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A Characterization of Boundedness of Fractional Maximal Operator with Variable Kernel on Herz-Morrey Spaces

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Abstract. A significant number of studies have been carried out on the generalized Lebesgue spaces $L^{p(x)}$, Sobolev spaces $W^{1,p(x)}$ and Herz spaces. In this paper, we demonstrated a characterization of boundedness of the fractional maximal operator with variable kernel on Herz-Morrey spaces.

Key Words: Variable exponent, herz space, operator theory.

AMS Subject Classifications: 42B25, 42B35, 47B38

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1 Introduction

In 1991, Kovacik and Rakosnik introduced variable exponent Lebesgue spaces and Sobolev spaces as a new method for dealing with nonlinear Dirichet boundary value problem [9]. Then, variable problem and differential equation with variable exponent are intensively developed. In recent years, many researchers have been interested by the theory of the variable exponent function space and its applications [16,17]. For example, the compactness of Hardy spaces with weighted and variable exponent Lebesgue spaces was introduced by author [11]. Fractional integral on Herz-Morrey spaces with variable exponent were introduced by authors [7] and [10]. The boundedness of fractional integral with variable kernel on variable exponent Herz-Morrey spaces was given by [1]. Boundedness of fractional integral with variable kernel and their commutators on variable exponent Herz spaces was given by [2]. Boundedness of fractional Marcinkiewicz integral with variable kernel on variable exponent Morrey-Herz spaces was given by [3]. We also note that Herz-Morrey spaces with variable exponent are generalization of Morrey-Herz spaces [8] and Herz spaces with variable exponent [8, 13]. Our main goal is to give a characterization on boundedness of the fractional maximal operator with variable kernel from $MH^{\beta,\alpha}_{q_1,p_1(\cdot)}(R^n)$ to $MH^{\beta,\alpha}_{q_2,p_2(\cdot)}(R^n)$.

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Let $0<\rho<m$, $\Theta\in L^{\infty}(R^m)\times L^n(S^{m-1})$ and for $x\in R^m$, $\Theta(X,\cdot)\in L^n(S^{m-1})$, $(n\geq 1)$ is homogenous on R^m, S^{m-1} denote the unit sphere in R^m. If

a) For $x,t\in R^m$, $\Theta(x,\alpha t)=\Theta(x,t)$,

b) $d\nu(t^{'})$ is an element of area in the S^{m-1} and

$$\|\Theta\|_{L^{\infty}(R^m xL^n(S^{m-1}))}=\sup_{x\in R^m}\Big(\int_{S^{m-1}}|\Theta(x,t^{'})|^n d\nu(t^{'})\Big)^{\frac{1}{n}}<\infty.$$

We need the further assumption for $\Theta(x,t)$. For $t\in R$, $\Theta(\cdot,t)\in L^{\infty}(R^m)$. It satisfies

$$\int_{S^{m-1}}|\Theta(x,t^{'})|^n d\nu(t^{'})=0,\quad \forall x\in R^m.$$

For $n\geq 1$, we say $\Theta(x,t)$ satisfies

$$\int_0^1\frac{w_n(\lambda)}{\lambda}d\lambda<\infty,$$

where $w_r(\lambda)$ denote the integral modulus of continuity of order n of Θ defined by

$$w_n(\lambda)=\sup_{x\in R^m,\ |\sigma|<\lambda}\Big(\int_{S^{m-1}}|\Theta(x,\sigma t^{'})-\Theta(x,t^{'})|^n d\nu(t^{'})\Big)^{\frac{1}{n}},$$

where σ is a rotation in R^m,

$$|\sigma|=\sup_{t^{'}\in S^{m-1}}|\sigma t^{'}-t^{'}|.$$

The fractional maximal operator with variable kernel is defined by

$$M_{\Theta,\rho}f(x)=\sup_{n>0}\frac{1}{n^{m-\rho}}\int_{|x-z|<n}f(z)|\Theta(x,x-z)|dz.$$

The space $L^{p(\cdot)}_{loc}(E)$ is defined by

$$L^{p(\cdot)}_{loc}(E)=\{f\text{ is measurable}:\ f\in L^{p(\cdot)}(N)\text{ for all compact }N\subset E\}.$$

We denote

$$p_-=ess\inf\{p(x):x\in E\},\quad p_+=ess\sup\{p(x):x\in E\}.$$

Denote $\Gamma(E)$ set of allmeasurable functions $p(\cdot)$ satisfying $p_->1$ and $p_+<\infty$. Also denote $\Pi(E)$ the set of all function $p(\cdot)\in\Gamma(E)$ satisfying the M is bounded on $L^{p(\cdot)}(E)$. Let $B_i=\{x\in R^m:|x|\leq 2^i\}$, $C_i=B_i\backslash B_{i-1}$, $\chi_i=\chi_{C_i}$, $i\in Z$.

