Analysis in Theory and Applications
Anal. Theory Appl., Vol. **36**, No. 1 (2020), pp. 99-110
DOI: 10.4208/ata.OA-2017-0055

Boundedness of High Order Commutators of Riesz Transforms Associated with Schrödinger Type Operators

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Received 27 August 2017; Accepted (in revised version) 8 June 2018

Abstract. Let $\mathcal{L}_2 = (-\Delta)^2 + V^2$ be the Schrödinger type operator, where $V \neq 0$ is a nonnegative potential and belongs to the reverse Hölder class RH_{q_1} for $q_1 > n/2, n \geq 5$. The higher Riesz transform associated with $\mathcal{L}_2$ is denoted by $\mathcal{R} = \nabla^2 \mathcal{L}_2^{-\frac{1}{2}}$ and its dual is denoted by $\mathcal{R}^* = \mathcal{L}_2^{-\frac{1}{2}} \nabla^2$. In this paper, we consider the m-order commutators $[b^m, \mathcal{R}]$ and $[b^m, \mathcal{R}^*]$, and establish the (L^p, L^q)-boundedness of these commutators when b belongs to the new Campanato space $\Lambda_\beta^\theta(\rho)$ and $1/q = 1/p - m\beta/n$.

Key Words: Schrödinger operator, Campanato space, Riesz transform, commutator.

AMS Subject Classifications: 42B25, 35J10, 42B35

1 Introduction

In this paper, we consider the Schrödinger type operator

$$\mathcal{L}_2 = (-\Delta)^2 + V^2 \quad \text{on } \mathbb{R}^n, \quad n \geq 5,$$

where V is nonnegative, $V \neq 0$, and belongs to the reverse Hölder class RH_q for some $q \geq n/2$, i.e., there exists a constant C such that

$$\left(\frac{1}{|B|} \int_B V(y)^q dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) dy$$

for every ball $B \subset \mathbb{R}^n$.

The higher Riesz transform associated with $\mathcal{L}_2$ is defined by $\mathcal{R} = \nabla^2 \mathcal{L}_2^{-1/2}$, and its dual is defined by $\mathcal{R}^* = \mathcal{L}_2^{-1/2} \nabla^2$. The L^p-boundedness of the higher Riesz transforms

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have been obtained in [1] by Liu and Dong: Suppose $V \in RH_{q_1}$ with $n/2 < q_1 < n$. Let $1/p_1 = 2/q_1 - 2/n$, $p_1' = p_1/(p_1-1)$. If $1 < p < p_1$, then for all $f \in L^p(\mathbb{R}^n)$,

$$\|\mathcal{R}f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

If $p_1' < p < \infty$, then for all $f \in L^p(\mathbb{R}^n)$,

$$\|\mathcal{R}^*f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

As in [2], for a given potential $V \in RH_q$ with $q > n/2$, we define the auxiliary function

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leq 1\right\}, \quad x \in \mathbb{R}^n.$$

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

Let $\theta > 0$ and $0 < \beta < 1$, in view of [3], the new Campanato class $\Lambda_\beta^\theta(\rho)$ consists of the locally integrable functions b such that

$$\frac{1}{|B(x,r)|^{1+\beta/n}}\int_{B(x,r)} |b(y) - b_B|dy \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda_\beta^\theta(\rho)$, denoted by $[b]_\beta^\theta$, is given by the infimum of the constants in the inequalities above.

Note that if $\theta = 0$, $\Lambda_\beta^\theta(\rho)$ is the classical Campanato space; If $\beta = 0$, $\Lambda_\beta^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$ introduced in [4].

We denote by $\mathcal{K}$ and $\mathcal{K}^*$ the kernels of $\mathcal{R}$ and $\mathcal{R}^*$, respectively. Let b be a locally integrable function, m be a positive integer. The m-order commutators generated by higher Riesz transform and b are defined by

$$[b^m, \mathcal{R}]f(x) = \int_{R^n} \mathcal{K}(x,y)(b(x) - b(y))^m f(y)dy$$

and

$$[b^m, \mathcal{R}^*]f(x) = \int_{R^n} \mathcal{K}^*(x,y)(b(x) - b(y))^m f(y)dy.$$

In this paper, we are interested in the boundedness of $[b^m, \mathcal{R}]$ and $[b^m, \mathcal{R}^*]$ on Lebesgue space when b belongs to the new Campanato class $\Lambda_\beta^\theta(\rho)$. The main result of this paper is as follows.

