\Delta t,\quad (\mathbf{u}_0,T_0)\in V\times W,\\
&2|\tilde{b}(\mathbf{u}_h^0,T_h^0,A_h^rT_h^i)|\Delta t\\
&\le 2|(((\mathbf{u}_h^0-\mathbf{u}_0)\cdot\nabla)T_h^0,A_h^rT_h^i)|\Delta t+|(\mathrm{div}\mathbf{u}_h^0(T_h^0-T_0),A_h^rT_h^i)|\Delta t\\
&\qquad+2|((\mathbf{u}_0\cdot\nabla)T_h^0,A_h^rT_h^i)|\Delta t+|(\mathrm{div}\mathbf{u}_h^0\cdot T_0,A_h^rT_h^i)|\Delta t\\
&\le c\|\mathbf{u}_h^0-\mathbf{u}_0\|_0\|T_h^0\|_1\|A_h^rT_h^i\|_{L^\infty}\Delta t+c\|\mathbf{u}_0\|_{L^\infty}\|T_h^0\|_1\|A_h^rT_h^i\|_0\Delta t\\
&\le\frac{Pr^{-1}\nu}{8}\|T_h^i\|_{r+1}^2\Delta t+c(\|\nabla\mathbf{u}_0\|_0^2+\|\mathbf{u}_0\|_{L^\infty}^2)\|T_h^0\|_1^2\Delta t,\quad \mathbf{u}_0\in V\cap L^\infty(\Omega)^2,\quad T_0\in W\cap L^\infty(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
&2|\tilde{b}(\mathbf{u}_h^1,T_h^1,A_h^rT_h^2)|\Delta t\\
\le&\frac{3}{2}\widetilde{C}\gamma_0^{1-r}\|\mathbf{u}_h^1\|_0^{\frac12}\|T_h^1\|_1\|A_h\mathbf{u}_h^1\|_0^{\frac12}\|T_h^2\|_{r+1}\Delta t\\
\le&\frac{Pr^{-1}\nu}{8}\|T_h^2\|_{r+1}^2\Delta t+\frac{Pr^{-1}\nu}{8}\|A_h\mathbf{u}_h^1\|_0^2\Delta t+c\|\mathbf{u}_h^1\|_0^2\|T_h^1\|_1^4\Delta t.
\end{aligned}
$$

Combining above inequalities with (3.6a) and (3.6b), one finds

$$
\begin{cases}
\|\mathbf{u}_h^1\|_r^2+\|d_t\mathbf{u}_h^1\|_r^2\Delta t^2+\nu\|\mathbf{u}_h^1\|_{r+1}^2\Delta t\\
\le\|\mathbf{u}_h^0\|_r^2+c\|\mathbf{f}(t_1)\|_0^2\Delta t+c\|T_h^0\|_0^2\Delta t+cM_1\Delta t,\\
\|T_h^1\|_r^2+\|d_tT_h^1\|_r^2\Delta t^2+Pr^{-1}\nu\|T_h^1\|_{r+1}^2\Delta t\\
\le\|T_h^0\|_r^2+c\|g(t_1)\|_0^2\Delta t+cM_1'\Delta t,
\end{cases}
\tag{3.7a}
$$

$$
\begin{cases}
\|\mathbf{u}_h^2\|_r^2+\|d_t\mathbf{u}_h^2\|_r^2\tau^2+\frac{\nu}{4}\|\mathbf{u}_h^2\|_{r+1}^2\tau\\
\le\|\mathbf{u}_h^1\|_r^2+\frac{\nu}{2}\|\mathbf{u}_h^1\|_{r+1}^2+\nu\|A_h\mathbf{u}_h^1\|_0^2\tau+c\|\bar{\mathbf{f}}(t_2)\|_0^2\tau\\
\qquad+c\|\overline{T}_h^2\|_0^2\tau+c\|\mathbf{u}_h^1\|_0^2\|\mathbf{u}_h^1\|_1^4\tau+cM_1\tau,\\
\|T_h^2\|_r^2+\|d_tT_h^2\|_r^2\tau^2+\frac{Pr^{-1}\nu}{4}\|T_h^2\|_{r+1}^2\tau\\
\le\|T_h^1\|_r^2+\frac{Pr^{-1}\nu}{2}\|T_h^1\|_{r+1}^2\tau+Pr^{-1}\nu\|A_h\mathbf{u}_h^1\|_0^2\tau\\
\qquad+c\|\bar{g}(t_2)\|_0^2\tau+c\|\mathbf{u}_h^1\|_0^2\|T_h^1\|_1^4\tau+cM_1'\tau,
\end{cases}
\tag{3.7b}
$$

where

$$
M_1=\begin{cases}
(\|\nabla\mathbf{u}_0\|_0^2+\|\mathbf{u}_0\|_{L^\infty}^2)\|\mathbf{u}_h^0\|_1^2, & \mathbf{u}_0\in V\cap L^\infty(\Omega)^2,\\
|\log h|^r\|\mathbf{u}_h^0\|_1^4, & \mathbf{u}_0\in V,
\end{cases}
$$

$$
M_1'=\begin{cases}
|\log h|^r\|\mathbf{u}_h^0\|_1^2\|T_h^0\|_1^2, & (\mathbf{u}_0,T_0)\in V\times W,\\
(\|\nabla\mathbf{u}_0\|_0^2+\|\mathbf{u}_0\|_{L^\infty}^2)\|T_h^0\|_1^2, & \mathbf{u}_0\in V\cap L^\infty(\Omega)^2,\quad T_0\in W\cap L^\infty(\Omega),
\end{cases}
$$

for $r=0,1$.

By the assumption (A3), applying Lemma 2.2, (2.4), (3.1) and (3.2), we obtain

$$
\|p_h^1\|_0^2\Delta t\le c(\|d_t\mathbf{u}_h^1\|_0^2+\nu^2\|A_h\mathbf{u}_h^1\|_0^2+\|\mathbf{f}(t_1)\|_0^2+\|T_h^0\|_0^2+M_1)\Delta t,\tag{3.8a}
$$

$$
\begin{aligned}
\|p_h^2\|_0^2\Delta t\le& c(\|d_t\mathbf{u}_h^2\|_0^2+\nu^2\|A_h\mathbf{u}_h^2\|_0^2+\nu^2\|A_h\mathbf{u}_h^1\|_0^2)\Delta t\\
&+c(\|\bar{\mathbf{f}}(t_2)\|_0^2+\|\overline{T}_h^2\|_0^2+\|\mathbf{u}_h^1\|_0^2\|\mathbf{u}_h^1\|_1^4+M_1)\Delta t.
\end{aligned}\tag{3.8b}
$$

Furthermore, by Lemma 2.2, (2.4), (3.1) and (3.2), we deduce that

$$
\begin{aligned}
&\|d_tT_h^1\|_0^2\Delta t\le c(Pr^{-2}\nu^2\|A_hT_h^1\|_0^2+\|g(t_1)\|_0^2+M_1')\Delta t,\\
&\|d_tT_h^2\|_0^2\Delta t\le c(Pr^{-2}\nu^2\|A_hT_h^2\|_0^2+\|A_hT_h^1\|_0^2+\|\bar{g}(t_2)\|_0^2+\|\mathbf{u}_h^1\|_0^2\|T_h^1\|_1^4+M_1')\Delta t.
\end{aligned}
$$

As a consequence, under the condition (3.4), we complete these proof. □

Theorem 3.2. *Suppose that the assumptions* (A1)-(A3) *hold and the time step* Δt *satisfies*

$$\begin{cases} \max\{\widetilde{C}_3,\widehat{C}_3\}\Delta t \leq 1, & \boldsymbol{u}_0\in V\cap L^\infty, \quad T_0\in W\cap L^\infty, \\ \max\{\widetilde{C}_3,\widehat{C}_3\}|\log h|\Delta t \leq \min\{1,C_2^{-2}\}, & \boldsymbol{u}_0\in V, \quad T_0\in W, \end{cases} \tag{3.9}$$

where $\widehat{C}_3 = 128^2\gamma_0^2\widehat{C}\nu^{-5}C_1C_2(1+C_2^2)$, $\widetilde{C}_3 = Pr^5\widehat{C}_3$. *For all* $1\leq m\leq N$, *we have*

$$\|\boldsymbol{u}_h^m,T_h^m\|_0^2+\min\{\nu,Pr^{-1}\nu\}\Delta t\sum_{n=1}^m\|\overline{\boldsymbol{u}}_h^n,\overline{T}_h^n\|_1^2\leq C_0, \tag{3.10a}$$

$$\begin{aligned}\min\{\nu,Pr^{-1}\nu\}\|\boldsymbol{u}_h^m,T_h^m\|_1^2+\Delta t\sum_{n=1}^m(\|d_t\boldsymbol{u}_h^1,d_tT_h^1\|_0^2+\|p_h^n\|_0^2 \\ +\min\{\nu^2,Pr^{-2}\nu^2\}\|A_h\overline{\boldsymbol{u}}_h^n,A_h\overline{T}_h^n\|_0^2)\leq C_1,\end{aligned} \tag{3.10b}$$

$$\min\{\nu^2,Pr^{-2}\nu^2\}\|A_h\boldsymbol{u}_h^m,A_hT_h^m\|_0^2\Delta t+\min\{\nu,Pr^{-1}\nu\}\Delta t^2\sum_{n=1}^m\|d_t\boldsymbol{u}_h^n,d_tT_h^n\|_1^2\leq C_2, \tag{3.10c}$$

where $C_i\geq C_i'$, $(i=0,1,2)$ *are some constants depending on the data* Pr^{-1}, ν, κ, Ω, T_{time}^{final}, $\boldsymbol{u}_0$, T_0, $\boldsymbol{f}$, g.

Proof. We inductive method to prove (3.10a)-(3.10c). From Theorem 3.1, we know that (3.10a)-(3.10c) hold with $m=1,2$. Then we assume that (3.10a)-(3.10c) hold for $m=3,\cdots,J-1$. We need to prove that (3.10a)-(3.10c) hold for $m=J$.

Taking $\psi_h=2T_h^n\Delta t\in W_h$, $\mathbf{v}_h=0$, $q_h=0$ in (3.2), using Lemma 2.2 and the following facts

$$\begin{aligned}&\phi_h^n=\overline{\phi}_h^n+\frac{1}{2}d_t\phi_h^n\Delta t, \quad \phi_h^n=2\overline{\phi}_h^n-\phi_h^{n-1}, \quad \phi_h^n=\overline{\phi}_h^{n-1}+d_t\phi_h^n\Delta t+\frac{1}{2}d_t\phi_h^{n-1}\Delta t,\\ &2(d_t\phi_h^n,\phi_h^n)\Delta t=\|\phi_h^n\|_0^2-\|\phi_h^{n-1}\|_0^2+\|d_t\phi_h^n\|_0^2\Delta t^2,\\ &2a(\overline{\phi}_h^n,\phi_h^n)\Delta t=\frac{\nu}{2}(\|\phi_h^n\|_1^2-\|\phi_h^{n-1}\|_1^2+4\|\overline{\phi}_h^n\|_1^2)\Delta t, \ \phi \text{ can take } \mathbf{u} \text{ or } T,\end{aligned}$$

we have

$$\begin{aligned}&\Big(\|T_h^n\|_0^2+\frac{Pr^{-1}\nu}{2}\|T_h^n\|_1^2\Delta t\Big)-\Big(\|T_h^{n-1}\|_0^2+\frac{Pr^{-1}\nu}{2}\|T_h^{n-1}\|_1^2\Delta t\Big)\\ &\quad+\|d_tT_h^n\|_0^2\Delta t^2+2Pr^{-1}\nu\|\overline{T}_h^n\|_1^2\Delta t\\ &=(\overline{g}(t_n),2\overline{T}_h^n+d_tT_h^n\Delta t)\Delta t+\tilde{b}(\mathbf{u}_h^{n-1},T_h^{n-1},2d_tT_h^n)\Delta t^2.\end{aligned} \tag{3.11}$$

Thanks to the Cauchy inequality and Lemma 2.2, one finds

$$\begin{aligned}&|(\overline{g}(t_n),2\overline{T}_h^n+d_tT_h^n\Delta t)|\Delta t\\ &\leq Pr^{-1}\nu\|\overline{T}_h^n\|_1^2\Delta t+\frac{1}{2}\|d_tT_h^n\|_0^2\Delta t^2+\Big(\frac{\gamma_0^2}{Pr^{-1}\nu}+\frac{1}{2}\Delta t\Big)\|\overline{g}(t_n)\|_0^2\Delta t,\\ &2|\tilde{b}(\mathbf{u}_h^{n-1},T_h^{n-1},d_tT_h^n)|\Delta t^2\leq\frac{1}{4}G^{\frac{1}{2}}(T_h^{n-1})\|\mathbf{u}_h^{n-1}\|_1\|d_tT_h^n\|_0\Delta t^2\\ &\leq CG(T_h^{n-1})\|\mathbf{u}_h^{n-1}\|_1^2\Delta t^2+\frac{1}{2}\|d_tT_h^n\|_0^2\Delta t^2,\end{aligned}$$

where $G(\phi_h^n)=4^2\gamma_0\widetilde{C}\|\phi_h^n\|_1\|A_h\phi_h^n\|_0$, ϕ can take $\mathbf{u}$ and T.

Under the inductive assumption with $m=1,\cdots,J-1$, we get

$$\begin{aligned}G(T_h^{n-1})\Delta t\leq&4^2\gamma_0\widetilde{C}\|T_h^{n-1}\|_1\|A_hT_h^{n-1}\|_0\Delta t\\ \leq&4^2Pr^{3/2}\nu^{-3/2}\gamma_0\widetilde{C}C_1^{1/2}C_2^{1/2}\Delta t^{1/2}\leq Pr^{-1}\nu.\end{aligned}$$

Combining above inequalities with (3.11), using the inductive assumption with $m=1,\cdots,J-1$ and summing the inequality from $n=3$ to $n=J$, we have

$$\begin{aligned}&\|T_h^J\|_0^2+Pr^{-1}\nu\Delta t\|T_h^J\|_1^2+Pr^{-1}\nu\Delta t\sum_{n=3}^J\|\bar{T}_h^n\|_1^2\\ \leq&\|T_h^2\|_0^2+\frac{Pr^{-1}\nu}{2}\|T_h^2\|_1^2\Delta t+\Big(\frac{\gamma_0^2}{Pr^{-1}\nu}+\frac{1}{2}\Delta t\Big)T_{time}^{final}\sup_{0\leq t\leq T_{time}^{final}}\|g(t)\|_0^2+C_1\Delta t.\end{aligned}\tag{3.12}$$

Choosing $\mathbf{v}_h=2\mathbf{u}_h^n\Delta t\in V_h$, $\psi_h=0$, $q_h=0$ in (3.2), we get that

$$(d_t\mathbf{u}_h^n,2\mathbf{u}_h^n\Delta t)+a(\overline{\mathbf{u}}_h^n,2\mathbf{u}_h^n\Delta t)+b(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},2\mathbf{u}_h^n\Delta t)=(\bar{\mathbf{f}}(t_n),2\mathbf{u}_h^n\Delta t)-\kappa\nu^2(j\overline{T}_h^n,2\mathbf{u}_h^n\Delta t).$$

Thanks to the Cauchy inequality and Lemma 2.2, we arrive at

$$\begin{aligned}&2|b(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},\mathbf{u}_h^n)|\Delta t=2|b(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},d_t\mathbf{u}_h^n)|\Delta t^2\\ &\leq\frac{1}{4}G^{\frac{1}{2}}(\mathbf{u}_h^{n-1})\|\mathbf{u}_h^{n-1}\|_1\|d_t\mathbf{u}_h^n\|_0\Delta t^2\leq CG(\mathbf{u}_h^{n-1})\|\mathbf{u}_h^{n-1}\|_1^2\Delta t^2+\frac{1}{4}\|d_t\mathbf{u}_h^n\|_0^2\Delta t^2,\\ &|(\bar{\mathbf{f}}(t_n),2\overline{\mathbf{u}}_h^n+d_t\mathbf{u}_h^n\Delta t)|\Delta t\leq\frac{\nu}{2}\|\overline{\mathbf{u}}_h^n\|_1^2\Delta t+\frac{1}{4}\|d_t\mathbf{u}_h^n\|_0^2\Delta t^2+\Big(\frac{2\gamma_0^2}{\nu}+\Delta t\Big)\|\bar{\mathbf{f}}(t_n)\|_0^2\Delta t,\\ &\kappa\nu^2|(j\overline{T}_h^n,2\overline{\mathbf{u}}_h^n+d_t\mathbf{u}_h^n\Delta t)|\Delta t\leq\frac{\nu}{2}\|\overline{\mathbf{u}}_h^n\|_1^2\Delta t+\frac{1}{4}\|d_t\mathbf{u}_h^n\|_0^2\Delta t^2+\Big(\frac{2\gamma_0^2}{\nu}+\Delta t\Big)\|\overline{T}_h^n\|_0^2\Delta t.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}&\Big(\|\mathbf{u}_h^n\|_0^2+\frac{\nu}{2}\|\mathbf{u}_h^n\|_1^2\Delta t\Big)-\Big(\|\mathbf{u}_h^{n-1}\|_0^2+\frac{\nu}{2}\|\mathbf{u}_h^{n-1}\|_1^2\Delta t\Big)+\nu\|\overline{\mathbf{u}}_h^n\|_1^2\Delta t\\ \leq&\Big(\frac{2\gamma_0^2}{\nu}+\Delta t\Big)\|\bar{\mathbf{f}}(t_n)\|_0^2\Delta t+\Big(\frac{2\gamma_0^2}{\nu}+\Delta t\Big)\|\overline{T}_h^n\|_0^2\Delta t+CG(\mathbf{u}_h^{n-1})\|\mathbf{u}_h^{n-1}\|_1^2\Delta t^2,\end{aligned}$$

where

$$G(\mathbf{u}_h^{n-1})\Delta t\leq4^2\gamma_0\widehat{C}\|\mathbf{u}_h^{n-1}\|_1\|A_h\mathbf{u}_h^{n-1}\|_0\Delta t\leq4^2\nu^{-3/2}\gamma_0\widehat{C}C_1^{1/2}C_2^{1/2}\Delta t^{1/2}\leq\nu.$$

Summing above inequalities from $n=3$ to $n=J$, we deduce that

$$\begin{aligned}&\|\mathbf{u}_h^J\|_0^2+\frac{\nu}{2}\|\mathbf{u}_h^J\|_1^2\Delta t+\nu\Delta t\sum_{n=3}^J\|\bar{\mathbf{u}}_h^n\|_1^2\\ \leq&\|\mathbf{u}_h^2\|_0^2+\frac{\nu}{2}\|\mathbf{u}_h^2\|_1^2\Delta t+\Big(\frac{2\gamma_0^2}{\nu}+\Delta t\Big)T_{time}^{final}\sup_{0\leq t\leq T_{time}^{final}}\|g(t)\|_0^2\end{aligned}\tag{3.13}$$

$$+\Big(\frac{2\gamma_0^2}{\nu}+\Delta t\Big)\sum_{n=3}^{J}\|\overline{T}_h^n\|_0^2\Delta t+C_1\Delta t. \tag{3.14}$$

Using Theorem 3.1, (3.9) in (3.12)-(3.13) to obtain (3.10a) with $m=J$.