2 Auxiliary statements and assertions

Definition 2.1. Let $\beta \in R$, $0 < q < \infty$, $p(\cdot) \in \Gamma(R^m)$ and $0 \le \alpha < \infty$. The Herz-Morrey space with variable exponent $MH^{\beta,\alpha}_{q,p(\cdot)}(R^m)$ is defined by

$$MH^{\beta,\alpha}_{q,p(\cdot)}(R^m) = \Big\{ f \in L^{p(\cdot)}_{Loc}(R^m \backslash (0)) : \|f\|_{MH^{\beta,\alpha}_{q,p(\cdot)}(R^m)} < \infty \Big\},$$

$$\|f\|_{MH^{\beta,\alpha}_{q,p(\cdot)}(R^m)} = \sup_{L \in Z} 2^{-L\alpha} \Big\{ \sum_{i=-\infty}^{L} 2^{i\beta p} \|f\chi_i\|^q_{L^{p(\cdot)}} \Big\}^{\frac{1}{q}}.$$

We set $H^{\beta,p}_{q(\cdot)}(R^m) = MH^{\beta,0}_{p,q(\cdot)}(R^m)$ (see [9]).

Similarly in [4], let $p(\cdot) \in \Gamma(R^m)$ be such that

$$|p(x) - p(z)| \le \frac{-C}{\log(|x-z|)}, \qquad |x - z| \le \frac{1}{2},$$

$$|p(x) - p(z)| \le \frac{C}{\log(e + |x|)}, \qquad |z| \ge |x|.$$

Proposition 2.1 ([14]). *Let $p_1(\cdot) \in \Pi(R^m)$, $\Theta \in L^\infty(R^m) \times L^n(S^{m-1})$. Let $0 < \rho \le \frac{m}{(p_1)}$, and define the variable exponent $p_2(\cdot)$ as*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\rho}{m}.$$

Then for all $f \in L^{p_1(\cdot)}(R^m)$, we have that

$$\|M_{\Theta,\rho} f\|_{L^{p_2(\cdot)}(R^m)} = \|f\|_{L^{p_1(\cdot)}(R^m)}.$$

Lemma 2.1 ([5]). *Let $0 < \rho < m$, $n > 1$, satisfies the L^n-Dini condition. Let there exists an $0 < \beta < \frac{1}{2}$ such that for $|z| < \beta_0 R$ it holds*

$$\Big(\int_{R<|x|<2R} \Big| \frac{\Theta(x, x-z)}{|x-z|^{m-\rho}} - \frac{\Theta(x,x)}{|x|^{m-\rho}} \Big|^n dx \Big)^{\frac{1}{n}} \le C R^{(\frac{m}{n} - m + \rho)} \Big(\frac{|z|}{R} + \int_{\frac{|z|}{2R}}^{\frac{|z|}{R}} \frac{w_n(\lambda)}{\lambda} d\lambda \Big).$$

Lemma 2.2 ([6]). *Let $p(\cdot) \in \Pi(R^m)$ and $0 < p^- \le p^+ < \infty$.*

a) For a cube with $|\Delta| \le 2^m$, and all $\chi \in \Delta$, it holds $\|\chi_\Delta\|_{L^{p(\cdot)}} = |\Delta|^{\frac{1}{p(x)}}$;

b) For a cube $|\Delta| \ge 1$, the $\|\chi_\Delta\|_{L^{p(\cdot)}} = |\Delta|^{\frac{1}{p_\infty}}$ and where $p_\infty = \lim_{x \longrightarrow \infty} p(x)$.