Theorem 1.1. *Suppose $V \in RH_{q_1}$ with $n/2 < q_1 < n, 1/p_1 = 2/q_1 - 2/n, p_1' = p_1/(p_1-1)$. Let $0 < \beta < 1$, and let $b \in \Lambda_\beta^\theta(\rho)$. If $p_1' < p < \infty$, then for all $f \in L^p(\mathbb{R}^n)$,*

$$\|[b^m, \mathcal{R}^*]f\|_{L^q(\mathbb{R}^n)} \leq C([b]_\beta^\theta)^m\|f\|_{L^p(\mathbb{R}^n)},$$

where $1/q = 1/p - m\beta/n$.

We immediately deduce the following result by duality.

Corollary 1.1. *Suppose $V \in RH_{q_1}$ with $n/2 < q_1 < n$, $\frac{1}{p_1} = \frac{2}{q_1} - \frac{2}{n}$. Let $0 < \beta < 1$, and let $b \in \Lambda_\beta^\theta(\rho)$. If $1 < p < p_1$, then for all $f \in L^p(\mathbb{R}^n)$,*

$$\|[b^m, \mathcal{R}]f\|_{L^q(\mathbb{R}^n)} \leq C([b]_\beta^\theta)^m \|f\|_{L^p(\mathbb{R}^n)},$$

where $1/q = 1/p - m\beta/n$.

We shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some preliminaries

We recall some important properties concerning the auxiliary function.

Proposition 2.1 ([2]). *Let $V \in RH_{n/2}$. For the function ρ there exist C and $k_0 \geq 1$ such that*

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}$$

for all $x, y \in \mathbb{R}^n$.

Assume that $Q = B(x_0, \rho(x_0))$, for $x \in Q$, Proposition 2.1 tell us that $\rho(x) \approx \rho(y)$, if $|x-y| < C\rho(x)$.

Lemma 2.1 ([5]). *Let $k \in \mathbb{N}$ and $x \in 2^{k+1}B(x_0, r) \setminus 2^k B(x_0, r)$. Then we have*

$$\frac{1}{\left(1+\frac{2^k r}{\rho(x)}\right)^N} \lesssim \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}.$$

Lemma 2.2 ([6]). *Suppose $V \in RH_{q_1}, q_1 \geq n/2$. Then there exists a constants $l_0 > 0$, such that*

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \lesssim \left(1+\frac{r}{\rho(x)}\right)^{l_0}.$$

The following finite overlapping property given by Dziubański and Zienkiewicz in [7].

Proposition 2.2. *There exists a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, so that the family of critical balls $Q_k = B(x_k, \rho(x_k))$, $k \geq 1$, satisfies*

(i) $\bigcup_k Q_k = \mathbb{R}^n$.

(ii) There exists $N = N(\rho)$ such that for every $k \in N$, $card\{j : 4Q_j \cap 4Q_k\} \leq N$.

For $\alpha > 0, g \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we introduce the following maximal functions

$$M_{\rho,\alpha}g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{|B|}\int_B |g(y)|dy,$$
$$M^\sharp_{\rho,\alpha}g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{|B|}\int_B |g(y)-g_B|dy,$$

where $\mathcal{B}_{\rho,\alpha} = \{B(z,r) : z \in \mathbb{R}^n \text{ and } r \leq \alpha\rho(y)\}$.

We have the following Fefferman-Stein type inequality.

Proposition 2.3 ([4]). *For $1 < p < \infty$, there exist δ and γ such that if $\{Q_k\}_{k=1}^\infty$ is a sequence of balls as in Proposition 2.2, then*

$$\int_{\mathbb{R}^n} |M_{\rho,\delta}g(x)|^p dx \lesssim \int_{\mathbb{R}^n} |M^\sharp_{\rho,\gamma}g(x)|^p dx + \sum_k |Q_k| \left(\frac{1}{|Q_k|}\int_{2Q_k} |g|\right)^p$$

for all $g \in L^1_{loc}(\mathbb{R}^n)$.

We give an inequality for the function $b \in \Lambda^\theta_\beta(\rho)$.

Lemma 2.3 ([3]). *Let $1 \leq s < \infty$, $b \in \Lambda^\theta_\beta(\rho)$, $0 < \beta < 1$, $k \in \mathbb{N}$, and $B = B(x,r)$. Then we have*

$$\left(\frac{1}{|2^kB|}\int_{2^kB} |b(y)-b_B|^s dy\right)^{1/s} \lesssim [b]^\theta_\beta (2^k r)^\beta \left(1+\frac{2^k r}{\rho(x)}\right)^{\theta'},$$

where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in Proposition 2.1.

Let K and K^* be the kernels of the Riesz transform $\mathcal{R}$ and its dual operator $\mathcal{R}^*$, respectively. Then $K(x,z) = K^*(z,x)$ and we have the following estimates.