Secondly, choosing $\psi_h=2A_h\overline{T}_h^n\Delta t\in W_h$, $\mathbf{v}_h=0$, $q_h=0$ in (3.2), we get

$$(d_tT_h^n,2A_h\overline{T}_h^n\Delta t)+\tilde{a}(\overline{T}_h^n,2A_h\overline{T}_h^n\Delta t)+\tilde{b}(\mathbf{u}_h^{n-1},T_h^{n-1},2A_h\overline{T}_h^n\Delta t)=(\overline{g}(t_n),2A_h\overline{T}_h^n\Delta t). \tag{3.15}$$

Thanks to Lemma 2.2 and the Cauchy inequality, one yields

$$\begin{aligned}
&2|\tilde{b}(\mathbf{u}_h^{n-1},T_h^{n-1},A_h\overline{T}_h^n)|\Delta t\le\frac{1}{4}G^{\frac12}(\mathbf{u}_h^{n-1})\|T_h^{n-1}\|_1\|A_h\overline{T}_h^n\|_0\Delta t\\
&\le\frac{Pr^{-1}\nu}{2}\|A_h\overline{T}_h^n\|_0^2\Delta t+\frac{1}{32Pr^{-1}\nu}G(\mathbf{u}_h^{n-1})\|T_h^{n-1}\|_1^2\Delta t,\\
&2|(\overline{g}(t_n),A_h\overline{T}_h^n\Delta t)|\le\frac{Pr^{-1}\nu}{2}\|A_h\overline{T}_h^n\|_0^2\Delta t+\frac{1}{2Pr^{-1}\nu}\|\overline{g}(t_n)\|_0^2\Delta t.
\end{aligned}$$

Combining above inequalities with (3.15), we obtain

$$\begin{aligned}
&\|T_h^n\|_1^2-\|T_h^{n-1}\|_1^2+Pr^{-1}\nu\|A_h\overline{T}_h^n\|_0^2\Delta t\\
\le&\frac{1}{32Pr^{-1}\nu}G(\mathbf{u}_h^{n-1})\|T_h^{n-1}\|_1^2\Delta t+\frac{1}{2Pr^{-1}\nu}\|\overline{g}(t_n)\|_0^2\Delta t.
\end{aligned}$$

Summing above inequality from $n=3$ to $n=J$ and using (3.12), we deduce that

$$\begin{aligned}
&\|T_h^J\|_1^2+Pr^{-1}\nu\Delta t\sum_{n=3}^{J}\|A_h\overline{T}_h^n\|_0^2\\
\le&\frac{1}{32}C_1Pr\nu^{-1}+\frac{1}{Pr^{-1}\nu}T_{time}^{final}\sup_{0\le t\le T_{time}^{final}}\|g(t)\|_0^2+\|T_h^2\|_1^2.
\end{aligned} \tag{3.16}$$

Thirdly, choosing $\mathbf{v}_h=2A_h\overline{\mathbf{u}}_h^n\Delta t\in V_h$, $q_h=0$, $\psi_h=0$ in (3.2), we get

$$\begin{aligned}
&(d_t\mathbf{u}_h^n,2A_h\overline{\mathbf{u}}_h^n\Delta t)+a(\overline{\mathbf{u}}_h^n,2A_h\overline{\mathbf{u}}_h^n\Delta t)+b(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},2A_h\overline{\mathbf{u}}_h^n\Delta t)\\
=&(\bar{\mathbf{f}}(t_n),2A_h\overline{\mathbf{u}}_h^n\Delta t)-\kappa\nu^2(j\overline{T}_h^n,2A_h\overline{\mathbf{u}}_h^n\Delta t).
\end{aligned}$$

With the help of Lemma 2.2 and the Cauchy inequality, one yields

$$\begin{aligned}
&2|b(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},A_h\overline{\mathbf{u}}_h^n)|\Delta t\le\frac{1}{4}G^{\frac12}(\mathbf{u}_h^{n-1})\|\mathbf{u}_h^{n-1}\|_1\|A_h\overline{\mathbf{u}}_h^n\|_0\Delta t\\
&\le\frac{\nu}{2}\|A_h\overline{\mathbf{u}}_h^n\|_0^2\Delta t+\frac{1}{32\nu}G(\mathbf{u}_h^{n-1})\|\mathbf{u}_h^{n-1}\|_1^2\Delta t,\\
&2|(\bar{\mathbf{f}}(t_n),A_h\overline{\mathbf{u}}_h^n\Delta t)|\le\frac{\nu}{4}\|A_h\overline{\mathbf{u}}_h^n\|_0^2\Delta t+\frac{4}{\nu}\|\bar{\mathbf{f}}(t_n)\|_0^2\Delta t,\\
&2|-\kappa\nu^2(j\overline{T}_h^n,A_h\overline{\mathbf{u}}_h^n)|\Delta t\le\frac{\nu}{4}\|A_h\overline{\mathbf{u}}_h^n\|_0^2\Delta t+4\kappa^2\nu^3\|\bar{T}_h^n\|_0^2\Delta t.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}&\|\mathbf{u}_h^n\|_1^2-\|\mathbf{u}_h^{n-1}\|_1^2+\nu\|A_h\overline{\mathbf{u}}_h^n\|_0^2\Delta t\\ \le&\frac{1}{32\nu}G(\mathbf{u}_h^{n-1})\|\mathbf{u}_h^{n-1}\|_1^2\Delta t+\frac{4}{\nu}\|\bar{\mathbf{f}}(t_n)\|_0^2\Delta t+4\kappa^2\nu^3\|\bar{T}_h^n\|_0^2\Delta t.\end{aligned}$$

Summing above inequality from $n=3$ to $n=J$ and using (3.13), we deduce

$$\begin{aligned}&\|\mathbf{u}_h^J\|_1^2+\nu\Delta t\sum_{n=3}^{J}\|A_h\overline{\mathbf{u}}_h^n\|_0^2\\ \le&\|\mathbf{u}_h^2\|_1^2+\frac{C_1}{32\nu}+\frac{4}{\nu}T_{time}^{final}\sup_{0\le t\le T_{time}^{final}}\|f(t)\|_0^2+4\kappa^2\nu^3\sum_{n=3}^{J}\|\bar{T}_h^n\|_0^2\Delta t.\end{aligned}\tag{3.17}$$

Thanks to (3.9) and (3.10a), using Theorem 3.1 to (3.16) and (3.17), we obtain

$$\|(\mathbf{u}_h^J,T_h^J)\|_1^2+\min\{\nu,Pr^{-1}\nu\}\Delta t\sum_{n=3}^{J}\|(A_h\overline{\mathbf{u}}_h^n,A_h\overline{T}_h^n)\|_0^2\le C,\quad\forall 1\le J\le N.\tag{3.18}$$

From (3.2), Lemma 2.2 and the assumption (A3), we deduce that

$$\begin{aligned}&(\|p_h^n\|_0^2+\|d_t\mathbf{u}_h^n\|_0^2)\Delta t\\ &\le\nu^2\|A_h\overline{\mathbf{u}}_h^n\|_0^2\Delta t+\widehat{C}\gamma_0^2\|\mathbf{u}_h^{n-1}\|_1^2\|A_h\mathbf{u}_h^{n-1}\|_0^2\Delta t+\kappa^2\nu^4\|\overline{T}_h^n\|_0^2\Delta t+\|\bar{\mathbf{f}}(t_n)\|_0^2\Delta t, &(3.19a)\\ &\|d_tT_h^n\|_0^2\Delta t\le Pr^{-2}\nu^2\|A_h\overline{T}_h^n\|_0^2\Delta t+\widetilde{C}\gamma_0^2\|\mathbf{u}_h^{n-1}\|_1^2\|A_hT_h^{n-1}\|_0^2\Delta t+\|\overline{g}(t_n)\|_0^2\Delta t. &(3.19b)\end{aligned}$$

Summing (3.19a) and (3.19b) from $n=3$ to J, using Lemma 2.1, (3.10a) and (3.18) and the induction assumption, we obtain (3.10b) with $m=J$.

Finally, choosing $\psi_h=2A_hd_tT_h^n\Delta t^2\in W_h$, $\mathbf{v}_h=0$, $q_h=0$ in (3.2) and noting the fact that $\phi_h^n-\phi_h^{n-1}=2(\overline{\phi}_h^n-\phi_h^{n-1})$, ϕ takes $\mathbf{u}$ and T, we get

$$\begin{aligned}&(d_tT_h^n,2A_hd_tT_h^n\Delta t^2)+\tilde{a}(\overline{T}_h^n,2A_hd_tT_h^n\Delta t^2)+\tilde{b}(\mathbf{u}_h^{n-1},T_h^{n-1},2A_hd_tT_h^n\Delta t^2)\\ =&(\overline{g}(t_n),2A_hd_tT_h^n\Delta t^2).\end{aligned}\tag{3.20}$$

Thanks to Lemma 2.2 and the Cauchy inequality, one finds

$$\begin{aligned}&2|\tilde{b}(\mathbf{u}_h^{n-1},T_h^{n-1},A_hd_tT_h^n\Delta t^2)|\le\widetilde{C}\|A_h\mathbf{u}_h^{n-1}\|_0\|A_hT_h^{n-1}\|_0\|d_tT_h^n\|_1\Delta t^2\\ &\le\frac{1}{2}\|d_tT_h^n\|_1^2\Delta t^2+2\widetilde{C}\gamma_0^2\|A_h\mathbf{u}_h^{n-1}\|_0^2\|A_hT_h^{n-1}\|_0^2\Delta t^2,\\ &|(\overline{g}(t_n),2A_hd_tT_h^n\Delta t^2)|\le\frac{1}{2}\|d_tT_h^n\|_1^2\Delta t^2+2\|\bar{g}(t_n)\|_1^2\Delta t^2.\end{aligned}$$

Combining above inequalities with (3.20), we get

$$\begin{aligned}&Pr^{-1}\nu(\|A_hT_h^n\|_0^2\Delta t-\|A_hT_h^{n-1}\|_0^2\Delta t)+\|d_tT_h^n\|_1^2\Delta t^2\\ \le&2\widetilde{C}\gamma_0^2\|A_h\mathbf{u}_h^{n-1}\|_0^2\|A_hT_h^{n-1}\|_0^2\Delta t^2+2\|\bar{g}(t_n)\|_1^2\Delta t^2.\end{aligned}$$

Summing above inequality from $n=3$ to J and using (3.10b), we deduce that

$$\begin{aligned}&Pr^{-1}\nu\|A_hT_h^J\|_0^2\Delta t+\Delta t^2\sum_{n=3}^{J}\|d_tT_h^n\|_1^2\\ \leq&2T_{time}^{final}\sup_{0\leq t\leq T_{time}^{final}}\|g(t)\|_1^2\Delta t^2+Pr^{-1}\nu\|A_hT_h^2\|_0^2\Delta t+C_2^2Pr^4\nu^{-4}.\end{aligned}\tag{3.21}$$

Choosing $\mathbf{v}_h=2A_hd_t\mathbf{u}_h^n\Delta t^2\in V_h$, $\psi_h=0$, $q_h=0$ in (3.2), we obtain

$$\begin{aligned}&(d_t\mathbf{u}_h^n,2A_hd_t\mathbf{u}_h^n\Delta t^2)+a(\overline{\mathbf{u}}_h^n,2A_hd_t\mathbf{u}_h^n\Delta t^2)+b(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},2A_hd_t\mathbf{u}_h^n\Delta t^2)\\ =&(\overline{\mathbf{f}}(t_n),2A_hd_t\mathbf{u}_h^n\Delta t^2)-\kappa\nu^2(j\overline{T}_h^n,2A_hd_t\mathbf{u}_h^n\Delta t^2).\end{aligned}\tag{3.22}$$

Applying Lemma 2.2 and the Cauchy inequality, one yields

$$\begin{aligned}&2|b(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},A_hd_t\mathbf{u}_h^n)|\Delta t^2\leq\frac{1}{2}\|d_t\mathbf{u}_h^n\|_1^2\Delta t^2+2\widehat{C}\gamma_0^2\|A_h\mathbf{u}_h^{n-1}\|_0^4\Delta t^2,\\ &2|(\overline{\mathbf{f}}(t_n),A_hd_t\mathbf{u}_h^n)|\Delta t^2\leq\frac{1}{4}\|d_t\mathbf{u}_h^n\|_1^2\Delta t^2+4\|\bar{f}(t_n)\|_1^2\Delta t^2,\\ &2|-\kappa\nu^2(j\overline{T}_h^n,A_hd_t\mathbf{u}_h^n)|\Delta t^2\leq\frac{1}{4}\|d_t\mathbf{u}_h^n\|_1^2\Delta t^2+4\kappa^2\nu^4\|\overline{T}_h^n\|_1^2\Delta t^2.\end{aligned}$$

Combing above inequalities with (3.22), we get

$$\begin{aligned}&\nu(\|A_h\mathbf{u}_h^n\|_0^2-\|A_h\mathbf{u}_h^{n-1}\|_0^2)\Delta t+\|d_t\mathbf{u}_h^n\|_1^2\Delta t^2\\ \leq&2\widehat{C}\gamma_0^2\|A_h\mathbf{u}_h^{n-1}\|_0^4\Delta t^2+4\|\bar{f}(t_n)\|_1^2\Delta t^2+4\kappa^2\nu^4\|\overline{T}_h^n\|_1^2\Delta t^2.\end{aligned}$$

Summing above inequality from $n=3$ to J and using (3.9), we arrive at

$$\begin{aligned}&\nu\|A_h\mathbf{u}_h^J\|_0^2\Delta t+\Delta t^2\sum_{n=3}^{J}\|d_t\mathbf{u}_h^n\|_1^2\\ \leq&\nu\|A_h\mathbf{u}_h^2\|_0^2\Delta t+4T_{time}^{final}\sup_{0\leq t\leq T_{time}^{final}}\|\mathbf{f}(t)\|_1^2\Delta t^2+4\kappa^2\nu^4\sum_{n=3}^{J}\|\overline{T}_h^n\|_1^2\Delta t^2+\frac{C_2^2}{\nu^2}.\end{aligned}\tag{3.23}$$

Thanks to (3.9), (3.10b), using Lemma 2.1 to (3.21) and (3.23), we complete the proof of (3.10c) with $m=J$. □

We end this section by recalling the following priori estimates provided in [31].