Lemma 2.3 ([15]). *Let $p(\cdot) \in \Pi(R^m)$, and there exist a constant $C > 0$ such that for all balls D in R^m, we get*

$$\frac{1}{|D|} \|\chi_D\|_{L^{p(\cdot)}(R^m)} \le C.$$

3 Main result

Theorem 3.1. *Let* $0<\rho<m$, $0<\mu\le 1$, $\alpha<\beta<m\lambda_1+\mu$, $0<q_1\le q_2<\infty$. *Suppose that* $t\in R$, $\Theta(\cdot,t)\in L^\infty(R^m)$ *and the integral modulus of continuity* $w_n(\lambda)$ *satisfies*

$$\int_0^1 \frac{w_n(\lambda)}{\lambda^{1+\mu}}d\lambda<\infty.$$

And let $p_1(\cdot)\in\Pi(R^m)$ *satisfy* $0<\rho\le\frac{m}{(p_1)_+}$ *and define the variable exponent* $p_2(x)$ *as* $\frac{1}{p_1(x)}-\frac{1}{p_2(x)}=\frac{\rho}{m}$, *then we have for all* f

$$\|M_{\Theta,\rho}f\|_{MH^{\beta,\alpha}_{q_2,p_2(\cdot)}(R^m)}\le C\|f\|_{MH^{\beta,\alpha}_{q_1,p_1(\cdot)}(R^m)}.$$

Proof. For $\|f\|_{MH^{\beta,\alpha}_{q_1,p_1(\cdot)}(R^m)}$, we apply inequality

$$\begin{aligned}
&\Big[\sum_{k=1}^{\infty}\beta_k\Big]^{\frac{q_1}{q_2}}\le\sum_{k=1}^{\infty}\beta_k^{\frac{q_1}{q_2}},\quad q_1,q_2>0,\\
&\|M_{\Theta,\rho}f\|^{q_1}_{MH^{\beta,\alpha}_{q_2,p_2(\cdot)}(R^m)}=\sup_{L\in Z}2^{-L\alpha q_1}\Big\{\sum_{i=-\infty}^{L}2^{i\beta q_2}\|M_{\Theta,\rho}(f)\chi_i\|^{q_2}_{L^{p_2(\cdot)}(R^m)}\Big\}^{\frac{q_1}{q_2}}\\
&\qquad\le\sup_{L\in Z}2^{-L\alpha q_1}\Big\{\sum_{i=-\infty}^{L}2^{i\beta q_1}\|M_{\Theta,\rho}(f)\chi_i\|^{q_1}_{L^{p_2(\cdot)}(R^m)}\Big\}.
\end{aligned}$$

If we denote

$$f(x)=\sum_{i=-\infty}^{\infty}f(x)\chi_i=\sum_{i=-\infty}^{\infty}f_i(x),$$

then we have

$$\begin{aligned}
\|M_{\Theta,\rho}f\|^{q_1}_{MH^{\beta,\alpha}_{q_2,p_2(\cdot)}(R^m)}&\le\sup_{L\in Z}2^{-L\alpha q_1}\sum_{i=-\infty}^{L}2^{i\beta q_1}\left(\sum_{k=-\infty}^{\infty}\|M_{\Theta,\rho}(f_k)\chi_i\|_{L^{p_2(\cdot)}(R^m)}\right)^{q_1}\\
&\le\sup_{L\in Z}2^{-L\alpha q_1}\sum_{i=-\infty}^{L}2^{i\beta q_1}\left(\sum_{k=-\infty}^{i-2}\|M_{\Theta,\rho}(f_k)\chi_i\|_{L^{p_2(\cdot)}(R^m)}\right)^{q_1}\\
&\quad+\sup_{L\in Z}2^{-L\alpha q_1}\sum_{i=-\infty}^{L}2^{i\beta q_1}\left(\sum_{k=i-1}^{\infty}\|M_{\Theta,\rho}(f_k)\chi_i\|_{L^{p_2(\cdot)}(R^m)}\right)^{q_1}\\
&=:J_1+J_2.
\end{aligned}$$