Lemma 2.4 ([8]). *Suppose $V \in RH_{q_1}$ with $n/2 < q_1 < n$.*

(i) For every N, there exists a constant $C_N > 0$ such that

$$|\mathcal{K}^*(x,z)| \leq \frac{C_N\left(1+\frac{|x-z|}{\rho(x)}\right)^{-N}}{|x-z|^{n-2}} \left(\int_{B(z,|x-z|/4)} \frac{V^2(u)}{|u-z|^{n-2}}du + \frac{1}{|x-z|^2}\right).$$

(ii) For every N and $0 < \delta < \min\{1, 2-n/q_1\}$, there exists a constant $C_N > 0$ such that

$$\begin{aligned}&|\mathcal{K}^*(x,z)-\mathcal{K}^*(y,z)|\\ \leq&\frac{C_N|x-y|^\delta\left(1+\frac{|x-z|}{\rho(x)}\right)^{-N}}{|x-z|^{n-2+\delta}} \left(\int_{B(z,|x-z|)} \frac{V^2(u)}{|u-z|^{n-2}}du + \frac{1}{|x-z|^2}\right),\end{aligned}$$

where $|x-y| < |x-z|/16$.

3 Proof of Theorem 1.1

To prove Theorem 1.1, we first prove the following lemmas.

Lemma 3.1. *Suppose $V \in RH_{q_1}$ with $n/2 < q_1 < n$. Let $1/p_1 = 2/q_1 - 2/n$, and $b \in \Lambda_\beta^\theta(\rho)$. If $p_1' < s < \infty$, then for all $f \in L^s_{loc}(\mathbb{R}^n)$ and every critical ball $Q = B(x_0, \rho(x_0))$,*

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |[b^m,\mathcal{R}^*]f(y)|dy\\ \lesssim&([b]_\beta^\theta)^m \inf_{x\in Q} M_{m\beta,s}(f)(x) + \sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{m-\gamma}\inf_{x\in Q} M_{(m-\gamma)\beta,s}([b^\gamma,\mathcal{R}^*]f)(x),\end{aligned}$$

where

$$M_{m\beta,s}(f)(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-m\beta s/n}}\int_B |f(y)|^s dy\right)^{1/s}.$$

Proof. By binomial theorem, we have

$$\begin{aligned}(b(y)-b(z))^m &= \sum_{l=1}^m C_{l,m}(b(y)-\lambda)^l(\lambda-b(z))^{m-l} + (\lambda-b(z))^m\\ &= \sum_{l=1}^m C_{l,m}(b(y)-\lambda)^l(\lambda-b(y)+b(y)-b(z))^{m-l} + (\lambda-b(z))^m\\ &= \sum_{l=1}^m\sum_{h=0}^{m-l} C_{l,m,h}(b(y)-\lambda)^{l+h}(b(y)-b(z))^{m-l-h} + (\lambda-b(z))^m\\ &= \sum_{\gamma=0}^{m-1} C_{\gamma,m}(b(y)-\lambda)^{m-\gamma}(b(y)-b(z))^\gamma + (\lambda-b(z))^m,\end{aligned}$$

then

$$\begin{aligned}|[b^m,\mathcal{R}^*]f(y)| &= \int_{R^n}|\mathcal{K}^*(y,z)(b(y)-b(z))^m f(z)|dz\\ &\lesssim \sum_{\gamma=0}^{m-1}|b(y)-\lambda|^{m-\gamma}|[b^\gamma,\mathcal{R}^*](f)(y)| + |\mathcal{R}^*((b-\lambda)^m f)(y)|.\end{aligned}$$

Let $\lambda = b_Q$. Then by Hölder's inequality and Lemma 2.3 we get

$$\begin{aligned}&\frac{1}{|Q|}\int_Q\left|\sum_{\gamma=0}^{m-1}|b(y)-\lambda|^{m-\gamma}|[b^\gamma,\mathcal{R}^*](f)(y)|dy\right.\\ \lesssim&\sum_{\gamma=0}^{m-1}\frac{1}{|Q|}\int_Q|b(y)-\lambda|^{m-\gamma}|[b^\gamma,\mathcal{R}^*](f)(y)|dy\\ \lesssim&\sum_{\gamma=0}^{m-1}\left(\frac{1}{|Q|}\int_Q|b(y)-b_Q|^{(m-\gamma)s'}dy\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q|[b^\gamma,\mu_j^L]f(y)|^s dy\right)^{1/s}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{m-\gamma}(\rho(x_0))^{\beta(m-\gamma)}\left(\frac{1}{|Q|}\int_Q |[b^\gamma,\mathcal{R}^*]f(y)|^s dy\right)^{1/s}\\
&\lesssim \sum_{\gamma=0}^{m-1}([b]_\beta^\theta)^{m-\gamma}\inf_{x\in Q} M_{(m-\gamma)\beta,s}([b^\gamma,\mathcal{R}^*]f)(x).
\end{aligned}
$$