Theorem 3.3. *Suppose that the assumptions* (A1)-(A3) *and* (3.9) *hold, it holds*

$$\begin{aligned}&\|(e_{\boldsymbol{u}}^1,e_T^1)\|_\alpha^2+\min\{\nu,Pr^{-1}\nu\}\|(e_{\boldsymbol{u}}^1,e_T^1)\|_{\alpha+1}^2\Delta t\leq C\Delta t^{1-\alpha},\quad \alpha=-2,-1,0,1,\\ &\|e_p^1\|_0^2\leq C\Delta t^{-1},\end{aligned}$$

where

$$e_\phi^0=0,\quad e_\phi^n=\phi_h(t_n)-\phi_h^n,\quad \phi \textit{ takes } \boldsymbol{u} \textit{ or } T,\quad e_p^n=\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}p_h(s)ds-p_h^n,\quad n=1,\cdots,N.$$

4 Error estimates

In this section, we present the errors of the numerical solutions $e_{\mathbf{u}}^n=\mathbf{u}_h(t_n)-\mathbf{u}_h^n$ and $e_T^n=T_h(t_n)-T_h^n$ in H^1-and L^2-norms and the error

$$e_p^n=\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}p_h(s)ds-p_h^n$$

in L^2-norm with nonsmooth initial data. For all $t\in[0,T_{time}^{final}]$ and $1\le n\le N$, integrate (2.3) from t_{n-1} to t_n, we obtain

$$\begin{cases}(d_t\mathbf{u}_h(t_n),\mathbf{v}_h)+\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}a(\mathbf{u}_h(s),\mathbf{v}_h)ds-\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}d(\mathbf{v}_h,p_h(s))ds\\ \quad+\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}b(\mathbf{u}_h(s),\mathbf{u}_h(s),\mathbf{v}_h)ds=\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}(\mathbf{f}(s),\mathbf{v}_h)ds-\dfrac{\kappa\nu^2}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}(T_h(s),\mathbf{v}_h)ds,\\ d(\overline{\mathbf{u}}_h(t_n),q_h)=0,\\ (d_tT_h(t_n),\psi_h)+\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}\tilde{a}(T_h(s),\psi_h)ds+\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}\tilde{b}(\mathbf{u}_h(s),T_h(s),\psi_h)ds\\ \quad=\dfrac{1}{\Delta t}\displaystyle\int_{t_{n-1}}^{t_n}(g(s),\psi_h)ds.\end{cases}\tag{4.1}$$

For all $\phi\in L^2(t_{n-1},t_n;H^2(t_{n-1},t_n))$, using the integral formula

$$\overline{\phi}(t_n)-\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}\phi(s)ds=\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)\phi_{ss}(s)ds.$$

Subtracting (3.2) from (4.1), we get

$$\begin{cases}(d_te_{\mathbf{u}}^n,\mathbf{v}_h)+a(\overline{e}_{\mathbf{u}}^n,\mathbf{v}_h)-d(\mathbf{v}_h,e_p^n)+b(\mathbf{u}_h^{n-1},e_{\mathbf{u}}^{n-1},\mathbf{v}_h)\\ \quad+b(e_{\mathbf{u}}^{n-1},\mathbf{u}_h(t_{n-1}),\mathbf{v}_h)=(e_n,\mathbf{v}_h),\\ d(\overline{e}_{\mathbf{u}}^n,q_h)=0,\\ (d_te_T^n,\psi_h)+\tilde{a}(\overline{e}_T^n,\psi_h)+\tilde{b}(\mathbf{u}_h^{n-1},e_T^{n-1},\psi_h)\\ \quad+\tilde{b}(e_{\mathbf{u}}^{n-1},T_h(t_{n-1}),\psi_h)=(e_n',\psi_h),\end{cases}\tag{4.2}$$

where

$$\begin{aligned}(e_n,\mathbf{v}_h)=&\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)(\mathbf{f}_{ss},\mathbf{v}_h)ds-\frac{\kappa\nu^2}{2\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)(jT_{hss},\mathbf{v}_h)ds\\ &+\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)a(\mathbf{u}_{hss},\mathbf{v}_h)ds+\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})b_s(\mathbf{u}_h(s),\mathbf{u}_h(s),\mathbf{v}_h)ds\\ &+\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(t_n-s)b_s(\mathbf{u}_h(s),\mathbf{u}_h(s),\mathbf{v}_h)ds-\frac{2\kappa\nu^2}{\Delta t}\int_{t_{n-1}}^{t_n}(jT_h(s),\mathbf{v}_h)ds\\ &+\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)b_{ss}(\mathbf{u}_h(s),\mathbf{u}_h(s),\mathbf{v}_h)ds,\end{aligned}$$

$$\begin{aligned}(e_n',\psi_h)=&\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)(g_{ss},\psi_h)ds+\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)\tilde{a}(T_{hss},\psi_h)ds\\&+\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})\tilde{b}_s(\mathbf{u}_h(s),T_h(s),\psi_h)ds+\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(t_n-s)\tilde{b}_s(\mathbf{u}_h(s),T_h(s),\psi_h)ds\\&+\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)\tilde{b}_{ss}(\mathbf{u}_h(s),T_h(s),\psi_h)ds,\end{aligned}$$

furthermore, we know that

$$\begin{aligned}&b_{tt}(\mathbf{u}_h(t),\mathbf{u}_h(t),\mathbf{v}_h)=b(\mathbf{u}_{htt}(t),\mathbf{u}_h(t),\mathbf{v}_h)+b(\mathbf{u}_h(t),\mathbf{u}_{htt}(t),\mathbf{v}_h)+2b(\mathbf{u}_{ht},\mathbf{u}_{ht},\mathbf{v}_h),\\&\tilde{b}_{tt}(\mathbf{u}_h(t),T_h(t),\psi_h)=\tilde{b}(\mathbf{u}_{htt}(t),T_h(t),\psi_h)+\tilde{b}(\mathbf{u}_h(t),T_{htt}(t),\psi_h)+2\tilde{b}(\mathbf{u}_{ht},T_{ht},\psi_h).\end{aligned}$$

Lemma 4.1 ([31]). *Suppose that the assumptions* (A1)-(A3) *and* (3.9) *hold, for all* $3\le m\le N$, *the errors* e_n *and* e_n' *satisfy*

$$\begin{aligned}&\Delta t\sum_{n=3}^{m}\sigma^i(t_n)\|A_h^{-1}P_he_n,A_h^{-1}P_he_n'\|_0^2\le C\Delta t^{2+i}, && i=0,1,2,\\&\Delta t\sum_{n=3}^{m}\sigma^i(t_n)\|A_h^{-1/2}P_he_n,A_h^{-1/2}P_he_n'\|_0^2\le C\Delta t^{1+i}, && i=0,1,2,3,\\&\Delta t\sum_{n=3}^{m}\sigma^i(t_n)\|P_he_n,P_he_n'\|_0^2\le C\Delta t^{i}, && i=0,1,2,3,4,\\&\Delta t\sum_{n=3}^{m}\sigma^i(t_n)\|A_h^{1/2}P_he_n,A_h^{1/2}P_he_n'\|_0^2\le C\Delta t^{i-1}, && i=0,1,2,3,4,\end{aligned}$$

where

$$\sigma(t_n)\le\sigma(t_{n-1})+\Delta t\le 2\sigma(t),\quad \sigma(t_{n-1})\le\sigma(t),\quad t\in[t_{n-1},t_n],\quad 2\le n\le N.$$

Theorem 4.1. *Under the conditions of* (A1)-(A3) *and* (3.9), *it holds*

$$\begin{aligned}&\|e_{\boldsymbol{u}}^m,e_T^m\|_\alpha^2+\min\{\nu,Pr^{-1}\nu\}\|e_{\boldsymbol{u}}^m,e_T^m\|_{\alpha+1}^2\Delta t+\Delta t\sum_{n=1}^{m}(\|d_te_{\boldsymbol{u}}^n,d_te_T^n\|_\alpha^2\Delta t\\&\quad+\min\{\nu,Pr^{-1}\nu\}\|\bar{e}_{\boldsymbol{u}}^n,\bar{e}_T^n\|_{\alpha+1}^2)\le C\Delta t^{1-\alpha},\quad \forall 1\le m\le N,\quad \alpha=-1,0,1, && (4.3a)\\&\|e_{\boldsymbol{u}}^m,e_T^m\|_{-2}^2+\min\{\nu,Pr^{-1}\nu\}\|e_{\boldsymbol{u}}^m,e_T^m\|_{-1}^2\Delta t+\Delta t\sum_{n=1}^{m}(\|d_te_{\boldsymbol{u}}^n,d_te_T^n\|_{-2}^2\Delta t\\&\quad+\min\{\nu,Pr^{-1}\nu\}\|\bar{e}_{\boldsymbol{u}}^n,\bar{e}_T^n\|_{-1}^2)\le C\Delta t^3,\quad m=1,2. && (4.3b)\end{aligned}$$

Proof. From Theorem 3.3, we know that (4.3b) and (4.3a) hold with $m=1$.

Taking $\mathbf{v}_h=2A_h^\alpha e_{\mathbf{u}}^n\Delta t\in V_h$, $\psi_h=0$ with $\alpha=-1,0,1$ in (4.2), we get

$$\begin{aligned}&(d_te_{\mathbf{u}}^n,2A_h^\alpha e_{\mathbf{u}}^n\Delta t)+a(\bar{e}_{\mathbf{u}}^n,2A_h^\alpha e_{\mathbf{u}}^n\Delta t)+b(\mathbf{u}_h^{n-1},e_{\mathbf{u}}^{n-1},2A_h^\alpha e_{\mathbf{u}}^n\Delta t)\\&\quad+b(e_{\mathbf{u}}^{n-1},\mathbf{u}_h(t_{n-1}),2A_h^\alpha e_{\mathbf{u}}^n\Delta t)=(e_n,2A_h^\alpha e_{\mathbf{u}}^n\Delta t).\end{aligned}$$

Using

$$e_\phi^n=\bar{e}_\phi^n+\frac{1}{2}d_te_\phi^n\Delta t \quad \text{and} \quad e_\phi^{n-1}=\bar{e}_\phi^n-\frac{1}{2}d_te_\phi^n\Delta t,$$

where ϕ can take $\mathbf{u}$ and T, we obtain

$$\begin{aligned}&\|e_\mathbf{u}^n\|_\alpha^2-\|e_\mathbf{u}^{n-1}\|_\alpha^2+\|d_te_\mathbf{u}^n\|_\alpha^2\Delta t^2+\frac{\nu}{2}(\|e_\mathbf{u}^n\|_{\alpha+1}^2-\|e_\mathbf{u}^{n-1}\|_{\alpha+1}^2+4\|\bar{e}_\mathbf{u}^n\|_{\alpha+1}^2)\Delta t\\&\quad+2b(e_\mathbf{u}^{n-1},\mathbf{u}_h(t_{n-1}),A_h^\alpha e_\mathbf{u}^n)\Delta t+2b(\mathbf{u}_h(t_{n-1}),e_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n)\Delta t-2b(\bar{e}_\mathbf{u}^{n-1},\bar{e}_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n)\Delta t\\&\quad-b(d_te_\mathbf{u}^{n-1},\bar{e}_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n)\Delta t^2-b(\bar{e}_\mathbf{u}^{n-1},d_te_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n)\Delta t^2-\frac{1}{2}b(d_te_\mathbf{u}^{n-1},d_te_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n)\Delta t^3\\&\leq\frac{\nu}{8}\|\bar{e}_\mathbf{u}^n\|_{\alpha+1}^2\Delta t+\frac{1}{16}\|d_te_\mathbf{u}^n\|_\alpha^2\Delta t^2+8\nu^{-1}\|A_h^{\frac{\alpha-1}{2}}P_he_n\|_0^2\Delta t+16\|A_h^{\frac{\alpha}{2}}e_n\|_0^2\Delta t^2.\end{aligned}\tag{4.4}$$

Thanks to Lemma 2.2 and (2.4), we have

$$\begin{aligned}&2|b(e_\mathbf{u}^{n-1},\mathbf{u}_h(t_{n-1}),A_h^\alpha e_\mathbf{u}^n)|\Delta t+2\Big|b\Big(\mathbf{u}_h(t_{n-1}),e_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n\Big)\Big|\Delta t\\=&2\Big|b\Big(e_\mathbf{u}^{n-1},\mathbf{u}_h(t_{n-1}),A_h^\alpha\bar{e}_\mathbf{u}^n+\frac{1}{2}A_h^\alpha d_te_\mathbf{u}^n\Delta t\Big)\Big|\Delta t+2\Big|b\Big(\mathbf{u}_h(t_{n-1}),e_\mathbf{u}^{n-1},A_h^\alpha\bar{e}_\mathbf{u}^n+\frac{1}{2}A_h^\alpha d_te_\mathbf{u}^n\Delta t\Big)\Big|\Delta t\\\leq&2\widehat{C}\gamma_0\|e_\mathbf{u}^{n-1}\|_\alpha\|A_h\mathbf{u}_h(t_{n-1})\|_0\|\bar{e}_\mathbf{u}^n\|_{\alpha+1}\Delta t+\widehat{C}\gamma_0\|e_\mathbf{u}^{n-1}\|_{\alpha+1}\|A_h\mathbf{u}_h(t_{n-1})\|_0\|d_te_\mathbf{u}^n\|_\alpha\Delta t^2\\\leq&\frac{\nu}{16}\|\bar{e}_\mathbf{u}^n\|_{\alpha+1}^2\Delta t+\frac{1}{16}\|d_te_\mathbf{u}^n\|_\alpha^2\Delta t^2+c\|A_h\mathbf{u}_h(t_{n-1})\|_0^2\Big(\|e_\mathbf{u}^{n-1}\|_\alpha^2+\frac{\nu}{2}\|e_\mathbf{u}^{n-1}\|_{\alpha+1}^2\Delta t\Big)\Delta t,\\&2|b(\bar{e}_\mathbf{u}^{n-1},\bar{e}_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n)|\Delta t\\\leq&\frac{\nu}{16}\|\bar{e}_\mathbf{u}^n\|_{\alpha+1}^2\Delta t+\frac{1}{16}\|d_te_\mathbf{u}^n\|_\alpha^2\Delta t^2+c\|A_h\bar{e}_\mathbf{u}^{n-1}\|_0^2(\|e_\mathbf{u}^{n-1}+e_\mathbf{u}^{n-2}\|_\alpha^2+\|e_\mathbf{u}^{n-1}+e_\mathbf{u}^{n-2}\|_{\alpha+1}^2\Delta t)\Delta t,\\&|b(d_te_\mathbf{u}^{n-1},\bar{e}_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n)|\Delta t^2+|b(\bar{e}_\mathbf{u}^{n-1},d_te_\mathbf{u}^{n-1},A_h^\alpha e_\mathbf{u}^n)|\Delta t^2\\\leq&\frac{\nu}{16}\|\bar{e}_\mathbf{u}^n\|_{\alpha+1}^2\Delta t+\frac{1}{16}\|d_te_\mathbf{u}^n\|_\alpha^2\Delta t^2+c\|A_h\bar{e}_\mathbf{u}^{n-1}\|_0^2(\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_\alpha^2+\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_{\alpha+1}^2\Delta t)\Delta t,\\&\frac{1}{2}|b(d_te_\mathbf{u}^{n-1},d_te_\mathbf{u}^{n-1},A_h^\alpha\bar{e}_\mathbf{u}^n)|\Delta t^3\\\leq&\frac{\nu}{16}\|\bar{e}_\mathbf{u}^n\|_{\alpha+1}^2\Delta t+\frac{1}{16}\|d_te_\mathbf{u}^{n-1}\|_\alpha^2\Delta t^2+c\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_1^4\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_{\alpha+1}^2\Delta t^2,\quad\alpha=0,1,\\&\frac{1}{2}|b(d_te_\mathbf{u}^{n-1},d_te_\mathbf{u}^{n-1},A_h^{-1}\bar{e}_\mathbf{u}^n)|\Delta t^3\\\leq&\frac{\nu}{16}\|\bar{e}_\mathbf{u}^n\|_0^2\Delta t+c\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_1^2\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_0^2\Delta t^2,\\&\frac{1}{4}|b(d_te_\mathbf{u}^{n-1},d_te_\mathbf{u}^{n-1},A_h^\alpha d_te_\mathbf{u}^n)|\Delta t^4\\\leq&\frac{1}{16}\|d_te_\mathbf{u}^n\|_\alpha^2\Delta t^2+\frac{1}{16}\|d_te_\mathbf{u}^{n-1}\|_\alpha^2\Delta t^2+c\|A_h(e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2})\|_0^4\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_{\alpha+1}^2\Delta t^4,\quad\alpha=0,1,\\&\frac{1}{4}|b(d_te_\mathbf{u}^{n-1},d_te_\mathbf{u}^{n-1},A_h^{-1}d_te_\mathbf{u}^n)|\Delta t^4\\\leq&\frac{1}{16}\|d_te_\mathbf{u}^n\|_{-1}^2\Delta t^2+\frac{1}{16}\|d_te_\mathbf{u}^{n-1}\|_0^2\Delta t^3+c\|A_h(e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2})\|_0^2\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_1^2\|e_\mathbf{u}^{n-1}-e_\mathbf{u}^{n-2}\|_0^2\Delta t^3.\end{aligned}$$

For all $3\leq n\leq N$ and $\alpha=0,1$, combining above inequalities with (4.4), we obtain

$$\begin{aligned}&\Big(\|e_{\mathbf{u}}^n\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_{\alpha+1}^2\Delta t\Big)-\Big(\|e_{\mathbf{u}}^{n-1}\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_{\alpha+1}^2\Delta t\Big)+\frac{5}{8}\|d_te_{\mathbf{u}}^n\|_\alpha^2\Delta t^2\\&\quad-\frac{1}{8}\|d_te_{\mathbf{u}}^{n-1}\|_\alpha^2\Delta t^2+\nu\|\bar{e}_{\mathbf{u}}^n\|_{\alpha+1}^2\Delta t\\&\leq\frac{1}{2}\widehat{b}_{n-1}\Big(\|e_{\mathbf{u}}^{n-1}\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_{\alpha+1}^2\Delta t\Big)\Delta t+\frac{1}{2}\widehat{c}_{n-2}\Big(\|e_{\mathbf{u}}^{n-2}\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^{n-2}\|_{\alpha+1}^2\Delta t\Big)\Delta t\\&\quad+8\nu^{-1}\|A_h^{\frac{\alpha-1}{2}}P_he_n\|_0^2\Delta t+16\|A_h^{\frac{\alpha}{2}}P_he_n\|_0^2\Delta t^2.\end{aligned}\tag{4.5}$$

As $\alpha=-1$, we have

$$\begin{aligned}&\Big(\|e_{\mathbf{u}}^n\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_0^2\Delta t\Big)-\Big(\|e_{\mathbf{u}}^{n-1}\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_0^2\Delta t\Big)\\&\quad+\frac{5}{8}\|d_te_{\mathbf{u}}^n\|_{-1}^2\Delta t^2-\frac{1}{8}\|d_te_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t^2+\nu\|\bar{e}_{\mathbf{u}}^n\|_0^2\Delta t\\&\leq\frac{1}{2}\widehat{b}_{n-1}\Big(\|e_{\mathbf{u}}^{n-1}\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_0^2\Delta t\Big)\Delta t+\frac{1}{2}\widehat{c}_{n-2}\Big(\|e_{\mathbf{u}}^{n-2}\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^{n-2}\|_0^2\Delta t\Big)\Delta t\\&\quad+16\|A_h^{\frac{\alpha}{2}}P_he_n\|_0^2\Delta t^2+c\|d_te_{\mathbf{u}}^{n-1}\|_0^2\Delta t^3\\&\quad+c\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^2\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_0^2\Delta t+8\nu^{-1}\|A_h^{\frac{\alpha-1}{2}}P_he_n\|_0^2\Delta t,\end{aligned}\tag{4.6}$$

where

$$\begin{aligned}\widehat{b}_{n-1}=&c\|A_h\mathbf{u}_h(t_{n-1})\|_0^2+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2+c\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^4+c\|A_h(e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2})\|_0^4\Delta t^2,\\\widehat{c}_{n-2}=&c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2+c\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^4+cH(\alpha)\|A_h(e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2})\|_0^4\Delta t^2\\&+cH(-\alpha)\|A_h(e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2})\|_0^2\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^2\Delta t.\end{aligned}$$

Here and below, $H(\alpha)=1$ for all $\alpha>0$ and $H(\alpha)=0$ for all $\alpha<0$.

