Analysis in Theory and Applications
Anal. Theory Appl., Vol. **36**, No. 1 (2020), pp. 52-59

DOI: 10.4208/ata.OA-2017-0074

# A Note on Rough Parametric Marcinkiewicz Functions

Laith Hawawsheh[1], Ahmad Al-Salman[2,*] and Shaher Momani[3,4]

[1] *School of Basic Sciences and Humanities, German Jordanian University, Amman, Jordan*
[2] *Department of Mathematics, Sultan Qaboos University, P.O. Box 36, Al-Khod 123 Muscat, Sultanate of Oman*
[3] *Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan*
[4] *Department of Mathematics and Sciences, College of Humanities and Sciences, Ajman University, Ajman, UAE*

Received 23 November 2017; Accepted (in revised version) 13 April 2018

**Abstract.** In this note, we obtain sharp $L^p$ estimates of parametric Marcinkiewicz integral operators. Our result resolves a long standing open problem. Also, we present a class of parametric Marcinkiewicz integral operators that are bounded provided that their kernels belong to the sole space $L^1\left(\mathbb{S}^{n-1}\right)$.

**Key Words**: Marcinkiewicz integrals, parametric Marcinkiewicz functions, rough kernels, Fourier transform, Marcinkiewicz interpolation theorem.

**AMS Subject Classifications**: 42B20, 42B15, 42B25

## 1 Introduction

Let $n \geq 2$ and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Suppose that $\Omega$ is a homogeneous function of degree zero on $\mathbb{R}^n$ that satisfies $\Omega \in L^1\left(\mathbb{S}^{n-1}\right)$ and

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.1}$$

In 1960, Hörmander (see [6]) introduced the following parametric Marcinkiewicz function $\mu_\Omega^\rho$ of higher dimension by

$$\mu_\Omega^\rho f(x) = \Big( \int_{-\infty}^{\infty} \left| 2^{-\rho t} \int_{|y|\leq 2^t} f(x-y)\,|y|^{-n+\rho}\,\Omega(y)dy \right|^2 dt \Big)^{\frac{1}{2}}, \tag{1.2}$$

*Corresponding author. *Email addresses:* `Laith.hawawsheh@gju.edu.jo` (L. Hawawsheh), `alsalman@squ.edu.om` (A. Al-Salman), `s.momani@ju.edu.jo` (S. Momani)

 

where $\rho > 0$. When $\rho = 1$, the corresponding operator $\mu_\Omega = \mu_\Omega^1$ is the classical Marcinkiewicz integral operator introduced by Stein (see [7]). When $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})$, $(0 < \alpha \le 1)$, Stein proved that $\mu_\Omega$ is bounded on $L^p$ for all $1 < p \le 2$. Subsequently, Benedek-Calderón-Panzone proved the $L^p$ boundedness of $\mu_\Omega$ for all $1 < p < \infty$ under the condition $\Omega \in C^1\left(\mathbb{S}^{n-1}\right)$ (see [4]). Since then, the $L^p$ boundedness of $\mu_\Omega$ has been investigated by several authors. For background information, we advise readers to consult [1–3,7], among others.

Concerning the problem whether there are some $L^p$ results on $\mu_\Omega^\rho$ similar to those on $\mu_\Omega$ when $\Omega$ satisfies only some size conditions, Ding, Lu, and Yabuta (see [5]) studied the general operator

$$\mu^\rho_{\Omega,h} f(x) = \Big( \int_{-\infty}^{\infty} \left| 2^{-\rho t} \int_{|y|\le 2^t} f(x-y)\,|y|^{-n+\rho}\, h(|y|)\Omega(y)dy \right|^2 dt \Big)^{\frac{1}{2}}, \tag{1.3}$$

where $h$ is a radial function on $\mathbb{R}^n$ satisfying $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 \le q \le \infty$, where the class $l^\infty(L^q)(\mathbb{R}^+)$ is defined by

$$l^\infty(L^q)(\mathbb{R}^+) = \Big\{ h : |h|_{l^\infty(L^q)(\mathbb{R}^+)} = \sup_{j\in\mathbb{Z}} \Big( \int_{2^{j-1}}^{2^j} |h(r)|^q \frac{dr}{r} \Big)^{\frac{1}{q}} < \infty \Big\}.$$

For $q = \infty$, we set $l^\infty(L^\infty)(\mathbb{R}^+) = L^\infty(\mathbb{R}^+)$. It is clear that

$$l^\infty(L^\infty)(\mathbb{R}^+) \subset l^\infty(L^r)(\mathbb{R}^+) \subset l^\infty(L^q)(\mathbb{R}^+) \subset l^\infty(L^1)(\mathbb{R}^+),$$