To the second term, we split $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$. Let $p_1' < \tilde{s} < s < \infty$, and $\nu = s\tilde{s}/(s-\tilde{s})$, by Hölder's inequality, the boundedness of $\mathcal{R}^*$ on $L^{\tilde{s}}(\mathbb{R}^n)$, and Lemma 2.3 we obtain

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q |\mathcal{R}^*((b-b_{2Q})^m f_1)(y)|dy\\
\le&\left(\frac{1}{|Q|}\int_Q |\mathcal{R}^*((b-b_Q)^m f_1)(y)|^{\tilde{s}}\right)^{1/\tilde{s}}\\
\lesssim&\left(\frac{1}{|Q|}\int_{2Q} |(b(y)-b_Q)^m f(y)|^{\tilde{s}}dy\right)^{1/\tilde{s}}\\
\lesssim&\left(\frac{1}{|Q|}\int_{2Q} |f(y)|^s dy\right)^{1/s}\left(\frac{1}{|Q|}\int_{2Q}|b(y)-b_Q|^{m\nu}dy\right)^{1/\nu}\\
\lesssim&([b]_\beta^\theta)^m \inf_{x\in Q} M_{m\beta,s}f(x).
\end{aligned}
$$

For the remaining term, note that $\rho(y)\approx\rho(x_0)$ for any $y\in Q$, by Lemma 2.4 and decomposing $(2Q)^c$ into annuli $2^kQ\setminus 2^{k-1}Q$, $k\ge 2$, we get

$$
\begin{aligned}
&|\mathcal{R}^*((b-b_Q)^m f_2)(y)|\\
\le&\int_{(2Q)^c}|\mathcal{K}^*(y,z)(b(z)-b_Q)^m f(z)|dz\\
\lesssim&\int_{(2Q)^c}\frac{\left(1+\frac{|y-z|}{\rho(y)}\right)^{-N}}{|y-z|^n}|(b(z)-b_Q)^m f(z)|dz\\
&+\int_{(2Q)^c}\frac{\left(1+\frac{|y-z|}{\rho(y)}\right)^{-N}}{|y-z|^{n-2}}\int_{B(z,|y-z|/4)}\frac{V^2(u)}{|u-z|^{n-2}}du|(b(z)-b_Q)^m f(z)|dz\\
\lesssim&\sum_{k\ge 2}\frac{2^{-kN}}{|2^kQ|}\int_{2^kQ}|(b(z)-b_Q)^m f(z)|dz\\
&+\sum_{k\ge 2}\frac{2^{-kN}}{|2^kQ|^{1-2/n}}\int_{2^kQ}|(b(z)-b_Q)^m f(z)||I_2(V^2\chi_{2^{k+2}Q})(z)|dz,
\end{aligned}
$$

where

$$
I_2(f)(y)=\int_{R^n}\frac{f(z)}{|y-z|^{n-2}}dz
$$

is the Riesz potential. By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}
&\sum_{k\geq 2}\frac{2^{-kN}}{|2^kQ|}\int_{2^kQ}|(b(z)-b_Q)^m f(z)|dz\\
\lesssim&\sum_{k=2}^{\infty}2^{-kN}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|b(z)-b_{2Q}|^{ms'}dz\right)^{1/s'}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|f(z)|^s dz\right)^{1/s}\\
\lesssim&([b]_\beta^\theta)^m\sum_{k=2}^{\infty}2^{-kN}\inf_{x\in Q}M_{m\beta,s}(f)(x)\\
\lesssim&([b]_\beta^\theta)^m\inf_{x\in Q}M_{m\beta,s}(f)(x).
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\sum_{k\geq 2}\frac{2^{-kN}}{|2^kQ|^{1-2/n}}\int_{2^kQ}|(b(z)-b_Q)^m f(z)||I_2(V^2\chi_{2^{k+3}Q})(z)dz\\
\lesssim&\sum_{k\geq 2}2^{-kN}(2^k\rho(x_0))^{2-n/\tilde{s}'}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|(b(z)-b_Q)^m f(z)|^{\tilde{s}}dz\right)^{1/\tilde{s}}\|I_2(V^2\chi_{2^{k+3}Q})\|_{L^{\tilde{s}'}(\mathbb{R}^n)}.
\end{aligned}$$

Let $v=s\tilde{s}/(s-\tilde{s})$. Then by Hölder's inequality again, we get

$$\begin{aligned}
&\left(\frac{1}{|2^kQ|}\int_{2^kQ}|(b(z)-b_Q)^m f(z)|^{\tilde{s}}dz\right)^{1/\tilde{s}}\\
\leq&\left(\frac{1}{|2^kQ|}\int_{2^kQ}|f(z)|^s dz\right)^{1/s}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|b(z)-b_Q|^{mv}dz\right)^{1/v}\\
\lesssim&([b]_\beta^\theta)^m\inf_{x\in Q}M_{m\beta,s}(f)(x).
\end{aligned}$$