Noting the fact that $\widehat{d}_n=\frac{1}{2}(\widehat{b}_n+\widehat{c}_n)$, by Theorems 2.2 and 3.2, we have

$$\Delta t\sum_{n=1}^N\widehat{d}_n\leq C.\tag{4.7}$$

Thanks to Lemma 4.1 and summing (4.5) from $n=3$ to m, we get

$$\begin{aligned}&\|e_{\mathbf{u}}^m\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^m\|_{\alpha+1}^2\Delta t+\Delta t\sum_{n=3}^m\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_\alpha^2\Delta t+\nu\|\bar{e}_{\mathbf{u}}^n\|_{\alpha+1}^2\Big)\\&\leq\|e_{\mathbf{u}}^2\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^2\|_{\alpha+1}^2\Delta t+\frac{1}{8}\|d_te_{\mathbf{u}}^2\|_\alpha^2\Delta t^2+\Delta t\sum_{n=1}^{m-1}\widehat{d}_n\Big(\|e_{\mathbf{u}}^n\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_{\alpha+1}^2\Delta t\Big)\\&\quad+\nu^{-1}\Delta t\sum_{n=3}^m(8\|A_h^{\frac{\alpha-1}{2}}P_he_n\|_0^2+16\nu\|A_h^{\frac{\alpha}{2}}P_he_n\|_0^2\Delta t)\\&\leq\Delta t\sum_{n=1}^{m-1}\widehat{d}_n\Big(\|e_{\mathbf{u}}^n\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_{\alpha+1}^2\Delta t\Big)+C\Delta t^{1-\alpha}.\end{aligned}$$

For all $3\leq m\leq N$ with $\alpha=0,1$. Using (4.7) and Lemma 2.2, we deduce that

$$\|e_{\mathbf{u}}^m\|_\alpha^2+\frac{\nu}{2}\|e_{\mathbf{u}}^m\|_{\alpha+1}^2\Delta t+\Delta t\sum_{n=3}^m\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_\alpha^2\Delta t+\nu\|\bar{e}_{\mathbf{u}}^n\|_{\alpha+1}^2\Big)\leq C\Delta t^{1-\alpha}, \tag{4.8}$$

with $\alpha=-1,0,1$, choosing $\psi_h=2A_h^\alpha e_T^n\Delta t\in W_h$, $\mathbf{v}_h=0$, $q_h=0$ in (4.2), it yields

$$\begin{aligned}
&(d_te_T^n,2A_h^\alpha e_T^n\Delta t)+\tilde{a}(\bar{e}_T^n,2A_h^\alpha e_T^n\Delta t)+2\tilde{b}(e_{\mathbf{u}}^{n-1},T_h(t_{n-1}),A_h^\alpha e_T^n)\Delta t\\
&\quad+2\tilde{b}(\mathbf{u}_h(t_{n-1}),e_T^{n-1},A_h^\alpha e_T^n)\Delta t-2\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^\alpha e_T^n)\Delta t-\tilde{b}(d_te_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^\alpha e_T^n)\Delta t^2\\
&\quad-\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^\alpha e_T^n)\Delta t^2-\frac{1}{2}\tilde{b}(d_te_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^\alpha e_T^n)\Delta t^3\\
=&(e_n',2A_h^\alpha e_T^n\Delta t).
\end{aligned}$$

By Lemma 2.2, the Cauchy inequality and (2.2b), we obtain

$$\begin{aligned}
&2|\tilde{b}(e_{\mathbf{u}}^{n-1},T_h(t_{n-1}),A_h^\alpha e_T^n)|\Delta t+2|\tilde{b}(\mathbf{u}_h(t_{n-1}),e_T^{n-1},A_h^\alpha e_T^n)|\Delta t\\
=&2\Big|\tilde{b}\Big(e_{\mathbf{u}}^{n-1},T_h(t_{n-1}),A_h^\alpha\bar{e}_T^n+\frac{1}{2}A_h^\alpha d_te_T^n\Delta t\Big)\Big|\Delta t+2\Big|\tilde{b}\Big(\mathbf{u}_h(t_{n-1}),e_T^{n-1},A_h^\alpha\bar{e}_T^n+\frac{1}{2}A_h^\alpha d_te_T^n\Delta t\Big)\Big|\Delta t\\
\leq&2\widetilde{C}\gamma_0\|e_{\mathbf{u}}^{n-1}\|_\alpha\|A_hT_h(t_{n-1})\|_0\|\bar{e}_T^n\|_{\alpha+1}\Delta t+\widetilde{C}\gamma_0\|e_{\mathbf{u}}^{n-1}\|_{\alpha+1}\|A_hT_h(t_{n-1})\|_0\|d_te_T^n\|_\alpha\Delta t^2\\
&\quad+2\widetilde{C}\gamma_0\|e_T^{n-1}\|_\alpha\|A_h\mathbf{u}_h(t_{n-1})\|_0\|\bar{e}_T^n\|_{\alpha+1}\Delta t+\widetilde{C}\gamma_0\|e_T^{n-1}\|_{\alpha+1}\|A_h\mathbf{u}_h(t_{n-1})\|_0\|d_te_T^n\|_\alpha\Delta t^2\\
\leq&Pr^{-1}\nu/16\|\bar{e}_T^n\|_{\alpha+1}^2\Delta t+1/16\|d_te_T^n\|_\alpha^2\Delta t^2+c\|A_h\mathbf{u}_h(t_{n-1})\|_0^2\Big(\|e_T^{n-1}\|_\alpha^2\\
&\quad+\frac{Pr^{-1}\nu}{2}\|e_T^{n-1}\|_{\alpha+1}^2\Delta t\Big)\Delta t+c\|A_hT_h(t_{n-1})\|_0^2\Big(\|e_{\mathbf{u}}^{n-1}\|_\alpha^2+\frac{Pr^{-1}\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_{\alpha+1}^2\Delta t\Big)\Delta t,\\
&|\tilde{b}(d_te_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^\alpha e_T^n)|\Delta t^2+|\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^\alpha e_T^n)|\Delta t^2\\
\leq&Pr^{-1}\nu/16\|\bar{e}_T^n\|_{\alpha+1}^2\Delta t+c\|A_h\bar{e}_T^{n-1}\|_0^2(\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_\alpha^2+\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_{\alpha+1}^2\Delta t)\Delta t\\
&\quad+1/16\|d_te_T^n\|_\alpha^2\Delta t^2+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2(\|e_T^{n-1}-e_T^{n-2}\|_\alpha^2+\|e_T^{n-1}-e_T^{n-2}\|_{\alpha+1}^2\Delta t)\Delta t,\\
&2|\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^\alpha e_T^n)|\Delta t\\
\leq&Pr^{-1}\nu/16\|\bar{e}_T^n\|_{\alpha+1}^2\Delta t+1/16\|d_te_T^n\|_\alpha^2\Delta t^2\\
&\quad+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2(\|e_T^{n-1}+e_T^{n-2}\|_\alpha^2+\|e_T^{n-1}+e_T^{n-2}\|_{\alpha+1}^2\Delta t)\Delta t,\\
&\frac{1}{2}|\tilde{b}(d_te_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^\alpha\bar{e}_T^n)|\Delta t^3\\
\leq&Pr^{-1}\nu/16\|\bar{e}_T^n\|_{\alpha+1}^2\Delta t+1/16\|d_te_T^{n-1}\|_\alpha^2\Delta t^2+c\|e_{\mathbf{u}}^{n-1}\\
&\quad-e_{\mathbf{u}}^{n-2}\|_1^4\|e_T^{n-1}-e_T^{n-2}\|_{\alpha+1}^2\Delta t^2,\quad\alpha=0,1,\\
&\frac{1}{4}|\tilde{b}(d_te_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^\alpha d_te_T^n)|\Delta t^4\\
\leq&1/16\|d_te_T^n\|_\alpha^2\Delta t^2+1/16\|d_te_T^{n-1}\|_\alpha^2\Delta t^2+c\|A_h(e_{\mathbf{u}}^{n-1}\\
&\quad-e_{\mathbf{u}}^{n-2})\|_0^4\|e_T^{n-1}-e_T^{n-2}\|_{\alpha+1}^2\Delta t^4,\quad\alpha=0,1,\\
&\frac{1}{2}|\tilde{b}(d_te_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^{-1}\bar{e}_T^n)|\Delta t^3
\end{aligned}$$

$$
\begin{aligned}
&\le Pr^{-1}\nu/16\|\bar{e}_T^n\|_0^2\Delta t + c\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^2\|e_T^{n-1}-e_T^{n-2}\|_0^2\Delta t^2,\\
&\frac{1}{4}|\tilde{b}(d_te_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^{-1}d_te_T^n)|\Delta t^4\\
&\le 1/16\|d_te_T^n\|_{-1}^2\Delta t^2 + 1/16\|d_te_T^{n-1}\|_0^2\Delta t^3\\
&\quad + c\|A_h(e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2})\|_0^2\|e_T^{n-1}-e_T^{n-2}\|_1^2\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_0^2\Delta t^3,\\
&2|(e_n',A_h^{\alpha}e_T^n)|\Delta t\\
&\le \frac{Pr^{-1}\nu}{8}\|\bar{e}_T^n\|_{\alpha+1}^2\Delta t + \frac{1}{16}\|d_te_T^n\|_{\alpha}^2\Delta t^2\\
&\quad + 8Pr\nu^{-1}\|A_h^{\frac{\alpha-1}{2}}P_he_n'\|_0^2\Delta t + 16\|A_h^{\frac{\alpha}{2}}e_n'\|_0^2\Delta t^2.
\end{aligned}
$$

Then, for all $3\le n\le N$ with $\alpha=0,1$, one finds

$$
\begin{aligned}
&\Big(\|e_T^n\|_\alpha^2 + \frac{Pr^{-1}\nu}{2}\|e_T^n\|_{\alpha+1}^2\Delta t\Big) - (\|e_T^{n-1}\|_\alpha^2 + \frac{Pr^{-1}\nu}{2}\|e_T^{n-1}\|_{\alpha+1}^2\Delta t)\\
&\quad + \frac{5}{8}\|d_te_T^n\|_\alpha^2\Delta t^2 - \frac{1}{8}\|d_te_T^{n-1}\|_\alpha^2\Delta t^2 + Pr^{-1}\nu\|\bar{e}_T^n\|_{\alpha+1}^2\Delta t\\
&\le \frac{1}{2}\widetilde{b}_{n-1}\Big(\|e_T^{n-1}\|_\alpha^2 + \frac{Pr^{-1}\nu}{2}\|e_T^{n-1}\|_{\alpha+1}^2\Delta t\Big)\Delta t + \frac{1}{2}\widetilde{c}_{n-2}\Big(\|e_T^{n-2}\|_\alpha^2 + \frac{Pr^{-1}\nu}{2}\|e_T^{n-2}\|_{\alpha+1}^2\Delta t\Big)\Delta t\\
&\quad + c\|A_hT_h(t_{n-1})\|_0^2\Big(\|e_{\mathbf{u}}^{n-1}\|_\alpha^2 + \frac{Pr^{-1}\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_{\alpha+1}^2\Delta t\Big)\Delta t + 8Pr\nu^{-1}\|A_h^{\frac{\alpha-1}{2}}P_he_n'\|_0^2\Delta t\\
&\quad + c\|A_h\bar{e}_T^{n-1}\|_0^2\Big(\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_\alpha^2 + \|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_{\alpha+1}^2\Delta t\Big)\Delta t + 16\|A_h^{\frac{\alpha}{2}}P_he_n'\|_0^2\Delta t^2,
\end{aligned}
\tag{4.9}
$$

and with $\alpha=-1$, we have

$$
\begin{aligned}
&\Big(\|e_T^n\|_{-1}^2 + \frac{Pr^{-1}\nu}{2}\|e_T^n\|_0^2\Delta t\Big) - \Big(\|e_T^{n-1}\|_{-1}^2 + \frac{Pr^{-1}\nu}{2}\|e_T^{n-1}\|_0^2\Delta t\Big)\\
&\quad + \frac{5}{8}\|d_te_T^n\|_{-1}^2\Delta t^2 - \frac{1}{8}\|d_te_T^{n-1}\|_{-1}^2\Delta t^2 + Pr^{-1}\nu\|\bar{e}_T^n\|_0^2\Delta t\\
&\le \frac{1}{2}\widetilde{b}_{n-1}\Big(\|e_T^{n-1}\|_{-1}^2 + \frac{Pr^{-1}\nu}{2}\|e_T^{n-1}\|_0^2\Delta t\Big)\Delta t + \frac{1}{2}\widetilde{c}_{n-2}\Big(\|e_T^{n-2}\|_{-1}^2 + \frac{Pr^{-1}\nu}{2}\|e_T^{n-2}\|_0^2\Delta t\Big)\Delta t\\
&\quad + c\|d_te_T^{n-1}\|_0^2\Delta t^3 + c\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^2\|e_T^{n-1}-e_T^{n-2}\|_0^2\Delta t\\
&\quad + c\|A_hT_h(t_{n-1})\|_0^2\Big(\|e_{\mathbf{u}}^{n-1}\|_{-1}^2 + \frac{Pr^{-1}\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_0^2\Delta t\Big)\Delta t + 8Pr\nu^{-1}\|A_h^{\frac{\alpha-1}{2}}P_he_n'\|_0^2\Delta t\\
&\quad + c\|A_h\bar{e}_T^{n-1}\|_0^2\Big(\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_{-1}^2 + \|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_0^2\Delta t\Big)\Delta t + 16\|A_h^{\frac{\alpha}{2}}P_he_n'\|_0^2\Delta t^2,
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
\widetilde{b}_{n-1} &= c\|A_h\mathbf{u}_h(t_{n-1})\|_0^2 + c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2 + c\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^4 + c\|A_h(e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2})\|_0^4\Delta t^2,\\
\widetilde{c}_{n-2} &= c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2 + c\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^4 + cH(\alpha)\|A_h(e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2})\|_0^4\Delta t^2\\
&\quad + cH(-\alpha)\|A_h(e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2})\|_0^2\|e_T^{n-1}-e_T^{n-2}\|_1^2\Delta t.
\end{aligned}
$$

Noting $\widetilde{d}_n=\frac{1}{2}(\widetilde{b}_n+\widetilde{c}_n)$, using Theorems 2.2 and 3.2, we have

$$\Delta t\sum_{n=1}^{N}\widetilde{d}_n\le C. \tag{4.11}$$

Applying Lemma 4.1 and summing (4.9) from $n=3$ to m, we get

$$\begin{aligned}
&\|e_T^m\|_\alpha^2+\frac{Pr^{-1}\nu}{2}\|e_T^m\|_{\alpha+1}^2\Delta t+\Delta t\sum_{n=3}^{m}\Big(\frac{1}{2}\|d_te_T^n\|_\alpha^2\Delta t+Pr^{-1}\nu\|\bar{e}_T^n\|_{\alpha+1}^2\Big)\\
\le&\|e_T^2\|_\alpha^2+\frac{Pr^{-1}\nu}{2}\|e_T^2\|_{\alpha+1}^2\Delta t+\frac{1}{8}\|d_te_T^2\|_\alpha^2\Delta t^2+\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\Big(\|e_T^n\|_\alpha^2+\frac{Pr^{-1}\nu}{2}\|e_T^n\|_{\alpha+1}^2\Delta t\Big)\\
&+c\sum_{n=3}^{m}\|A_hT_h(t_{n-1})\|_0^2\Big(\|e_{\mathbf{u}}^{n-1}\|_\alpha^2+\frac{Pr^{-1}\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_{\alpha+1}^2\Delta t\Big)\Delta t\\
&+c\sum_{n=3}^{m}\|A_h\bar{e}_T^{n-1}\|_0^2\Big(\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_\alpha^2+\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_{\alpha+1}^2\Delta t\Big)\Delta t\\
&+Pr\nu^{-1}\Delta t\sum_{n=3}^{m}\Big(8\|A_h^{\frac{\alpha-1}{2}}P_he_n'\|_0^2+16Pr^{-1}\nu\|A_h^{\frac{\alpha}{2}}P_he_n'\|_0^2\Delta t\Big)\\
\le&\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\Big(\|e_T^n\|_\alpha^2+\frac{Pr^{-1}\nu}{2}\|e_T^n\|_{\alpha+1}^2\Delta t\Big)+C\Delta t^{1-\alpha}.
\end{aligned}$$

For all $3\le m\le N$ with $\alpha=0,1$. By Lemma 2.2 and (4.11) to above inequality, we get

$$\|e_T^m\|_\alpha^2+\frac{Pr^{-1}\nu}{2}\|e_T^m\|_{\alpha+1}^2\Delta t+\Delta t\sum_{n=3}^{m}\Big(\frac{1}{2}\|d_te_T^n\|_\alpha^2\Delta t+Pr^{-1}\nu\|\bar{e}_T^n\|_{\alpha+1}^2\Big)\le C\Delta t^{1-\alpha}. \tag{4.12}$$