Let as derive an J_1 using condition on f_k. Using Minkowski's inequality for $k\le i-1$, we get

$$
\begin{aligned}
\|M_{\Theta,\rho}(f_k)\chi_i\|_{L^{p_2(\cdot)}(R^m)}&= \Big(\int_{C_i}|M_{\Theta,\rho}(f_k)|^{p_2(\cdot)}dx\Big)^{\frac{1}{p_2(\cdot)}},\\
|M_{\Theta,\rho}(f_k)|&= \Big|\int_{R^m}\frac{\Theta(x,x-z)}{|x-z|^{m-\rho}}f_k(z)dz\Big|.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\|M_{\Theta,\rho}(f_k)\chi_i\|_{L^{p_2(\cdot)}(R^m)}\\
=&\int_{D_k}f_k(y)\Big\|\Big|\frac{\Theta(x,x-z)}{|x-z|^{m-\rho}}-\frac{\Theta(x,x)}{|x|^{m-\rho}}\Big|\chi_i\Big\|_{L^{p_2(\cdot)}(R^m)}dz.
\end{aligned}
$$

Since $n>p_2^+$, $\frac{1}{p_2(x)}=\frac{1}{n}+\frac{1}{p_2'(x)}$ by see [1, Lemma 3.3] and we get

$$
\begin{aligned}
&\Big\|\Big|\frac{\Theta(x,x-z)}{|x-z|^{m-\rho}}-\frac{\Theta(x,x)}{|x|^{m-\rho}}\Big|\chi_i\Big\|_{L^{p_2(\cdot)}(R^m)}\\
\leq&\Big\|\frac{\Theta(x,x-z)}{|x-z|^{m-\rho}}-\frac{\Theta(x,x)}{|x|^{m-\rho}}\Big\|_{L^n(R^m)}\|\chi_i\|_{L^{p_2'(\cdot)}(R^m)}\\
\leq&\Big\|\frac{\Theta(x,x-z)}{|x-z|^{m-\rho}}-\frac{\Theta(x,x)}{|x|^{m-\rho}}\Big\|_{L^n(R^m)}\|\chi_{D_i}\|_{L^{p_2'(\cdot)}(R^m)}.
\end{aligned}
$$

According to Lemma 2.2 and $\frac{1}{p_2(x)}-\frac{1}{n}=\frac{1}{p_2'(x)}$, then we have

$$
\|\chi_{D_i}\|_{L^{p_2'(\cdot)}(R^m)}\approx\|\chi_{D_i}\|_{L^{p_1(\cdot)}(R^m)}|D|^{\frac{-1}{n}\frac{\rho}{m}}.
$$

Combining Lemma 2.1 and $2^{k-i}\leq 2^{(k-i)\mu}$, we get

$$
\begin{aligned}
&\Big\|\frac{\Theta(x,x-z)}{|x-z|^{m-\rho}}-\frac{\Theta(x,x)}{|x|^{m-\rho}}\Big\|_{L^n(R^m)}\\
\leq&CR^{(\frac{m}{n}-m+\rho)}\Big(\frac{|z|}{2^{i-1}}+\int_{2^i}^{2^{i-1}}\frac{w_n(\lambda)}{\lambda}d\lambda\Big)\\
\leq&CR^{(\frac{m}{n}-m+\rho)}\Big(2^{k-i}+2^{(k-i)\mu}\int_0^1\frac{w_n(\lambda)}{\lambda^{1+\mu}}d\lambda\Big)\\
\leq&CR^{(\frac{m}{n}-m+\rho)}2^{(k-i)\mu}\Big(1+\int_0^1\frac{w_n(\lambda)}{\lambda^{1+\mu}}d\lambda\Big)\\
\leq&C2^{(i-1)(\frac{m}{n}-m+\rho)}2^{(k-i)\mu}.
\end{aligned}
$$

It follows that

$$
\|M_{\Theta,\rho}(f_k)\chi_i\|_{L^{p_2(\cdot)}(R^m)}\leq C2^{-im+(k-i)\mu}\int_{D_k}f_k(z)dz\|\chi_{D_i}\|_{L^{p_1(x)}(R^m)}.
$$