Since $p_1'<\tilde{s}$, we have $\tilde{s}'<p_1$, then $1/\tilde{s}'>1/p_1=2/q_1-2/n$. We can choose $n/2<\tilde{t}<q_1$ such that $1/\tilde{s}'=2/\tilde{t}-2/n$. By $(L^{\tilde{t}/2},L^{\tilde{s}'})$-boundedness of I_2, $V\in RH_{\tilde{t}}$, and Lemma 2.2, we get

$$\begin{aligned}
&\|I_2(V^2\chi_{2^{k+3}Q})\|_{L^{\tilde{s}'}(\mathbb{R}^n)}\lesssim\|V^2\chi_{2^{k+3}Q}\|_{L^{\tilde{t}/2}(\mathbb{R}^n)}\\
\lesssim&\left(\int_{2^{k+3}Q}V^{\tilde{t}}\right)^{2/\tilde{t}}\lesssim\left(\frac{1}{|2^{k+3}Q|}\int_{2^{k+3}Q}V^{\tilde{t}}\right)^{2/\tilde{t}}(2^k\rho(x_0))^{\frac{2n}{\tilde{t}}}\\
\lesssim&\left(\frac{1}{|2^{k+3}Q|}\int_{2^{k+3}Q}V\right)^{2}(2^k\rho(x_0))^{\frac{2n}{\tilde{t}}}\\
\lesssim&\left(1+\frac{2^k\rho(x_0)}{\rho(x_0)}\right)^{2l_0}(2^k\rho(x_0))^{\frac{2n}{\tilde{t}}-4}\lesssim 2^{2kl_0}(2^k\rho(x_0))^{\frac{2n}{\tilde{t}}-4}.
\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{k\geq 2}\frac{2^{-kN}}{|2^kQ|^{1-2/n}}\int_{2^kQ}|(b(z)-b_Q)^m f(z)||I_2(V^2\chi_{2^{k+3}Q})(z)dz\\ \lesssim&([b]_\beta^\theta)^m\inf_{x\in Q}M_{m\beta,s}(f)(x).\end{aligned}$$

So we get

$$|\mathcal{R}^*((b-b_Q)^m f_2)(y)|\lesssim([b]_\beta^\theta)^m\inf_{x\in Q}M_{\beta,s}(f)(x).$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *Suppose $V\in RH_{q_1}$ with $q_1>n/2$. Let $b\in\Lambda_\beta^\theta$, let $B=B(x_0,r)$ with $r\leq\gamma\rho(x_0)$ and let $x\in B$, then for any $y,z\in B$ we have*

$$\int_{(2B)^c}|\mathcal{K}^*(y,u)-\mathcal{K}^*(z,u)||b(u)-b_B|^m|f(u)|du\lesssim([b]_\beta^\theta)^m M_{m\beta,s}(f)(x),$$

where $s>p_1'$ and $\gamma>1$.

Proof. Setting $Q=B(x_0,\gamma\rho(x_0))$, due to the fact $\rho(y)\approx\rho(z)\approx\rho(x_0)$ and $|y-u|\approx|z-u|\approx|x_0-u|$, then by Lemma 2.4 we get

$$\int_{(2B)^c}|\mathcal{K}^*(y,u)-\mathcal{K}^*(z,u)||b(u)-b_B|^m|f(u)|du\lesssim J_1+J_2+J_3+J_4,$$

where

$$\begin{aligned}J_1&=r^\delta\int_{Q\setminus 2B}\frac{|f(u)(b(u)-b_B)^m|}{|x_0-u|^{n+\delta}}du,\\ J_2&=r^\delta\rho(x_0)^N\int_{Q^c}\frac{|f(u)(b(u)-b_B)^m|}{|x_0-u|^{n+N+\delta}}du,\\ J_3&=r^\delta\int_{Q\setminus 2B}\frac{|f(u)(b(u)-b_B)^m|}{|x_0-u|^{n+\delta-2}}\int_{B(x_0,4|x_0-u|)}\frac{V^2(\omega)}{|\omega-u|^{n-2}}d\omega du,\\ J_4&=r^\delta\rho(x_0)^N\int_{Q^c}\frac{|f(u)(b(u)-b_B)^m|}{|x_0-u|^{n+\delta+N-2}}\int_{B(x_0,4|x_0-u|)}\frac{V^2(\omega)}{|\omega-u|^{n-2}}d\omega du.\end{aligned}$$