Substituting (4.8) into (4.12) obtains (4.3a) with $\alpha=0,1$. From (4.3a) with $\alpha=0$, one finds

$$\Delta t^3\sum_{n=2}^{m}\|d_te_{\mathbf{u}}^n,d_te_T^n\|_0^2\le C\Delta t^2.$$

Finally, we deduce (4.3a) for $\alpha=-1$ by using (4.6) and (4.10). □

Theorem 4.2. *Under the assumptions of* (A1)-(A3) *and* (3.9), *for all* $1\le m\le N$, *we have*

$$\begin{aligned}
&\|e_{\boldsymbol{u}}^m,e_T^m\|_{-2}^2+\min\{\nu,Pr^{-1}\nu\}\|e_{\boldsymbol{u}}^m,e_T^m\|_{-1}^2\Delta t\\
&+\Delta t\sum_{n=1}^{m}\Big(\frac{1}{2}\|d_te_{\boldsymbol{u}}^n,d_te_T^n\|_{-2}^2\Delta t+\min\{\nu,Pr^{-1}\nu\}\|\bar{e}_{\boldsymbol{u}}^n,\bar{e}_T^n\|_{-1}^2\Big)\le C\Delta t^3.
\end{aligned}$$

Proof. Firstly, taking $\mathbf{v}_h = 2A_h^{-2}e_{\mathbf{u}}^n\Delta t \in V_h$, $q_h=0$, $\psi_h=0$ in (4.2), we obtain

$$\begin{aligned}
&\|e_{\mathbf{u}}^n\|_{-2}^2-\|e_{\mathbf{u}}^{n-1}\|_{-2}^2+\|d_te_{\mathbf{u}}^n\|_{-2}^2\Delta t^2+\frac{\nu}{2}(\|e_{\mathbf{u}}^n\|_{-1}^2-\|e_{\mathbf{u}}^{n-1}\|_{-1}^2)\Delta t\\
&\quad+2\nu\|\bar{e}_{\mathbf{u}}^n\|_{-1}^2\Delta t+2b(e_{\mathbf{u}}^{n-1},\mathbf{u}_h(t_{n-1}),A_h^{-2}e_{\mathbf{u}}^n)\Delta t+2b(\mathbf{u}_h(t_{n-1}),e_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^n)\Delta t\\
&\quad-2b(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^n)\Delta t-b(\bar{e}_{\mathbf{u}}^{n-1},d_te_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^n)\Delta t^2\\
&\quad-b(d_te_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^n)\Delta t^2-\frac{1}{2}b(d_te_{\mathbf{u}}^{n-1},d_te_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^n)\Delta t^3\\
=&2(e_n,A_h^{-2}e_{\mathbf{u}}^n)\Delta t. \qquad (4.13)
\end{aligned}$$

Noting the fact that

$$e_\phi^n=d_te_\phi^n\Delta t+e_\phi^{n-1},\quad e_\phi^n=\bar{e}_\phi^n+\frac{1}{2}d_te_\phi^n\Delta t,\quad e_\phi^{n-1}=\bar{e}_\phi^n-\frac{1}{2}d_te_\phi^n\Delta t,\quad \phi \text{ takes } \mathbf{u} \text{ or } T,$$

and using Lemma 2.2, the Cauchy inequality and (2.2b), we have

$$\begin{aligned}
&2|b(e_{\mathbf{u}}^{n-1},\mathbf{u}_h(t_{n-1}),A_h^{-2}d_te_{\mathbf{u}}^n)|\Delta t^2+2|b(\mathbf{u}_h(t_{n-1}),e_{\mathbf{u}}^{n-1},A_h^{-2}d_te_{\mathbf{u}}^n)|\Delta t^2\\
\le&2\widehat{C}\gamma_0\|e_{\mathbf{u}}^{n-1}\|_{-1}\|A_h\mathbf{u}_h(t_{n-1})\|_0\|d_te_{\mathbf{u}}^n\|_{-2}\Delta t^2\\
\le&\frac{1}{16}\|d_te_{\mathbf{u}}^n\|_{-2}^2\Delta t^2+c\|A_h\mathbf{u}_h(t_{n-1})\|_0^2\|e_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t^2,\\
&2|b(e_{\mathbf{u}}^{n-1},\mathbf{u}_h(t_{n-1}),A_h^{-2}e_{\mathbf{u}}^{n-1})|\Delta t+2|b(\mathbf{u}_h(t_{n-1}),e_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^{n-1})|\Delta t\\
\le&\frac{\nu}{8}\|\overline{e_{\mathbf{u}}}^n\|_{-1}^2\Delta t+\frac{1}{8}\|d_te_{\mathbf{u}}^n\|_{-1}^2\Delta t^3+c\|A_h\mathbf{u}_h(t_{n-1})\|_0^2\|e_{\mathbf{u}}^{n-1}\|_{-2}^2\Delta t,\\
&2|b(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^{n-1},A_h^{-2}d_te_{\mathbf{u}}^n)|\Delta t^2\le\frac{1}{16}\|d_te_{\mathbf{u}}^n\|_{-2}^2\Delta t^2+c\|\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|\bar{e}_{\mathbf{u}}^{n-1}\|_1^2\Delta t^2,\\
&2|b(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^{n-1})|\Delta t\le\widehat{C}\gamma_0\|\bar{e}_{\mathbf{u}}^{n-1}\|_{-1}\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0\|e_{\mathbf{u}}^{n-1}\|_{-2}\Delta t\\
\le&\frac{\nu}{8}\|\bar{e}_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|e_{\mathbf{u}}^{n-1}\|_{-2}^2\Delta t,\\
&|b(d_te_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^{n-1},A_h^{-2}d_te_{\mathbf{u}}^n)|\Delta t^3+|b(\bar{e}_{\mathbf{u}}^{n-1},d_te_{\mathbf{u}}^{n-1},A_h^{-2}d_te_{\mathbf{u}}^n)|\Delta t^3\\
\le&\frac{1}{16}\|d_te_{\mathbf{u}}^n\|_{-2}^2\Delta t^2+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_{-1}^2\Delta t^2,\\
&|b(d_te_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^{n-1})|\Delta t^2+|b(\bar{e}_{\mathbf{u}}^{n-1},d_te_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^{n-1})|\Delta t^2\\
\le&\frac{1}{8}\|d_te_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t^3+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|e_{\mathbf{u}}^{n-1}\|_{-2}^2\Delta t,\\
&\frac{1}{2}|b(d_te_{\mathbf{u}}^{n-1},d_te_{\mathbf{u}}^{n-1},A_h^{-2}d_te_{\mathbf{u}}^n)|\Delta t^4\le\frac{1}{4}\widehat{C}\gamma_0\|d_te_{\mathbf{u}}^{n-1}\|_0\|d_te_{\mathbf{u}}^{n-1}\|_1\|d_te_{\mathbf{u}}^n\|_{-2}\Delta t^4\\
\le&\frac{1}{16}\|d_te_{\mathbf{u}}^n\|_{-2}^2\Delta t^2+c\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^2\|d_te_{\mathbf{u}}^{n-1}\|_0^2\Delta t^4,\\
&\frac{1}{2}|b(d_te_{\mathbf{u}}^{n-1},d_te_{\mathbf{u}}^{n-1},A_h^{-2}e_{\mathbf{u}}^{n-1})|\Delta t^3\le\frac{1}{4}\widehat{C}\gamma_0\|d_te_{\mathbf{u}}^{n-1}\|_0\|d_te_{\mathbf{u}}^{n-1}\|_1\|e_{\mathbf{u}}^{n-1}\|_{-2}\Delta t^3
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{8}\|d_t e_{\mathbf{u}}^{n-1}\|_0^2 \Delta t^4 + c\|d_t e_{\mathbf{u}}^{n-1}\|_1^2 \|e_{\mathbf{u}}^{n-1}\|_{-2}^2 \Delta t^2,\\
&\int_{t_{n-1}}^{t_n} (s-t_{n-1})(t_n-s)(\mathbf{f}_{ss}, A_h^{-2}(d_t e_{\mathbf{u}}^n \Delta t + e_{\mathbf{u}}^{n-1}))ds\\
&\leq \frac{1}{16}\|d_t e_{\mathbf{u}}^n\|_{-2}^2 \Delta t^2 + \|e_{\mathbf{u}}^{n-1}\|_{-2}^2 \Delta t + c\Delta t^3 \int_{t_{n-1}}^{t_n} (\|\mathbf{f}_{ss}\|_0^2 + \Delta t\|\mathbf{f}_{ss}\|_0^2) ds,\\
&\kappa\nu^2 \int_{t_{n-1}}^{t_n} (s-t_{n-1})(t_n-s)(jT_{hss}, A_h^{-2}(d_t e_{\mathbf{u}}^n \Delta t + e_{\mathbf{u}}^{n-1}))ds\\
&\leq \frac{1}{16}\|d_t e_{\mathbf{u}}^n\|_{-2}^2 \Delta t^2 + \|e_{\mathbf{u}}^{n-1}\|_{-2}^2 \Delta t + c\kappa^2\nu^4 \Delta t^3 \int_{t_{n-1}}^{t_n} (\|T_{hss}\|_0^2 + \Delta t\|T_{hss}\|_0^2) ds,\\
&\kappa\nu^2 \int_{t_{n-1}}^{t_n} (jT_h, A_h^{-2}(d_t e_{\mathbf{u}}^n \Delta t + e_{\mathbf{u}}^{n-1}))ds\\
&\leq \frac{1}{16}\|d_t e_{\mathbf{u}}^n\|_{-2}^2 \Delta t^2 + \|e_{\mathbf{u}}^{n-1}\|_{-2}^2 \Delta t + c\kappa^2\nu^4 \Delta t^3 \int_{t_{n-1}}^{t_n} (\|T_h\|_0^2 + \Delta t\|T_h\|_0^2) ds,\\
&\Delta t^2 \int_{t_{n-1}}^{t_n} |b_s(\mathbf{u}_h(s), \mathbf{u}_h(s), A_h^{-2}(d_t e_{\mathbf{u}}^n \Delta t + e_{\mathbf{u}}^{n-1}))| ds\\
&\leq \widehat{C}\gamma_0 \Delta t^2 \int_{t_{n-1}}^{t_n} (\|\mathbf{u}_{hs}\|_{-1}\|A_h\mathbf{u}_h\|_0 + \|\mathbf{u}_h\|_0\|\mathbf{u}_{hs}\|_1) ds (\|d_t e_{\mathbf{u}}^n\|_{-2} + \|e_{\mathbf{u}}^{n-1}\|_{-2})\\
&\leq \frac{1}{16}\|d_t e_{\mathbf{u}}^n\|_{-2}^2 \Delta t^2 + c\int_{t_{n-1}}^{t_n} (\|A_h\mathbf{u}_h\|_0^2 + \|\mathbf{u}_h\|_0^2) ds \|e_{\mathbf{u}}^{n-1}\|_{-2}^2\\
&\quad + c\Delta t^3 \int_{t_{n-1}}^{t_n} \big[(\Delta t + \Delta t^2\|A_h\mathbf{u}_h\|_0^2)\|\mathbf{u}_{hs}\|_{-1}^2 + (\Delta t + \Delta t^2\|\mathbf{u}_h\|_0^2)\|\mathbf{u}_{hs}\|_1^2\big] ds,\\
&\int_{t_{n-1}}^{t_n} (s-t_{n-1})(t_n-s) a\Big(\mathbf{u}_{hss}, A_h^{-2}\big(\frac{1}{2} d_t e_{\mathbf{u}}^n \Delta t + \bar{e}_{\mathbf{u}}^n\big)\Big) ds\\
&\leq \frac{1}{16}\|d_t e_{\mathbf{u}}^n\|_{-2}^2 \Delta t^2 + \frac{\nu}{8}\|\overline{e_{\mathbf{u}}}^n\|_{-1}^2 \Delta t + c\int_{t_{n-1}}^{t_n} (\Delta t^2\|\mathbf{u}_{hss}\|_0^2 + \Delta t\|\mathbf{u}_{hss}\|_{-1}^2) ds,\\
&\Delta t^2 \int_{t_{n-1}}^{t_n} |b_{ss}(\mathbf{u}_h, \mathbf{u}_h, A_h^{-2}(d_t e_{\mathbf{u}}^n \Delta t + e_{\mathbf{u}}^{n-1}))| ds\\
&\leq \widehat{C}\gamma_0 \Delta t^2 \int_{t_{n-1}}^{t_n} (\|\mathbf{u}_{hss}\|_{-1}\|A_h\mathbf{u}_h\|_0 + \|\mathbf{u}_{hs}\|_0\|\mathbf{u}_{hs}\|_1) ds (\|d_t e_{\mathbf{u}}^n\|_{-2} + \|e_{\mathbf{u}}^{n-1}\|_{-2})\\
&\leq \frac{1}{16}\|d_t e_{\mathbf{u}}^n\|_{-2}^2 \Delta t^2 + c\int_{t_{n-1}}^{t_n} (\|A_h\mathbf{u}_h\|_0^2 + \|\mathbf{u}_{hs}\|_0^2) ds \|e_{\mathbf{u}}^{n-1}\|_{-2}^2\\
&\quad + c\Delta t^3 \int_{t_{n-1}}^{t_n} \big[(\Delta t + \Delta t^2\|A_h\mathbf{u}_h\|_0^2)\|\mathbf{u}_{hss}\|_{-1}^2 + (\Delta t + \Delta t^2\|\mathbf{u}_{hs}\|_0^2)\|\mathbf{u}_{hs}\|_1^2\big] ds.
\end{aligned}
$$

For all $3 \leq n \leq N$ and using the above inequalities with (4.13), we get

$$
\Big(\|e_{\mathbf{u}}^n\|_{-2}^2 + \frac{\nu}{2}\|e_{\mathbf{u}}^n\|_{-1}^2 \Delta t\Big) - \Big(\|e_{\mathbf{u}}^{n-1}\|_{-2}^2 + \frac{\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_{-1}^2 \Delta t\Big) + \frac{1}{2}\|d_t e_{\mathbf{u}}^n\|_{-2}^2 \Delta t^2 + \frac{3\nu}{2}\|\bar{e}_{\mathbf{u}}^n\|_{-1}^2 \Delta t
$$

$$
\begin{aligned}
\leq&\frac{1}{2}\widehat{b}'_{n-1}\Big(\|e_{\mathbf{u}}^{n-1}\|_{-2}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t\Big)\Delta t^2+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_{-1}^2\Delta t^2+c\|d_te_u^n\|_{-1}^2\Delta t^3\\
&+c\|\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|\bar{e}_{\mathbf{u}}^{n-1}\|_1^2\Delta t+c(1+\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^2)\|d_te_{\mathbf{u}}^{n-1}\|_0^2\Delta t^4+c\|d_te_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t^3\\
&+c\Delta t^3\int_{t_{n-1}}^{t_n}(\|\mathbf{f}_{ss}\|_0^2+\Delta t\|\mathbf{f}_{ss}\|_0^2)ds+c\kappa^2\nu^4\Delta t^3\int_{t_{n-1}}^{t_n}(\|T_{hss}\|_0^2+\Delta t\|T_{hss}\|_0^2)ds+\frac{\nu}{2}\|\bar{e}_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t\\
&+c\Delta t^2\int_{t_{n-1}}^{t_n}(\Delta t^2\|\mathbf{u}_{hss}\|_0^2+\Delta t\|\mathbf{u}_{hss}\|_{-1}^2)ds+c\kappa^2\nu^4\Delta t^3\int_{t_{n-1}}^{t_n}(\|T_h\|_0^2+\Delta t\|T_h\|_0^2)ds\\
&+c\Delta t^3\int_{t_{n-1}}^{t_n}\big[(\Delta t+\Delta t^2\|A_h\mathbf{u}_h\|_0^2)\|\mathbf{u}_{hss}\|_{-1}^2+(\Delta t+\Delta t^2\|\mathbf{u}_{hs}\|_0^2)\|\mathbf{u}_{hs}\|_1^2\big]ds,\\
&+c\Delta t^3\int_{t_{n-1}}^{t_n}\big[(\Delta t+\Delta t^2\|A_h\mathbf{u}_h\|_0^2)\|\mathbf{u}_{hs}\|_{-1}^2+(\Delta t+\Delta t^2\|\mathbf{u}_h\|_0^2)\|\mathbf{u}_{hs}\|_1^2\big]ds,
\end{aligned}\tag{4.14}
$$

where

$$
\widehat{b}'_{n-1}=c(\|d_te_{\mathbf{u}}^{n-1}\|_1^2\Delta t+\|A_h\mathbf{u}_h(t_{n-1})\|_0^2+\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2+\Delta t^{-1}\int_{t_{n-1}}^{t_n}(\|A_h\mathbf{u}_h\|_0^2+\|\mathbf{u}_{hs}\|_0^2)ds).
$$

Noting $\widehat{d}'_n=\frac{1}{2}\widehat{b}'_n$, by Theorems 2.2, 3.2 and 4.1, we have

$$
\Delta t\sum_{n=1}^{N}\widehat{d}'_n\leq C. \tag{4.15}
$$

Using Theorems 4.1, 4.2 and summing (4.14) from $n=3$ to m, we obtain

$$
\begin{aligned}
&\|e_{\mathbf{u}}^m\|_{-2}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^m\|_{-1}^2\Delta t+\Delta t\sum_{n=3}^{m}\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_{-2}^2\Delta t+\nu\|\bar{e}_{\mathbf{u}}^n\|_{-1}^2\Big)\\
\leq&\Delta t\sum_{n=2}^{m-1}\widehat{d}'_n\Big(\|e_{\mathbf{u}}^n\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_0^2\Delta t\Big)+C\Delta t^3.
\end{aligned}\tag{4.16}
$$