Using (see [1, Lemma 3.1 and Lemma 3.5]) and Lemma 2.3, it follows that

$$
\begin{aligned}
&\|M_{\Theta,\rho}(f_k)\chi_i\|_{L^{p_2(\cdot)}(R^m)}\\
\le& C2^{-im+(k-i)\mu}\|f_k\|_{L^{p_1(x)}(R^m)}\|\chi_{D_k}\|_{L^{p_1'(\cdot)}(R^m)}\|\chi_{D_i}\|_{L^{p_1(x)}(R^m)}\\
\le& C2^{(k-i)\mu}\|f_k\|_{L^{p_1(x)}(R^m)}\|\chi_{D_k}\|_{L^{p_1'(\cdot)}(R^m)}2^{-im}\|\chi_{D_i}\|_{L^{p_1(x)}(R^m)}\\
\le& C2^{(k-i)\mu}\|f_k\|_{L^{p_1(x)}(R^m)}\frac{\|\chi_{D_k}\|_{L^{p_1'(\cdot)}(R^m)}}{\|\chi_{D_i}\|_{L^{p_1(x)}(R^m)}}\\
\le& C2^{(k-i)\mu}2^{(k-i)m\lambda_1}\|f_k\|_{L^{p_1(x)}(R^m)}\\
\le& 2^{(k-i)(\mu+m\lambda_1)}\|f_k\|_{L^{p_1(x)}(R^m)}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
J_1 \le& C\sup_{L\in Z}2^{-L\alpha q_1}\sum_{i=-\infty}^{L}\left(\sum_{k=-\infty}^{i-2}2^{(k-i)(\mu+m\lambda_1)}\|f_k\|_{L^{p_1(x)}(R^m)}\right)^{q_1}\\
\le& C\sup_{L\in Z}2^{-L\alpha q_1}\sum_{i=-\infty}^{L}\left(\sum_{k=-\infty}^{i-2}2^{\beta k}2^{(k-i)(\mu+m\lambda_1-\beta)}\|f_k\|_{L^{p_1(x)}(R^m)}\right)^{q_1}.
\end{aligned}
$$

We consider the cases $1<q_1<\infty$ and $0<q_1\le 1$.

If $0<q_1\le 1$, using the Holder's inequality, we have

$$
\begin{aligned}
J_1 \le& C\sup_{L\in Z}2^{-L\alpha q_1}\sum_{i=-\infty}^{L}\Big(\sum_{k=-\infty}^{i-2}2^{\beta k q_1}2^{(k-i)(\mu+m\lambda_1-\beta)\frac{q_1}{2}}\|f_k\|^{q_1}_{L^{p_1(x)}(R^m)}\Big)\\
&\times\Big(\sum_{k=-\infty}^{i-2}2^{\frac{(k-i)(\mu+m\lambda_1-\beta)q'}{2}}\Big)^{\frac{q_1}{q_1'}}\\
\le& C\sup_{L\in Z}2^{-L\alpha q_1}\sum_{i=-\infty}^{L}\sum_{k=-\infty}^{i-2}2^{\beta k q_1}2^{(k-i)(\mu+m\lambda_1-\beta)\frac{q_1}{2}}\|f_k\|^{q_1}_{L^{p_1(x)}(R^m)}\\
\le& C\sup_{L\in Z}2^{-L\alpha q_1}\sum_{k=-\infty}^{i-2}2^{\beta k q_1}\|f_k\|^{q_1}_{L^{p_1(x)}(R^m)}\sum_{i=k+2}^{L}2^{(k-i)(\mu+m\lambda_1-\beta)\frac{q_1}{2}}\\
\le& C\sup_{L\in Z}2^{-L\alpha q_1}\sum_{k=-\infty}^{i-2}2^{\beta k q_1}\|f_k\|^{q_1}_{L^{p_1(x)}(R^m)}\\
\le& C\|f\|^{q_1}_{MH^{\beta,\alpha}_{q_1,p_1(\cdot)}(R^m)}.
\end{aligned}
$$

If $0 < q_1 \leq 1$, $\beta < m\lambda_1 + \mu$, we get

$$\begin{aligned}
J_1 \leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} \sum_{k=-\infty}^{i-2} 2^{\beta k q_1} 2^{(k-i)(\mu+m\lambda_1-\beta)^{q_1}} \|f_k\|^{q_1}_{L^{p_1(x)}(R^m)} \\
\leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{k=-\infty}^{i-2} 2^{\beta k q_1} \|f_k\|^{q_1}_{L^{p_1(x)}(R^m)} \sum_{i=k+2}^{L} 2^{(k-i)(\mu+m\lambda_1-\beta)^{q_1}} \\
\leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{k=-\infty}^{i-2} 2^{\beta k q_1} \|f_k\|^{q_1}_{L^{p_1(x)}(R^m)} \\
\leq & C \|f\|^{q_1}_{MH^{\beta,\alpha}_{q_1,p_1(\cdot)}(R^m)}.
\end{aligned}$$