Let j_0 be the least integer such that $2^{j_0}\geq\gamma\rho(x_0)/r$. Splitting into annuli, we have

$$J_1\leq\sum_{j=2}^{j_0}2^{-\delta j}\frac{1}{|2^jB|}\int_{2^jB}|f(u)||b(u)-b_B|^m du.$$

By Hölder's inequality and Lemma 2.3 we obtain for $j \le j_0$

$$\begin{aligned}
&\frac{1}{|2^jB|}\int_{2^jB}|f(u)||b(u)-b_B|^m du\\
\le&\left(\frac{1}{|2^jB|}\int_{2^jB}|f(u)|^s du\right)^{1/s}\left(\frac{1}{|2^jB|}\int_{2^jB}|b(u)-b_B|^{ms'}du\right)^{1/s'}\\
\lesssim&([b]_\beta^\theta)^m M_{m\beta,s}(f)(x).
\end{aligned}$$

Then

$$J_1 \lesssim ([b]_\beta^\theta)^m M_{m\beta,s}(f)(x).$$

To deal with J_2, splitting into annuli and using Lemma 2.3 we have

$$\begin{aligned}
J_2 \lesssim&\rho(x_0)^N\sum_{j\ge j_0}2^{-j\delta}\frac{1}{(2^jr)^N}\frac{1}{|2^jB|}\int_{2^jB}|f(u)||b(u)-b_B|^m du\\
\lesssim&([b]_\beta^\theta)^m\sum_{j\ge j_0}2^{-j\delta}\left(\frac{2^jr}{\rho(x_0)}\right)^{-N}\left(1+\frac{2^jr}{\rho(x_0)}\right)^{m\theta'}\left(\frac{1}{|2^jB|^{1-m\beta/n}}\int_{2^jB}|f(u)|^s du\right)^{1/s}\\
\lesssim&([b]_\beta^\theta)^m\sum_{j\ge j_0}2^{-j\delta}\left(\frac{2^jr}{\rho(x_0)}\right)^{-(N-m\theta')}M_{ms\beta,s}(f)(x).
\end{aligned}$$

Since $\frac{2^jr}{\rho(x_0)} \ge \gamma > 1$, taking $N \ge m\theta'$, we get

$$J_2 \lesssim([b]_\beta^\theta)^m M_{m\beta,s}(f)(x).$$

Now we consider the term J_3. Let $p_1' < \tilde{s} < s$. Since $2^j < \gamma\rho(x_0)/r$ for $j < j_0$, then

$$\begin{aligned}
J_3 \lesssim&\sum_{j=2}^{j_0}2^{-\delta j}\frac{1}{(2^jr)^{n-2}}\int_{2^jB}|f(u)||b(u)-b_B||I_2(V^2\chi_{B(x_0,2^{j+2}r)})(u)|du\\
\lesssim&\sum_{j=2}^{j_0}2^{-\delta j}\frac{1}{(2^jr)^{n-2}}\left(\int_{2^jB}|(b(z)-b_B)f(z)|^{\tilde{s}}dz\right)^{1/\tilde{s}}\|I_2(V^2\chi_{B(x_0,2^{j+2}r)})\|_{L^{\tilde{s}'}(\mathbb{R}^n)}\\
\lesssim&([b]_\beta^\theta)^m\sum_{j=2}^{j_0}2^{-\delta j}(2^jr)^{n/s-n-2+2n/\tilde{t}}\left(1+\frac{2^jr}{\rho(x_0)}\right)^{m\theta'+2l_0}M_{m\beta,s}(f)(x)\\
\lesssim&([b]_\beta^\theta)^m M_{m\beta,s}(f)(x).
\end{aligned}$$

Note $2^jr/\rho(x_0) > \gamma > 1$, similar to the estimates for J_3 we have

$$\begin{aligned}
J_4 \lesssim&\sum_{j=j_0-1}^{\infty}2^{-\delta j}\left(\frac{\rho(x_0)}{2^jr}\right)^N\frac{1}{(2^jr)^{n-2}}\int_{2^jB}|f(u)||b(u)-b_B|^m|I_2(V^2\chi_{B(x_0,2^{j+2}r)})(u)|du\\
\lesssim&([b]_\beta^\theta)^m\sum_{j=j_0-1}^{\infty}2^{-\delta j}\left(\frac{2^jr}{\rho(x_0)}\right)^{-(N-m\theta'-2l_0)}M_{m\beta,s}(f)(x).
\end{aligned}$$