Secondly, choosing $\psi_h=2A_h^{-2}e_T^n\Delta t\in W_h$, $\mathbf{v}_h=0$, $q_h=0$ in (4.2), it yields

$$
\begin{aligned}
&\|e_T^n\|_{-2}^2-\|e_T^{n-1}\|_{-2}^2+\|d_te_T^n\|_{-2}^2\Delta t^2+\frac{Pr^{-1}\nu}{2}(\|e_T^n\|_{-1}^2-\|e_T^{n-1}\|_{-1}^2)\Delta t+2Pr^{-1}\nu\|\bar{e}_T^n\|_{-1}^2\Delta t\\
&\quad+2\tilde{b}(e_{\mathbf{u}}^{n-1},T_h(t_{n-1}),A_h^{-2}e_T^n)\Delta t+2\tilde{b}(\mathbf{u}_h(t_{n-1}),e_T^{n-1},A_h^{-2}e_T^n)\Delta t\\
&\quad-2\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^{-2}e_T^n)\Delta t-\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^{-2}e_T^n)\Delta t^2\\
&\quad-\tilde{b}(d_te_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^{-2}e_T^n)\Delta t^2-\frac{1}{2}\tilde{b}(d_te_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^{-2}e_T^n)\Delta t^3\\
=&2(e'_n,A_h^{-2}e_T^n)\Delta t.
\end{aligned}\tag{4.17}
$$

Using Lemma 2.2, the bilinear term properties and (2.2b), we have

$$
\begin{aligned}
&2|\tilde{b}(e_{\mathbf{u}}^{n-1},T_h(t_{n-1}),A_h^{-2}d_te_T^n)|\Delta t^2+2|\tilde{b}(\mathbf{u}_h(t_{n-1}),e_T^{n-1},A_h^{-2}d_te_T^n)|\Delta t^2\\
\leq&\frac{1}{16}\|d_te_T^n\|_{-2}^2\Delta t^2+c\|A_h\mathbf{u}_h(t_{n-1})\|_0^2\|e_T^{n-1}\|_{-1}^2\Delta t^2+c\|A_hT_h(t_{n-1})\|_0^2\|e_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t^2,
\end{aligned}
$$

$$
\begin{aligned}
&2|\tilde{b}(\mathbf{u}_h(t_{n-1}),e_T^{n-1},A_h^{-2}e_T^{n-1})|\Delta t\\
\le&\frac{Pr^{-1}\nu}{8}\|\bar{e}_T^n\|_{-1}^2\Delta t+\frac{1}{8}\|d_te_T^n\|_{-1}^2\Delta t^3+c\|A_h\mathbf{u}_h(t_{n-1})\|_0^2\|e_T^{n-1}\|_{-2}^2\Delta t,\\
&2|\tilde{b}(e_{\mathbf{u}}^{n-1},T_h(t_{n-1}),A_h^{-2}e_T^{n-1})|\Delta t\\
\le&\frac{Pr^{-1}\nu}{8}\|\bar{e}_{\mathbf{u}}^n\|_{-1}^2\Delta t+\frac{1}{8}\|d_te_{\mathbf{u}}^n\|_{-1}^2\Delta t^3+c\|A_hT_h(t_{n-1})\|_0^2\|e_T^{n-1}\|_{-2}^2\Delta t,\\
&2|\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^{-2}d_te_T^n)|\Delta t^2\le\frac{1}{16}\|d_te_T^n\|_{-2}^2\Delta t^2+c\|\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|\bar{e}_T^{n-1}\|_1^2\Delta t^2,\\
&2|\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^{-2}e_T^{n-1})|\Delta t\le\frac{Pr^{-1}\nu}{8}\|\bar{e}_T^{n-1}\|_{-1}^2\Delta t+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|e_T^{n-1}\|_{-2}^2\Delta t,\\
&|\tilde{b}(d_te_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^{-2}d_te_T^n)|\Delta t^3+|\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^{-2}d_te_T^n)|\Delta t^3\\
\le&\frac{1}{16}\|d_te_T^n\|_{-2}^2\Delta t^2+c\|A_h\bar{e}_T^{n-1}\|_0^2\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_{-1}^2\Delta t^2+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|e_T^{n-1}-e_T^{n-2}\|_{-1}^2\Delta t^2,\\
&|\tilde{b}(d_te_{\mathbf{u}}^{n-1},\bar{e}_T^{n-1},A_h^{-2}e_T^{n-1})|\Delta t^2\le\frac{1}{8}\|d_te_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t^3+c\|A_h\bar{e}_T^{n-1}\|_0^2\|e_T^{n-1}\|_{-2}^2\Delta t,\\
&|\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^{-2}e_T^{n-1})|\Delta t^2\le\frac{1}{8}\|d_te_T^{n-1}\|_{-1}^2\Delta t^3+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|e_T^{n-1}\|_{-2}^2\Delta t,\\
&\frac{1}{2}|\tilde{b}(d_te_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^{-2}d_te_T^n)|\Delta t^4\le\frac{1}{16}\|d_te_T^n\|_{-2}^2\Delta t^2+c\|e_T^{n-1}-e_T^{n-2}\|_1^2\|d_te_{\mathbf{u}}^{n-1}\|_0^2\Delta t^4,\\
&\frac{1}{2}|\tilde{b}(d_te_{\mathbf{u}}^{n-1},d_te_T^{n-1},A_h^{-2}e_T^{n-1})|\Delta t^3\le\frac{1}{8}\|d_te_T^{n-1}\|_0^2\Delta t^4+c\|d_te_{\mathbf{u}}^{n-1}\|_1^2\|e_T^{n-1}\|_{-2}^2\Delta t^2,\\
&\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)(g_{ss},A_h^{-2}(d_te_T^n\Delta t+e_T^{n-1}))ds\\
\le&\frac{1}{16}\|d_te_T^n\|_{-2}^2\Delta t^2+\|e_T^{n-1}\|_{-2}^2\Delta t+c\Delta t^3\int_{t_{n-1}}^{t_n}(\|g_{ss}\|_0^2+\Delta t\|g_{ss}\|_0^2)ds,\\
&\int_{t_{n-1}}^{t_n}(s-t_{n-1})(t_n-s)\tilde{a}(T_{hss}(s),A_h^{-2}(\frac{1}{2}d_te_T^n\Delta t+\bar{e}_T^n))ds\\
\le&\frac{1}{16}\|d_te_T^n\|_{-2}^2\Delta t^2+\frac{Pr^{-1}\nu}{8}\|\bar{e}_T^n\|_{-1}^2\Delta t+c\Delta t^3\int_{t_{n-1}}^{t_n}(\|T_{hss}\|_0^2+\Delta t\|T_{hss}\|_{-1}^2)ds,\\
&\Delta t^2\int_{t_{n-1}}^{t_n}|\tilde{b}_{ss}(\mathbf{u}_h(s),T_h(s),A_h^{-2}(d_te_T^n\Delta t+e_T^{n-1}))|ds\\
\le&\frac{1}{16}\|d_te_T^n\|_{-2}^2\Delta t^2+c\int_{t_{n-1}}^{t_n}(\|A_hT_h\|_0^2+\|T_{hs}\|_0^2)ds\|e_T^{n-1}\|_{-2}^2\\
&\quad+c\Delta t^3\int_{t_{n-1}}^{t_n}[(\Delta t+\Delta t^2\|A_hT_h\|_0^2)\|\mathbf{u}_{hss}\|_{-1}^2+(\Delta t+\Delta t^2\|T_{hs}\|_0^2)\|\mathbf{u}_{hs}\|_1^2]ds,\\
&\Delta t^2\int_{t_{n-1}}^{t_n}|\tilde{b}_s(\mathbf{u}_h(s),T_h(s),A_h^{-2}(d_te_T^n\Delta t+e_T^{n-1}))|ds\\
\le&\frac{1}{16}\|d_te_T^n\|_{-2}^2\Delta t^2+c\int_{t_{n-1}}^{t_n}(\|A_hT_h\|_0^2+\|T_h\|_0^2)ds\|e_T^{n-1}\|_{-2}^2\\
&\quad+c\Delta t^3\int_{t_{n-1}}^{t_n}[(\Delta t+\Delta t^2\|A_hT_h\|_0^2)\|\mathbf{u}_{hs}\|_{-1}^2+(\Delta t+\Delta t^2\|T_{hs}\|_0^2)\|\mathbf{u}_h\|_1^2]ds.
\end{aligned}
$$

Combining above inequalities with (4.17), for all $3\leq n\leq N$, we obtain

$$\begin{aligned}
&\Big(\|e_T^n\|_{-2}^2+\frac{Pr^{-1}\nu}{2}\|e_T^n\|_{-1}^2\Delta t\Big)-\Big(\|e_T^{n-1}\|_{-2}^2+\frac{Pr^{-1}\nu}{2}\|e_T^{n-1}\|_{-1}^2\Delta t\Big)\\
&\quad+\frac{1}{2}\|d_te_T^n\|_{-2}^2\Delta t^2+\frac{3Pr^{-1}\nu}{2}\|\bar{e}_T^n\|_{-1}^2\Delta t-\frac{Pr^{-1}\nu}{2}\|\bar{e}_T^{n-1}\|_{-1}^2\Delta t\\
\leq&\frac{1}{2}\widetilde{b}'_{n-1}\Big(\|e_T^{n-1}\|_{-2}^2+\frac{Pr^{-1}\nu}{2}\|e_T^{n-1}\|_{-1}^2\Delta t\Big)\Delta t+\frac{1}{2}\widetilde{c}'_{n-2}\Big(\|e_T^{n-2}\|_{-2}^2+\frac{Pr^{-1}\nu}{2}\|e_T^{n-2}\|_{-1}^2\Delta t\Big)\Delta t\\
&\quad+c\|\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|\bar{e}_T^{n-1}\|_1^2\Delta t^2+c\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\|e_T^{n-1}-e_T^{n-2}\|_{-1}^2\Delta t^2+cPr^{-1}\nu\|\bar{e}_{\mathbf{u}}^n\|_{-1}^2\\
&\quad+c\|A_h\bar{e}_T^{n-1}\|_0^2\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_{-1}^2\Delta t^2+c\|d_te_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t^3+c\|d_te_{\mathbf{u}}^n\|_{-1}^2\Delta t^3\\
&\quad+c\|d_te_T^{n-1}\|_0^2\Delta t^4+c\|A_hT_h(t_{n-1})\|_0^2\|e_{\mathbf{u}}^{n-1}\|_{-1}^2\Delta t^2\\
&\quad+c\|e_T^{n-1}-e_T^{n-2}\|_1^2\|d_te_{\mathbf{u}}^{n-1}\|_0^2\Delta t^4+c\|d_te_T^{n-1}\|_{-1}^2\Delta t^3+c\|d_te_T^n\|_{-1}^2\Delta t^3\\
&\quad+c\Delta t^3\int_{t_{n-1}}^{t_n}(\|g_{ss}\|_0^2+\Delta t\|g_{ss}\|_0^2)ds+c\Delta t^3\int_{t_{n-1}}^{t_n}(\|T_{hss}\|_0^2+\Delta t\|T_{hss}\|_{-1}^2)ds\\
&\quad+c\Delta t^3\int_{t_{n-1}}^{t_n}[(\Delta t+\Delta t^2\|A_hT_h\|_0^2)\|\mathbf{u}_{hss}\|_{-1}^2+(\Delta t+\Delta t^2\|T_{hs}\|_0^2)\|\mathbf{u}_{hs}\|_1^2]ds,\\
&\quad+c\Delta t^3\int_{t_{n-1}}^{t_n}[(\Delta t+\Delta t^2\|A_hT_h\|_0^2)\|\mathbf{u}_{hs}\|_{-1}^2+(\Delta t+\Delta t^2\|T_{hs}\|_0^2)\|\mathbf{u}_{h}\|_1^2]ds,
\end{aligned}\tag{4.18}$$

where $\widetilde{c}'_{n-2}=c\|A_h\mathbf{u}_h(t_{n-2})\|_0^2$,

$$\begin{aligned}
\widetilde{b}'_{n-1}=&c(\|d_te_{\mathbf{u}}^{n-1}\|_1^2\Delta t+\|A_h\mathbf{u}_h(t_{n-1})\|_0^2+\|A_h\bar{e}_T^{n-1}\|_0^2+\|A_h\bar{e}_{\mathbf{u}}^{n-1}\|_0^2\\
&+\int_{t_{n-1}}^{t_n}(\|A_hT_h\|_0^2+\|T_{hs}\|_0^2+\|T_h\|_0^2)ds+\|A_hT_h(t_{n-1})\|_0^2).
\end{aligned}$$

With Theorems 2.2, 3.2, 4.1 and noting the fact that $\widetilde{d}'_n=\frac{1}{2}(\widetilde{b}'_n+\widetilde{c}'_n)$, we have

$$\Delta t\sum_{n=1}^{N}\widetilde{d}'_n\leq C.\tag{4.19}$$

Summing (4.18) from $n=3$ to m, using Theorems 4.1 and 4.2, we get

$$\begin{aligned}
&\|e_T^m\|_{-2}^2+\frac{Pr^{-1}\nu}{2}\|e_T^m\|_{-1}^2\Delta t+\Delta t\sum_{n=3}^{m}\Big(\frac{1}{2}\|d_te_T^n\|_{-2}^2\Delta t+Pr^{-1}\nu\|\bar{e}_T^n\|_{-1}^2\Big)\\
\leq&\Delta t\sum_{n=2}^{m-1}\widetilde{d}'_n\Big(\|e_T^n\|_{-1}^2+\frac{Pr^{-1}\nu}{2}\|e_T^n\|_0^2\Delta t\Big)+C\Delta t^3.
\end{aligned}\tag{4.20}$$

Using (4.19) and applying Lemma 2.2 to (4.16) and (4.20), we complete the proof. $\square$

Theorem 4.3. *Under the assumptions of* (A1)-(A3) *and* (3.9), *for all* $1\leq m\leq N$, *we have*

$$\sigma^2(t_m)\|e_{\boldsymbol{u}}^m,e_T^m\|_0^2+\min\{\nu,Pr^{-1}\nu\}\Delta t\sum_{n=1}^{m}\sigma^2(t_n)\|\bar{e}_{\boldsymbol{u}}^n,\bar{e}_T^n\|_1^2\leq C\Delta t^3.$$

Proof. For all $3 \le m \le N$, summing from $n=3$ to m, using Theorems 2.2, 4.1, 4.2, 4.3, and multiplying by $\sigma(t_n)$ in (4.6), we get

$$\begin{aligned}
&\sigma(t_m)\Big(\|e_{\mathbf{u}}^m\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^m\|_0^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_{-1}^2\Delta t+\nu\|\bar{e}_{\mathbf{u}}^n\|_0^2\Big)\\
\le&\Delta t\sum_{n=2}^{m-1}\Big(\Big\|\bar{e}_{\mathbf{u}}^n+\frac{1}{2}d_te_{\mathbf{u}}^n\Delta t\Big\|_{-1}^2+\frac{\nu}{2}\Big\|\bar{e}_{\mathbf{u}}^n+\frac{1}{2}d_te_{\mathbf{u}}^n\Delta t\Big\|_0^2\Delta t\Big)+C\Delta t^3\\
&+2\Delta t\sum_{n=2}^{m-1}\widehat{d}_n\sigma(t_n)\Big(\|e_{\mathbf{u}}^n\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_0^2\Delta t\Big)\\
\le&C\Delta t^3+2\Delta t\sum_{n=2}^{m-1}\widehat{d}_n\sigma(t_n)\Big(\|e_{\mathbf{u}}^n\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_0^2\Delta t\Big). \qquad (4.21)
\end{aligned}$$

For all $1\le m\le N$, applying Lemma 2.2 to (4.21) and using (4.7), we have

$$\sigma(t_m)\Big(\|e_{\mathbf{u}}^m\|_{-1}^2+\frac{\nu}{2}\|e_{\mathbf{u}}^m\|_0^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_{-1}^2\Delta t+\nu\|\bar{e}_{\mathbf{u}}^n\|_0^2\Big)\le C\Delta t^3. \qquad (4.22)$$

Multiplying by $\sigma(t_n)$ in (4.10) with $\alpha=-1$. Summing from $n=3$ to m with $3\le m\le N$, we get

$$\begin{aligned}
&\sigma(t_m)\Big(\|e_T^m\|_{-1}^2+\frac{Pr^{-1}\nu}{2}\|e_T^m\|_0^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_T^n\|_{-1}^2\Delta t+Pr^{-1}\nu\|\bar{e}_T^n\|_0^2\Big)\\
\le&\Delta t\sum_{n=2}^{m-1}\Big(\Big\|\bar{e}_T^n+\frac{1}{2}d_te_T^n\Delta t\Big\|_{-1}^2+\frac{Pr^{-1}\nu}{2}\Big\|\bar{e}_T^n+\frac{1}{2}d_te_T^n\Delta t\Big\|_0^2\Delta t\Big)+C\Delta t^3\\
&+2\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\sigma(t_n)\Big(\|e_T^n\|_{-1}^2+\frac{Pr^{-1}\nu}{2}\|e_T^n\|_0^2\Delta t\Big)\\
\le&C\Delta t^3+2\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\sigma(t_n)\Big(\|e_T^n\|_{-1}^2+\frac{Pr^{-1}\nu}{2}\|e_T^n\|_0^2\Delta t\Big). \qquad (4.23)
\end{aligned}$$

Applying Lemma 2.1 to (4.23) and using (4.11), for $1\le m\le N$, we get

$$\begin{aligned}
&\sigma(t_m)\Big(\|e_T^m\|_{-1}^2+\frac{Pr^{-1}\nu}{2}\|e_T^m\|_0^2\Delta t\Big)\\
&\quad+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_T^n\|_{-1}^2\Delta t+Pr^{-1}\nu\|\bar{e}_T^n\|_0^2\Big)\le C\Delta t^3. \qquad (4.24)
\end{aligned}$$

Thanks to (4.22), multiplying by $\sigma(t_n)$ in (4.5) with $\alpha=0$, summing from $n=3$ to m with