Now, we estimate J_2, by using Proposition 2.1, we have

$$\begin{aligned}
J_2 \leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} 2^{\beta i q_1} \Big(\sum_{k=i-1}^{\infty} \|M_{\Theta,\rho}(f_k)\chi_i\|_{L^{p_2(\cdot)}(R^m)} \Big)^{q_1} \\
\leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} 2^{\beta i q_1} \Big(\sum_{k=i-1}^{\infty} (\|f_k\|_{L^{p_1(\cdot)}(R^m)} \Big)^{q_1} \\
\leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} \Big(\sum_{k=i-1}^{\infty} 2^{\beta k} 2^{(i-k)\beta} \|f_k\|_{L^{p_1(\cdot)}(R^m)} \Big)^{q_1} \\
\leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} \Big(\sum_{k=i-1}^{i+1} 2^{\beta k} 2^{(i-k)\beta} \|f_k\|_{L^{p_1(\cdot)}(R^m)} \Big)^{q_1} \\
& + C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} \Big(\sum_{k=i+2}^{\infty} 2^{\beta k} 2^{(i-k)\beta} \|f_k\|_{L^{p_1(\cdot)}(R^m)} \Big)^{q_1} \\
=: & J_1 + J_2.
\end{aligned}$$

Then we have,

$$\begin{aligned}
\leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} \Big(\sum_{k=i-1}^{i+1} 2^{\beta k} 2^{(i-k)\beta} \|f_k\|_{L^{p_1(\cdot)}(R^m)} \Big)^{q_1} \\
\leq & C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} 2^{\beta i q_1} \|f_k\|^{q_1}_{L^{p_1(\cdot)}(R^m)} \\
\leq & C \|f\|^{q_1}_{MH^{\beta,\alpha}_{q_1,p_1(\cdot)}(R^m)}.
\end{aligned}$$

Estimate J_2, for $\alpha < \beta$,

$$J_2 \leq C \sup_{L\in Z} 2^{-L\alpha q_1} \sum_{i=-\infty}^{L} \Big(\sum_{k=i+2}^{\infty} 2^{\beta k} 2^{(i-k)\beta} \|f_k\|_{L^{p_1(\cdot)}(R^m)} \Big)^{q_1}$$

$$
\begin{aligned}
&\leq C\sup_{L\in Z}\sum_{i=-\infty}^{L}2^{(i-L)\alpha q_1}\Big(\sum_{k=i+2}^{\infty}2^{\beta k}2^{(i-k)(\beta-\alpha)}2^{-k\alpha}\|f_k\|_{L^{p_1(\cdot)}(R^m)}\Big)^{q_1}\\
&\leq C\sup_{L\in Z}\sum_{i=-\infty}^{L}2^{(i-L)\alpha q_1}\Big(\sum_{k=i+2}^{\infty}2^{(i-k)(\beta-\alpha)}2^{-k\alpha}\Big\{\sum_{m=-\infty}^{i}2^{m\beta}\|f_k\|_{L^{p_1(\cdot)}(R^m)}\Big\}^{\frac{1}{q_1}}\Big)^{q_1}\\
&\leq C\sup_{L\in Z}\sum_{i=-\infty}^{L}2^{(i-L)\alpha q_1}\Big(\sum_{k=i+2}^{\infty}2^{(i-k)(\beta-\alpha)}\Big)^{q_1}\|f\|^{q_1}_{MH^{\beta,\alpha}_{q_1,p_1(\cdot)}(R^m)}\\
&\leq C\|f\|^{q_1}_{MH^{\beta,\alpha}_{q_1,p_1(\cdot)}(R^m)}.
\end{aligned}
$$

Proof of Theorem 3.1 was completed. □

4 Conclusions

We obtained the boundedness of fractional maximal operator with variable kernel on the Herz-Morrey spaces by a new method.

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