Then, taking $N > m\theta' + 2l_0$ we get

$$J_4 \lesssim \ ([b]_\beta^\theta)^m M_{m\beta,s}(f)(x).$$

Thus, we complete the proof. $\square$

Lemma 3.3. *Let $p_1' < s < \infty$, let $B = B(x_0, r)$ with $r \leq \gamma\rho(x_0)$ and let $x \in B$. Then*

$$M^\sharp_{\rho,\gamma}([b^m, \mathcal{R}^*]f)(x) \lesssim ([b]_\beta^\theta)^m \left(M_{m\beta,s}(f)(x) + M_{m\beta,s}(\mathcal{R}^* f)(x)\right).$$

Proof. Write

$$\begin{aligned}
&\frac{1}{|B|}\int_B |[b^m, \mathcal{R}^*]f(y) - ([b^m, \mathcal{R}^*]f)_B| dy \\
\leq &\frac{2}{|B|}\int_B |(b(y) - b_B)^m R^* f(y)| dy + \frac{2}{|B|}\int_B |\mathcal{R}^*((b - b_B)^m f_1)(y)| dy \\
&+ \frac{1}{|B|}\int_B |\mathcal{R}^*((b - b_B)^m f_2)(y) - (\mathcal{R}^*((b - b_B)^m f_2))_B| dy \\
= &K_1 + K_2 + K_3,
\end{aligned}$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$.

Since $r \leq \gamma\rho(x_0)$ and $\rho(x) \approx \rho(x_0)$, by Hölder's inequality and Lemma 2.3, we get

$$\begin{aligned}
K_1 \leq &\left(\frac{1}{|B|}\int_B |b(y) - b_B|^{ms'} dy\right)^{1/s'} \left(\frac{1}{|B|}\int_B |\mathcal{R}^* f(y)|^s dy\right)^{1/s} \\
\lesssim &([b]_\beta^\theta)^m r^{m\beta} \left(\frac{1}{|B|}\int_B |\mathcal{R}^* f(y)|^s dy\right)^{1/s} \lesssim ([b]_\beta^\theta)^m M_{m\beta,s}(\mathcal{R}^* f)(x).
\end{aligned}$$

Select $\tilde{s}$ so that $p_0' < \tilde{s} < s$. Then by the L^p-boundedness of $\mathcal{R}^*$ and Lemma 2.3,

$$\begin{aligned}
K_2 \lesssim &\left(\frac{1}{|B|}\int_B |\mathcal{R}^*((b - b_B)^m f_1)(y)|^{\tilde{s}} dy\right)^{1/\tilde{s}} \\
\lesssim &\left(\frac{1}{|B|}\int_{2B} |(b(y) - b_B)^f(y)|^{\tilde{s}} dy\right)^{1/\tilde{s}} \\
\lesssim &([b]_\beta^\theta)^m M_{m\beta,s}(\mathcal{R}^* f)(x).
\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}
K_3 \leq &\frac{1}{|B|^2}\int_B\int_B\int_{(2B)^c} |\mathcal{K}^*(y,u) - \mathcal{K}^*(z,u)||b(u) - b_B|^m |f(u)| du dz dy \\
\lesssim &\int_{(2B)^c} |\mathcal{K}^*(y,u) - \mathcal{K}^*(z,u)||b(u) - b_B|^m |f(u)| du \\
\lesssim &([b]_\beta^\theta)^m M_{m\beta,s}(\mathcal{R}^* f)(x).
\end{aligned}$$

So, we complete the proof. $\square$

Now let us prove Theorem 1.1.

Choose numbers t_α such that $\frac{1}{t_\alpha}=\frac{1}{p}-\frac{\alpha\beta}{n}$, $\alpha=0,1,\cdots,m-1$. Then $\frac{1}{q}=\frac{1}{t_\alpha}-\frac{(m-\alpha)\beta}{n}$. We need to prove the following inequality

$$\big\|[b^m,\mathcal{R}^*]f\big\|^q_{L^q(\mathbb{R}^n)}\lesssim([b]^\theta_\beta)^{mq}\|f\|^q_{L^p(\mathbb{R}^n)}+\sum_{\alpha=0}^{m-1}([b]^\theta_\beta)^{(m-\alpha)q}\big\|[b^\alpha,\mathcal{R}^*](f)\big\|^q_{L^{t_\alpha}(\mathbb{R}^n)}. \tag{3.1}$$

If (3.1) holds, then Theorem 1.1 will be proved by the mathematical induction. In fact, when $m=1$, we have $\alpha=0$ and $p=t_\alpha$. Note that $[b^0,\mathcal{R}^*]=\mathcal{R}^*$, by the boundedness of $\mathcal{R}^*$ on $L^p(\mathbb{R}^n)$ for $p_1'<p<\infty$, then $[b,\mathcal{R}^*]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Suppose that the (L^p,L^{t_α})- boundedness of $[b^\alpha,\mathcal{R}^*]$ hold for $\frac{1}{t_\alpha}=\frac{1}{p}-\frac{\alpha\beta}{n}$, that is

$$\big\|[b^\alpha,\mathcal{R}^*](f)\big\|_{L^{t_\alpha}(\mathbb{R}^n)}\lesssim([b]^\theta_\beta)^\alpha\|f\|_{L^p(\mathbb{R}^n)},$$

where $\alpha=2,3,\cdots,m-1$, then by (3.1) we get

$$\big\|[b^m,\mathcal{R}^*](f)\big\|_{L^q(\mathbb{R}^n)}\lesssim([b]^\theta_\beta)^m\|f\|_{L^p(\mathbb{R}^n)}.$$

In the following, we will focus on the proof of (3.1).