$3\leq m\leq N$, we obtain

$$\begin{aligned}
&\sigma(t_m)\Big(\|e_{\mathbf{u}}^m\|_0^2+\frac{\nu}{2}\|e_{\mathbf{u}}^m\|_1^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_0^2\Delta t+\nu\|\bar{e}_{\mathbf{u}}^n\|_1^2\Big)\\
\leq&\Delta t\sum_{n=2}^{m-1}\Big(\|\bar{e}_{\mathbf{u}}^n+\frac{1}{2}d_te_{\mathbf{u}}^n\Delta t\|_0^2+\frac{\nu}{2}\|\bar{e}_{\mathbf{u}}^n+\frac{1}{2}d_te_{\mathbf{u}}^n\Delta t\|_1^2\Delta t\Big)+C\Delta t^2\\
&+2\Delta t\sum_{n=2}^{m-1}\widehat{d}_n\sigma(t_n)\Big(\|e_{\mathbf{u}}^n\|_0^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_1^2\Delta t\Big)\\
\leq&C\Delta t^2+2\Delta t\sum_{n=2}^{m-1}\widehat{d}_n\sigma(t_n)\Big(\|e_{\mathbf{u}}^n\|_0^2+\frac{\nu}{2}\|e_{\mathbf{u}}^n\|_1^2\Delta t\Big).
\end{aligned}\tag{4.25}$$

For all $1\leq m\leq N$, by using Lemma 2.1 to (4.25) and (4.7), we have

$$\sigma(t_m)\Big(\|e_{\mathbf{u}}^m\|_0^2+\frac{\nu}{2}\|e_{\mathbf{u}}^m\|_1^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_0^2\Delta t+\nu\|\bar{e}_{\mathbf{u}}^n\|_1^2\Big)\leq C\Delta t^2.\tag{4.26}$$

For all $3\leq m\leq N$, applying (4.24), Theorems 2.2, 4.1, 4.2, 4.3, multiplying $\sigma(t_n)$ in (4.9) with $\alpha=0$ and summing from $n=3$ to m, one finds

$$\begin{aligned}
&\sigma(t_m)\Big(\|e_T^m\|_0^2+\frac{Pr^{-1}\nu}{2}\|e_T^m\|_1^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_T^n\|_0^2\Delta t+Pr^{-1}\nu\|\bar{e}_T^n\|_1^2\Big)\\
\leq&\Delta t\sum_{n=2}^{m-1}\Big(\|\bar{e}_T^n+\frac{1}{2}d_te_T^n\Delta t\|_0^2+\frac{Pr^{-1}\nu}{2}\|\bar{e}_T^n+\frac{1}{2}d_te_T^n\Delta t\|_1^2\Delta t\Big)+C\Delta t^2\\
&+2\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\sigma(t_n)\Big(\|e_T^n\|_0^2+\frac{Pr^{-1}\nu}{2}\|e_T^n\|_1^2\Delta t\Big)\\
\leq&C\Delta t^2+2\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\sigma(t_n)\Big(\|e_T^n\|_0^2+\frac{Pr^{-1}\nu}{2}\|e_T^n\|_1^2\Delta t\Big).
\end{aligned}\tag{4.27}$$

For all $1\leq m\leq N$, thanks to (4.11) and using Lemma 2.1 to (4.27), we have

$$\sigma(t_m)\Big(\|e_T^m\|_0^2+\frac{Pr^{-1}\nu}{2}\|e_T^m\|_1^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_T^n\|_0^2\Delta t+Pr^{-1}\nu\|\bar{e}_T^n\|_1^2\Big)\leq C\Delta t^2.\tag{4.28}$$

Secondly, taking $\mathbf{v}_h=2\bar{e}_{\mathbf{u}}^n\Delta t\in V_h$, $q_h=0$, $\psi_h=0$ in (4.2), we get

$$\begin{aligned}
&\|e_{\mathbf{u}}^n\|_0^2-\|e_{\mathbf{u}}^{n-1}\|_0^2+2\nu\|\bar{e}_{\mathbf{u}}^n\|_1^2\Delta t+2b(e_{\mathbf{u}}^{n-1},\mathbf{u}_h(t_{n-1}),\bar{e}_{\mathbf{u}}^n)\Delta t+2b(\mathbf{u}_h(t_{n-1}),e_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^n)\Delta t\\
&\quad-2b(\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^n)\Delta t-b(d_te_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^n)\Delta t^2\\
&\quad-b(\bar{e}_{\mathbf{u}}^{n-1},d_te_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^n)\Delta t^2-\frac{1}{2}b(d_te_{\mathbf{u}}^{n-1},d_te_{\mathbf{u}}^{n-1},\bar{e}_{\mathbf{u}}^n)\Delta t^3=2(e_n,\bar{e}_{\mathbf{u}}^n)\Delta t.
\end{aligned}$$

For all $3 \le n \le N$, by the similar proof as Theorem 4.2, we obtain

$$\begin{aligned}
&\|e_{\mathbf{u}}^n\|_0^2 - \|e_{\mathbf{u}}^{n-1}\|_0^2 + \nu\|\bar{e}_{\mathbf{u}}^n\|_1^2 \Delta t \\
\le& \frac{1}{2} b_{n-1}\|e_{\mathbf{u}}^{n-1}\|_0^2 \Delta t + \frac{1}{2} c_{n-2}\|e_{\mathbf{u}}^{n-2}\|_0^2 \Delta t \\
&+ c\|e_{\mathbf{u}}^{n-1} - e_{\mathbf{u}}^{n-2}\|_1^4 \|e_{\mathbf{u}}^{n-1} - e_{\mathbf{u}}^{n-2}\|_1^2 \Delta t^2 + 8\nu^{-1}\|A_h^{-1/2} P_h e_n\|_0^2 \Delta t.
\end{aligned}$$

Thanks to (4.22), (4.26), Theorems 2.2, 4.1, 4.2, 4.3, and multiplying by $\sigma^2(t_n)$ in above inequality, summing from $n=3$ to m for all $3 \le m \le N$, one gets

$$\begin{aligned}
&\sigma^2(t_m)\|e_{\mathbf{u}}^m\|_0^2 + \nu \Delta t \sum_{n=3}^{m} \sigma^2(t_n)\|\bar{e}_{\mathbf{u}}^n\|_1^2 \\
\le& 3\Delta t \sum_{n=2}^{m-1} \sigma(t_n) \left\| \bar{e}_{\mathbf{u}}^n + \frac{1}{2} d_t e_{\mathbf{u}}^n \Delta t \right\|_0^2 + 4\Delta t \sum_{n=2}^{m-1} \widehat{d}_n \sigma^2(t_n)\|e_{\mathbf{u}}^n\|_0^2 \\
&+ c\Delta t \sum_{n=3}^{m} \sigma^2(t_n) \Big[\|A_h^{-1/2} P_h e_n\|_0^2 + \|e_{\mathbf{u}}^{n-1} - e_{\mathbf{u}}^{n-2}\|_1^4 \|e_{\mathbf{u}}^{n-1} - e_{\mathbf{u}}^{n-2}\|_1^2 \Delta t \Big] \\
\le& C\Delta t^3 + 4\Delta t \sum_{n=2}^{m-1} \widehat{d}_n \sigma^2(t_n)\|e_{\mathbf{u}}^n\|_0^2.
\end{aligned} \tag{4.29}$$

In the same way, taking $\psi_h = 2\bar{e}_T^n \Delta t \in W_h$, $\mathbf{v}_h = 0$, $q_h = 0$ in (4.2), we have

$$\begin{aligned}
&\|e_T^n\|_0^2 - \|e_T^{n-1}\|_0^2 + 2Pr^{-1}\nu\|\bar{e}_T^n\|_1^2 \Delta t + 2\tilde{b}(e_{\mathbf{u}}^{n-1}, T_h(t_{n-1}), \bar{e}_T^n)\Delta t \\
&+ 2\tilde{b}(\mathbf{u}_h(t_{n-1}), e_T^{n-1}, \bar{e}_T^n)\Delta t - 2\tilde{b}(\bar{e}_{\mathbf{u}}^{n-1}, \bar{e}_T^{n-1}, \bar{e}_T^n)\Delta t - \tilde{b}(d_t e_{\mathbf{u}}^{n-1}, \bar{e}_T^{n-1}, \bar{e}_T^n)\Delta t^2 \\
&- \tilde{b}(\bar{e}_{\mathbf{u}}^{n-1}, d_t e_T^{n-1}, \bar{e}_T^n)\Delta t^2 - \frac{1}{2}\tilde{b}(d_t e_{\mathbf{u}}^{n-1}, d_t e_T^{n-1}, \bar{e}_T^n)\Delta t^3 = 2(e_n', \bar{e}_T^n)\Delta t.
\end{aligned}$$

For all $3 \le n \le N$, by the similar proof as Theorem 4.1, we obtain

$$\begin{aligned}
&\|e_T^n\|_0^2 - \|e_T^{n-1}\|_0^2 + Pr^{-1}\nu\|\bar{e}_T^n\|_1^2 \Delta t \\
\le& \frac{1}{2} b_{n-1}\|e_T^{n-1}\|_0^2 \Delta t + c\|e_{\mathbf{u}}^{n-1} - e_{\mathbf{u}}^{n-2}\|_1^4 \|e_T^{n-1} - e_T^{n-2}\|_1^2 \Delta t^2 \\
&+ \frac{1}{2} c_{n-2}\|e_T^{n-2}\|_0^2 \Delta t + 8Pr\nu^{-1}\|A_h^{-1/2} P_h e_n'\|_0^2 \Delta t.
\end{aligned}$$

Using (4.24), (4.28), Theorems 2.2, 4.1, 4.2, 4.3, and multiplying by $\sigma^2(t_n)$ in above inequality, summing from $n=3$ to m with $3 \le m \le N$, we have

$$\begin{aligned}
&\sigma^2(t_m)\|e_T^m\|_0^2 + Pr^{-1}\nu \Delta t \sum_{n=3}^{m} \sigma^2(t_n)\|\bar{e}_T^n\|_1^2 \\
\le& 3\Delta t \sum_{n=2}^{m-1} \sigma(t_n) \left\| \bar{e}_T^n + \frac{1}{2} d_t e_T^n \Delta t \right\|_0^2 + 4\Delta t \sum_{n=2}^{m-1} \widetilde{d}_n \sigma^2(t_n)\|e_T^n\|_0^2
\end{aligned}$$

$$
\begin{aligned}
&+c\Delta t\sum_{n=3}^{m}\sigma^2(t_n)\big[\|A_h^{-1/2}P_he_n'\|_0^2+\|e_{\mathbf{u}}^{n-1}-e_{\mathbf{u}}^{n-2}\|_1^4\|e_T^{n-1}-e_T^{n-2}\|_1^2\Delta t\big]\\
\le& C\Delta t^3+4\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\sigma^2(t_n)\|e_T^n\|_0^2. \qquad (4.30)
\end{aligned}
$$

Applying (4.7), Lemma 2.1 and combining (4.29) with (4.30), we complete the proof. □

Theorem 4.4. *Under the assumptions of* (A1)-(A3) *and* (3.9)*, for all* $1\le m\le N$ *we have*

$$
\begin{aligned}
&\sigma^2(t_m)\|e_{\boldsymbol{u}}^m,e_T^m\|_1^2+\min\{\nu,Pr^{-1}\nu\}\Delta t\sum_{n=1}^{m}\sigma^2(t_n)(\|d_te_{\boldsymbol{u}}^n,d_te_T^n\|_0^2\\
&+\|A_h\bar{e}_{\boldsymbol{u}}^n,A_h\bar{e}_T^n\|_0^2)\le C\Delta t^2. \qquad (4.31)
\end{aligned}
$$

Proof. For all $3\le m\le N$, using Theorems 2.2, 4.1, 4.2, 4.3 and multiplying by $\sigma(t_n)$ in (4.5) and (4.9) with $\alpha=1$, summing from $n=3$ to m, we have

$$
\begin{aligned}
&\sigma(t_m)\Big(\|e_{\mathbf{u}}^m\|_1^2+\frac{\nu}{2}\|A_he_{\mathbf{u}}^m\|_0^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_1^2\Delta t+\nu\|A_h\bar{e}_{\mathbf{u}}^n\|_0^2\Big)\\
\le&\Delta t\sum_{n=2}^{m-1}\Big(\Big\|\bar{e}_{\mathbf{u}}^n+\frac{1}{2}d_te_{\mathbf{u}}^n\Delta t\Big\|_1^2+\frac{\nu}{2}\|A_he_{\mathbf{u}}^n\|_0^2\Delta t\Big)+C\Delta t+2\Delta t\sum_{n=2}^{m-1}\widehat{d}_n\sigma(t_n)\Big(\|e_{\mathbf{u}}^n\|_1^2+\frac{\nu}{2}\|A_he_{\mathbf{u}}^n\|_0^2\Delta t\Big)\\
\le& C\Delta t+2\Delta t\sum_{n=2}^{m-1}\widehat{d}_n\sigma(t_n)\Big(\|e_{\mathbf{u}}^n\|_1^2+\frac{\nu}{2}\|A_he_{\mathbf{u}}^n\|_0^2\Delta t\Big),\\
&\sigma(t_m)\Big(\|e_T^m\|_1^2+\frac{Pr^{-1}\nu}{2}\|A_he_T^m\|_0^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_T^n\|_1^2\Delta t+Pr^{-1}\nu\|A_h\bar{e}_T^n\|_0^2\Big)\\
\le&\Delta t\sum_{n=2}^{m-1}\Big(\Big\|\bar{e}_T^n+\frac{1}{2}d_te_T^n\Delta t\Big\|_1^2+\frac{Pr^{-1}\nu}{2}\|A_he_T^n\|_0^2\Delta t\Big)+C\Delta t\\
&+2\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\sigma(t_n)\Big(\|e_T^n\|_1^2+\frac{Pr^{-1}\nu}{2}\|A_he_T^n\|_0^2\Delta t\Big)\\
\le& C\Delta t+2\Delta t\sum_{n=2}^{m-1}\widetilde{d}_n\sigma(t_n)\Big(\|e_T^n\|_1^2+\frac{Pr^{-1}\nu}{2}\|A_he_T^n\|_0^2\Delta t\Big).
\end{aligned}
$$

For all $1\le m\le N$, thanks to (4.7), (4.11) and Lemma 2.1, we deduce that

$$
\sigma(t_m)\Big(\|e_{\mathbf{u}}^m\|_1^2+\frac{\nu}{2}\|A_he_{\mathbf{u}}^m\|_0^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_{\mathbf{u}}^n\|_1^2\Delta t+\nu\|A_h\bar{e}_{\mathbf{u}}^n\|_0^2\Big)\le C\Delta t, \qquad (4.32a)
$$

$$
\begin{aligned}
&\sigma(t_m)\Big(\|e_T^m\|_1^2+\frac{Pr^{-1}\nu}{2}\|A_he_T^m\|_0^2\Delta t\Big)+\Delta t\sum_{n=3}^{m}\sigma(t_n)\Big(\frac{1}{2}\|d_te_T^n\|_1^2\Delta t\\
&+Pr^{-1}\nu\|A_h\bar{e}_T^n\|_0^2\Big)\le C\Delta t. \qquad (4.32b)
\end{aligned}
$$

Choosing $\mathbf{v}_h=2d_te_{\mathbf{u}}^n\Delta t\in V_h$, $q_h=0$, $\psi_h=0$ with $\alpha=1$ in (4.2), for all $3\le n\le N$, by the similar proof as Theorem 4.2, we have

$$\begin{aligned}
&\nu\|e_{\mathbf{u}}^n\|_1^2-\nu\|e_{\mathbf{u}}^{n-1}\|_1^2+\frac{5}{8}\|d_te_{\mathbf{u}}^n\|_0^2\Delta t-\frac{1}{8}\|d_te_{\mathbf{u}}^{n-1}\|_0^2\Delta t\\
\le&\frac{\nu}{2}b_{n-1}\|e_{\mathbf{u}}^{n-1}\|_1^2\Delta t+\frac{\nu}{2}c_{n-1}\|e_{\mathbf{u}}^{n-2}\|_1^2\Delta t+16\|P_he_n\|_0^2\Delta t.
\end{aligned}\tag{4.33}$$

In the same way, taking $\psi_h=2d_te_T^n\Delta t\in W_h$, $\mathbf{v}_h=0$, $q_h=0$ with $\alpha=1$ in (4.2), for all $3\le n\le N$, it holds

$$\begin{aligned}
&Pr^{-1}\nu\|e_T^n\|_1^2-Pr^{-1}\nu\|e_T^{n-1}\|_1^2+\frac{5}{8}\|d_te_T^n\|_0^2\Delta t-\frac{1}{8}\|d_te_T^{n-1}\|_0^2\Delta t\\
\le&\frac{Pr^{-1}\nu}{2}b_{n-1}\|e_T^{n-1}\|_1^2\Delta t+\frac{Pr^{-1}\nu}{2}c_{n-1}\|e_T^{n-2}\|_1^2\Delta t+16\|P_he_n'\|_0^2\Delta t.
\end{aligned}\tag{4.34}$$

Multiplying by $\sigma^2(t_n)$ in (4.33), using (4.26), (4.32a), Theorems 2.2, 4.1, 4.2, 4.3 and summing from $n=3$ to m, we have

$$\begin{aligned}
&\nu\sigma^2(t_m)\|e_{\mathbf{u}}^m\|_1^2+\Delta t\sum_{n=3}^{m}\sigma^2(t_n)\|d_te_{\mathbf{u}}^n\|_0^2\\
\le&4\Delta t\sum_{n=2}^{m-1}\widehat{d}_n\nu\sigma^2(t_n)\|e_{\mathbf{u}}^n\|_1^2+c\Delta t\sum_{n=2}^{m-1}\nu\sigma(t_n)\Big\|\bar{e}_{\mathbf{u}}^n+\frac{1}{2}d_te_{\mathbf{u}}^n\Delta t\Big\|_1^2\\
&+\nu\sigma^2(t_2)\|e_{\mathbf{u}}^2\|_1^2+\frac{1}{8}\sigma^2(t_2)\|d_te_{\mathbf{u}}^2\|_0^2\Delta t+c\Delta t\sum_{n=3}^{m}\sigma^2(t_n)\|P_he_n\|_0^2\\
\le&C\Delta t^2+4\Delta t\sum_{n=2}^{m-1}\widehat{d}_n\nu\sigma^2(t_n)\|e_{\mathbf{u}}^n\|_1^2.
\end{aligned}\tag{4.35}$$

Similarly, multiplying by $\sigma^2(t_n)$ in (4.34) and summing from $n=3$ to m, by (4.28), (4.32b), Theorems 2.2, 4.1, 4.2, 4.3, we have

$$\begin{aligned}
&\nu\sigma^2(t_m)\|e_T^m\|_1^2+\Delta t\sum_{n=3}^{m}\sigma^2(t_n)\|d_te_T^n\|_0^2\\
\le&c\Delta t\sum_{n=2}^{m-1}\nu\sigma(t_n)\Big\|\bar{e}_T^n+\frac{1}{2}d_te_T^n\Delta t\Big\|_1^2+Pr^{-1}\nu\sigma^2(t_2)\|e_T^2\|_1^2\\
&+\frac{1}{8}\sigma^2(t_2)\|d_te_T^2\|_0^2\Delta t+4\Delta t\sum_{n=2}^{m-1}\widetilde{d}_nPr^{-1}\nu\sigma^2(t_n)\|e_T^n\|_1^2+c\Delta t\sum_{n=3}^{m}\sigma^2(t_n)\|P_he_n'\|_0^2\\
\le&C\Delta t^2+4\Delta t\sum_{n=2}^{m-1}\widetilde{d}_nPr^{-1}\nu\sigma^2(t_n)\|e_T^n\|_1^2.
\end{aligned}\tag{4.36}$$

Applying Lemma 2.1 to (4.35) and (4.36), we deduce (4.31).