Let $p_1'<s<p<\infty, f\in L^p(\mathbb{R}^n)$. By Proposition 2.3 we have

$$\begin{aligned}\|[b^m,\mathcal{R}^*]f\|^q_{L^q(\mathbb{R}^n)}&\le\int_{\mathbb{R}^n}\big|M_{\rho,\delta}([b^m,\mathcal{R}^*]f)(x)\big|^q\,dx\\ &\le\int_{\mathbb{R}^n}\big|M^\sharp_{\rho,\gamma}([b^m,\mathcal{R}^*]f)(x)\big|^q dx+\sum_k|Q_k|\left(\frac{1}{|Q_k|}\int_{2Q_k}|[b^m,\mathcal{R}^*]f(x)|\,dx\right)^q.\end{aligned}$$

By Lemma 3.3,

$$\begin{aligned}&M^\sharp_{\rho,\gamma}([b^m,\mathcal{R}^*]f)(x)\\ \lesssim&([b]^\theta_\beta)^m M_{m\beta,s}(f)(x)+\sum_{\alpha=0}^{m-1}([b]^\theta_\beta)^{m-\alpha}M_{(m-\alpha)\beta,s}([b^\alpha,\mathcal{R}^*]f)(x).\end{aligned}$$

Since

$$\frac{1}{q}=\frac{1}{t_\alpha}-\frac{(m-\alpha)\beta}{n},$$

and $t_\alpha=p$ when $\alpha=0$, then

$$\begin{aligned}&\int_{\mathbb{R}^n}\big|M^\sharp_{\rho,\gamma}([b^m,\mathcal{R}^*]f)(x)\big|^q dx\\ \lesssim&([b]^\theta_\beta)^{mq}\int_{\mathbb{R}^n}\big|M_{m\beta,s}(f)(x)\big|^q dx\\ &+\sum_{\alpha=0}^{m-1}([b]^\theta_\beta)^{(m-\alpha)q}\int_{\mathbb{R}^n}\big|M_{(m-\alpha)\beta,s}([b^\alpha,\mathcal{R}^*]f)(x)\big|^q dx\\ \lesssim&([b]^\theta_\beta)^{mq}\|f\|^q_{L^p(\mathbb{R}^n)}+\sum_{\alpha=0}^{m-1}([b]^\theta_\beta)^{(m-\alpha)q}\big\|[b^\alpha,\mathcal{R}^*](f)\big\|^q_{L^{t_\alpha}(\mathbb{R}^n)}.\end{aligned}$$

By Proposition 2.2 and Lemma 3.1 we have

$$
\begin{aligned}
&\sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |[b^m, \mathcal{R}^*] f(x)|\, dx \right)^q \\
\lesssim & ([b]_\beta^\theta)^{mq} \sum_k \int_{2Q_k} |M_{m\beta,s}(f)|^q dx \\
&+ \sum_{\alpha=0}^{m-1} ([b]_\beta^\theta)^{(m-\alpha)p} \sum_k \int_{2Q_k} \left| M_{(m-\alpha)\beta,s}([b^\alpha, \mathcal{R}^*] f) \right|^q dx \\
\lesssim & ([b]_\beta^\theta)^{mq} \int_{\mathbb{R}^n} \left| M_{m\beta,s}(f)(x) \right|^q dx \\
&+ \sum_{\alpha=0}^{m-1} ([b]_\beta^\theta)^{(m-\alpha)q} \int_{\mathbb{R}^n} \left| M_{(m-\alpha)\beta,s}([b^\alpha, \mathcal{R}^*] f)(x) \right|^q dx \\
\lesssim & ([b]_\beta^\theta)^{mq} \|f\|^q_{L^p(\mathbb{R}^n)} + \sum_{\alpha=0}^{m-1} ([b]_\beta^\theta)^{(m-\alpha)q} \left\| [b^\alpha, \mathcal{R}^*](f) \right\|^q_{L^{t_\alpha}(\mathbb{R}^n)}.
\end{aligned}
$$

Then the proof of (3.1) is finished. □

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