Now, we provide the error estimate of p_h^n with nonsmooth initial data. By (A3), (2.4), (4.2) and Lemma 2.2, we have

$$\|e_p^n\|_0 \le c\|d_t e_{\mathbf{u}}^n\|_0 + \nu\|\bar{e}_{\mathbf{u}}^n\|_1 + c\|e_{\mathbf{u}}^{n-1}\|_1(\|\mathbf{u}_h(t_{n-1})\|_1 + \|\mathbf{u}_h^{n-1}\|_1) + c\|e_n\|_0.$$

Thanks to Theorems 2.2 and 3.2, one finds

$$\Delta t\sigma^2(t_n)\|e_p^n\|_0^2 \le C\Delta t\sigma^2(t_n)(\|d_t e_{\mathbf{u}}^n\|_0^2 + \|\bar{e}_{\mathbf{u}}^n\|_1^2 + \|e_{\mathbf{u}}^{n-1}\|_1^2 + \|e_n\|_0^2).$$

Summing above inequality from 3 to m, by Theorems 4.1, 4.2, and 4.3, we get

$$\Delta t\sum_{n=1}^{m}\sigma^2(t_n)\|e_p^n\|_0^2 \le C\Delta t^2. \tag{4.37}$$

Thanks to Theorem 2.2 and the integral by parts, for all $1\le n\le N$, we have

$$\begin{aligned}
&\sigma^2(t_n)\|p_h(t_n)-p_h^n\|_0^2\Delta t\\
\le&2\sigma^2(t_n)\|e_p^n\|_0^2\Delta t + 2\sigma^2(t_n)\Big\|p_h(t_n)-\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}p_h(s)ds\Big\|_0^2\Delta t\\
\le&2\sigma^2(t_n)\|e_p^n\|_0^2\Delta t + 8\Delta t^2\int_{t_{n-1}}^{t_n}\sigma^2(s)\|p_{hs}(s)\|_0 ds.
\end{aligned}$$

Finally, summing above inequality from 1 to m, for $1\le m\le N$, we deduce that

$$\Delta t\sum_{n=1}^{m}\sigma^2(t_n)\|p_h(t_n)-p_h^n\|_0^2 \le C\Delta t^2.$$

Thus, we complete the proof. □

Combining Theorems 2.3, 4.3 and 4.4, we obtain the following main theoretical results.

Theorem 4.5. *Under the conditions of* (A1)-(A3) *and* (3.9), *for all* $t_m\in(0,T_{time}^{final}]$ *it holds*

$$\begin{aligned}
&\|\boldsymbol{u}(t_m)-\boldsymbol{u}_h^m, T(t_m)-T_h^m\|_0 \le C(\sigma^{-1}(t_m)\Delta t^{\frac{3}{2}} + \sigma^{-\frac{1}{2}}(t_m)h^2), && 1\le m\le N,\\
&\|\boldsymbol{u}(t_m)-\boldsymbol{u}_h^m, T(t_m)-T_h^m\|_1 \le C(\sigma^{-1}(t_m)\Delta t + \sigma^{-\frac{1}{2}}(t_m)h), && 1\le m\le N,\\
&\Big(\Delta t\sum_{n=1}^{m}\sigma^2(t_n)\|p(t_m)-p_h^m\|_0^2\Big)^{1/2} \le C(\Delta t + h), && 1\le m\le N.
\end{aligned}$$

5 Numerical experiments

In this section, we present some numerical results to verify the performances of the developed numerical schemes. In all experiments, the time-dependent natural convection problem is defined on a convex domain $\Omega=[0,1]^2$. The mesh consists of triangular elements that obtained by dividing Ω into subsquares of equal size and drawing the diagonal in each sub-square. The finite element spaces are adopted the MINI element for velocity and pressure and linear polynomial for temperature. The UMFPACK routine is used to solve the linear systems arising from the discrete algebraic systems. We set the final time $T_{time}^{final}=1$, the parameters ν, k, Pr are 1 and $h=\Delta t$. For comparison, the numerical results of the following Crank-Nicolson scheme are provided.

Example 5.1 (The Crank-Nicolson scheme for the natural convection problem)**.** Set the initial solutions $\mathbf{u}_h^0=\mathbf{u}_0, T_h^0=T_0$, for all $(\mathbf{v}_h,q_h,\psi_h)\in X_h\times M_h\times W_h$. Define the numerical solution $(\mathbf{u}_h^n,p_h^n,T_h^n)\in X_h\times M_h\times W_h$, $n=1,\cdots,N$ by

$$\left\{\begin{aligned} &(d_t\mathbf{u}_h^n,\mathbf{v}_h)+a(\overline{\mathbf{u}}_h^n,\mathbf{v}_h)-d(\mathbf{v}_h,p_h^n)+d(\mathbf{u}_h^n,q_h)+b(\overline{\mathbf{u}}_h^n,\overline{\mathbf{u}}_h^n,\mathbf{v}_h)\\ &=(\bar{\mathbf{f}}(t_n),\mathbf{v}_h)-\kappa\nu^2(j\overline{T}_h^n,\mathbf{v}_h),\\ &(d_tT_h^n,\psi_h)+\tilde{a}(\overline{T}_h^n,\psi_h)+\tilde{b}(\overline{\mathbf{u}}_h^n,\overline{T}_h^n,\psi_h)=(\overline{g}(t_n),\psi_h). \end{aligned}\right. \tag{5.1}$$

It is well known that (5.1) is a classical second order nonlinear numerical scheme, we use the Newton iteration to treat the nonlinear terms. The iteration stop condition

$$\sqrt{\|\mathbf{u}_h^n-\widetilde{\mathbf{u}}_h^n\|_0+\|T_h^n-\widetilde{T}_h^n\|_0}\leq 10^{-3}$$

is used, $\widetilde{\mathbf{u}}_h^n$ and $\widetilde{T}_h^n$ are the nth-level iterative solutions.

Firstly, we verify the established theoretical results of Theorem 4.5 with H^1-initial data. The boundary and initial conditions and body forces $\mathbf{f}$, g are given by the following exact solutions

$$\begin{aligned} u_1(x,y,t)&=\frac{5}{2}\pi\sin^{\frac{5}{2}}(\pi x)\sin^{\frac{3}{2}}(\pi y)\cos(\pi y)\cos(t),\\ u_2(x,y,t)&=-\frac{5}{2}\pi\sin^{\frac{3}{2}}(\pi x)\cos(\pi x)\sin^{\frac{5}{2}}(\pi y)\cos(t),\\ p(x,y,t)&=10\cos(\pi x)\cos(\pi y)\cos(t),\\ T(x,y,t)&=\frac{5}{2}\pi\sin^{\frac{5}{2}}(\pi x)\sin^{\frac{3}{2}}(\pi y)\cos(\pi y)\cos(t)\\ &\quad-\frac{5}{2}\pi\sin^{\frac{3}{2}}(\pi x)\cos(\pi x)\sin^{\frac{5}{2}}(\pi y)\cos(t), \end{aligned}$$

where the velocity $\mathbf{u}=(u_1,u_2)$ and the initial data $(\mathbf{u}_0(x,y),T_0(x,y))\in H^1$ (see [18]).

In Table 1, we present the relative errors between the exact solutions and numerical solutions of scheme (3.1)-(3.2). From these data, we can see that the relative errors of velocity and temperature in H^1-norm become smaller and smaller as the mesh refines, and the convergence orders are $\mathcal{O}(h)$. The error orders of pressure are nearly $\mathcal{O}(h^{1.6})$, which show some superconvergences, the reason may lie in the smoothness of pressure. Moreover, the error order for $\mathbf{u}$ and T in L^2-norm is of the order of $\mathcal{O}(h^{1.6})$ as the mesh

Table 1: The numerical results of the Crank-Nicolson/Explicit scheme with H^1-initial data.

Δt	$\frac{\lVert\mathbf{u}-\mathbf{u}_h^n\rVert_0}{\lVert\mathbf{u}\rVert_0}$	Rate	$\frac{\lVert\nabla(\mathbf{u}-\mathbf{u}_h^n)\rVert_0}{\lVert\nabla\mathbf{u}\rVert_0}$	Rate	$\frac{\lVert p-p_h^n\rVert_0}{\lVert p\rVert_0}$	Rate
0.1	0.067019	-	0.362041	-	0.277928	-
0.05	0.0184294	1.8626	0.187497	0.9493	0.0900259	1.6263
0.025	0.00565494	1.7044	0.0939339	0.9971	0.0302961	1.5712
0.0125	0.00186265	1.6022	0.0468949	1.0022	0.0104063	1.5417
0.00625	0.000674092	1.4663	0.0234641	0.9990	0.00373707	1.4775
Δt	$\frac{\lVert\nabla(T-T_h^n)\rVert_0}{\lVert\nabla T\rVert_0}$	Rate	$\frac{\lVert T-T_h^n\rVert_0}{\lVert T\rVert_0}$	Rate	CPU(S)	
0.1	0.0326752	-	0.181239	-	1.45	
0.05	0.00945539	1.7890	0.0906668	0.9992	7.03	
0.025	0.00294663	1.6821	0.0453439	0.9997	58.87	
0.0125	0.001018157	1.5331	0.0226746	0.9998	472.86	
0.00625	0.000371619	1.4541	0.0113403	0.9996	4584.74	

Table 2: The numerical results of the Crank-Nicolson scheme with H^1-initial data.

Δt	$\frac{\lVert\mathbf{u}-\mathbf{u}_h^n\rVert_0}{\lVert\mathbf{u}\rVert_0}$	Rate	$\frac{\lVert\nabla(\mathbf{u}-\mathbf{u}_h^n)\rVert_0}{\lVert\nabla\mathbf{u}\rVert_0}$	Rate	$\frac{\lVert p-p_h^n\rVert_0}{\lVert p\rVert_0}$	Rate
0.1	0.0675168	-	0.323773	-	0.280633	-
0.05	0.0187475	1.8485	0.161803	1.0007	0.0891139	1.6550
0.025	0.00543686	1.7859	0.0837612	0.9449	0.0300184	1.5698
0.0125	0.00176032	1.6269	0.0429238	0.9645	0.0104925	1.5165
0.00625	0.000633639	1.4741	0.0228389	0.9103	0.00377609	1.4744
Δt	$\frac{\lVert\nabla(T-T_h^n)\rVert_0}{\lVert\nabla T\rVert_0}$	Rate	$\frac{\lVert T-T_h^n\rVert_0}{\lVert T\rVert_0}$	Rate	CPU(S)	
0.1	0.0321728	-	0.181049	-	1.83	
0.05	0.00936417	1.7806	0.0906609	0.9978	9.89	
0.025	0.00284727	1.7176	0.0453435	0.9996	169.71	
0.0125	0.000981062	1.5372	0.0226746	0.9998	1191.89	
0.00625	0.000341612	1.5220	0.0113402	0.9996	16858.5	

sizes reduce with H^1-initial data, which confirms the Theorem 4.5 well. In contrast, the relative errors obtained from the Crank-Nicolson (CN) scheme (5.1) are presented in Table 2. Compared with Table 1, we can see that the accuracy of Crank-Nicolson/Explicit scheme is comparable with that of the CN scheme (5.1) with the same time steps, but the Crank-Nicolson/Explicit scheme can save 60% CPU time than the Crank-Nicolson scheme.

Next, we consider the influences of the initial data for the numerical results, and choose the following H^2-smooth initial data.

$$
\begin{aligned}
&u_1(x,y,t)=2\pi\sin^2(\pi x)\sin(\pi y)\cos(\pi y)\cos(t),\\
&u_2(x,y,t)=-2\pi\sin(\pi x)\cos(\pi x)\sin^2(\pi y)\cos(t),\\
&p(x,y,t)=10\cos(\pi x)\cos(\pi y)\cos(t),
\end{aligned}
$$

$$
\begin{aligned}
T(x,y,t)=&2\pi\sin^2(\pi x)\sin(\pi y)\cos(\pi y)\cos(t)\\
&-2\pi\sin(\pi x)\cos(\pi x)\sin^2(\pi y)\cos(t).
\end{aligned}
$$

We present the relative errors of the Crank-Nicolson/Explicit scheme and the Crank-Nicolson scheme for the natural convection problem with H^2-initial data in Tables 3-4. From these data, we know that the time convergence orders of velocity and temperature in L^2-norm of both numerical schemes are 2, which confirm the well-known theoretical analysis [19] well.

Finally, combining the data provided in Tables 1-2, we can see that the relative errors become smaller and smaller as the time step decreases. The higher smoothness of the initial data, the more accurate of numerical solution is obtained. The error decreases more slowly as the time steps less than 0.025 with nonsmooth initial data, while there are no influences for the accuracy of numerical solutions with H^2-initial data. Another interesting

Table 3: The numerical results of the Crank-Nicolson/Explicit scheme with H^2-initial data.

Δt	$\frac{\|\mathbf{u}-\mathbf{u}_h^n\|_0}{\|\mathbf{u}\|_0}$	Rate	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^n)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h^n\|_0}{\|p\|_0}$	Rate
0.1	0.0690326	-	0.361623	-	0.280436	-
0.05	0.0172267	2.0026	0.181268	0.9964	0.0890037	1.6557
0.025	0.00420379	2.0349	0.0915812	0.9850	0.0299049	1.5735
0.0125	0.00107255	1.9706	0.0468918	0.9657	0.0102821	1.5402
0.00625	0.000267906	2.0012	0.0234581	0.9992	0.00360688	1.5113
Δt	$\frac{\|\nabla(T-T_h^n)\|_0}{\|\nabla T\|_0}$	Rate	$\frac{\|T-T_h^n\|_0}{\|T\|_0}$	Rate	CPU(S)	
0.1	0.0374768	-	0.181012	-	1.78	
0.05	0.00938669	1.9973	0.0906547	0.9976	8.72	
0.025	0.00224005	2.0671	0.0453404	0.9996	79.11	
0.0125	0.000582718	1.9437	0.0226721	0.9999	624.07	
0.00625	0.000145659	2.0002	0.0113362	1.0001	5226.82	

Table 4: The numerical results of the Crank-Nicolson scheme with H^2-initial data.

Δt	$\frac{\|\mathbf{u}-\mathbf{u}_h^n\|_0}{\|\mathbf{u}\|_0}$	Rate	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h^n)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h^n\|_0}{\|p\|_0}$	Rate
0.1	0.0706045	-	0.323331	-	0.280751	-
0.05	0.0175661	2.0070	0.161824	0.9986	0.0889595	1.6581
0.025	0.00441545	1.9922	0.0821415	0.9782	0.0298910	1.5734
0.0125	0.00110449	1.9992	0.0424163	0.9535	0.0103573	1.5291
0.00625	0.000266768	2.0497	0.0218331	0.9581	0.00362797	1.5134
Δt	$\frac{\|\nabla(T-T_h^n)\|_0}{\|\nabla T\|_0}$	Rate	$\frac{\|T-T_h^n\|_0}{\|T\|_0}$	Rate	CPU(S)	
0.1	0.0368658	-	0.181034	-	13.627	
0.05	0.00930105	1.9868	0.0906586	0.9977	222.52	
0.025	0.00232981	1.9972	0.0453414	0.9996	1101.86	
0.0125	0.000582625	1.9996	0.0226721	0.9999	6460.91	
0.00625	0.000145651	2.0001	0.0113362	1.0001	21095.01	

phenomenon is the computational cost of solving the discrete algebraic equations. The used CPU time with H^1-initial data less than that with H^2-initial data in both the Crank-Nicolson/Explicit scheme and the Crank-Nicolson scheme. Therefore, we conclude that the smoothness of the initial data has important influences on both the theoretical results and the computational cost.

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