$1 < q < r < \infty$. Ding, Lu, and Yabuta (see [5]) proved the following result:

**Theorem 1.1** ([5]). *Suppose that $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying* (1.1) *and $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$ for some $1 < q \le \infty$. If $Re(\rho) = \alpha > 0$, then $\left|\mu^\rho_{\Omega,h} f\right|_2 \le C\alpha^{-\frac{1}{2}} |f|_2$, where $C$ is independent of $\rho$ and $f$.*

In [1], Al-Salman and Al-Qassem considered the $L^p$ boundedness of $\mu^\rho_{\Omega,h}$ for $p \neq 2$. which was left open in [5]. They proved the following result:

**Theorem 1.2** ([1]). *Suppose that $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying* (1.1). *If $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 < q \le \infty$, and $\alpha = Re(\rho) > 0$, then $\left|\mu^\rho_{\Omega,h} f\right|_p \le C\alpha^{-1} |f|_p$ for all $1 < p < \infty$, where $C$ is independent of $\rho$ and $f$.*

In light of Theorem 1.1, it is clear that the dependence of the $L^p$ bounds on $\alpha$ in Theorem 1.2 is not sharp. More precisely, we have the following long standing natural open problem:

**Problem**:

(a) Is the power $(-1/2)$ of $\alpha$ in Theorem 1.1 sharp?

(b) Does the result in Theorem 1.2 hold with power of $\alpha$ greater than $(-1)$?

It is our aim in this note to consider this problem. In fact, we shall prove the following result which completely resolves the above problem:

**Theorem 1.3.** *Suppose that $\Omega \in L(\log^+ L)(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying* (1.1). *If $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 < q \le \infty$ , and $\alpha = Re(\rho) > 0$, then*

$$\left|\mu^{\rho}_{\Omega,h} f\right|_p \le C\alpha^{-\frac{1}{p}} |f|_p \quad \textit{for all } \ 1 < p < \infty,$$

*where $C$ is independent of $\rho$ and $f$. Moreover, the power $(-1/p)$ is sharp in the sense that it can not be replaced by larger power.*

It is clear that Theorem 1.3 substantially improves Theorem 1.2 as far as the power of $\alpha$ is concerned. Concerning the function $\Omega$, we present in Section 3 of this note a subclass of the class $l^\infty(L^q)(\mathbb{R}^+)$ where the corresponding operator $\mu^{\rho}_{\Omega,h}$ is bounded on $L^2$ under the sole integrability condition $\Omega \in L^1(\mathbb{S}^{n-1})$.

Throughout the rest of the paper the letter $C$ will stand for a constant but not necessarily the same one in each occurrence.

## 2 Proof of main result

This section is devoted to present a proof of Theorem 1.3. We start by recalling the following well known interpolation theorem:

**Theorem 2.1** ([8]). *Let $T$ be a sublinear operator satisfying*

$$|T(f)|_{L^{p_1}(\mathbb{R}^n)} \le C_{p_1} |f|_{L^{p_1}(\mathbb{R}^n)}$$

*and*

$$|T(f)|_{L^{p_2}(\mathbb{R}^n)} \le C_{p_2} |f|_{L^{p_2}(\mathbb{R}^n)}$$

*for some $1 \le p_1, p_2 \le \infty$ and $C_{p_1}, C_{p_2} > 0$ . Then for all $\theta \in [0,1]$, we have*

$$|T(f)|_{L^{p_\theta}(\mathbb{R}^n)} \le C_{p_\theta} |f|_{L^{p_\theta}(\mathbb{R}^n)},$$

*where $p_\theta$ satisfies $\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $C_{p_\theta} = C^{\theta}_{p_1} C^{1-\theta}_{p_2}$.*

*Proof* of Theorem 1.3. The proof is based on an interpolation argument. By Theorem 1.1 and Theorem 1.2, we have

$$\left|\mu^{\rho}_{\Omega,h} f\right|_2 \le \frac{C}{\sqrt{\alpha}} |f|_2 \tag{2.1}$$

and

$$\left|\mu_{\Omega,h}^{\rho}f\right|_{1+\epsilon}\leq\frac{C}{\alpha}\left|f\right|_{1+\epsilon},\tag{2.2}$$

for any $\epsilon>0$. Thus, (2.1) and (2.2) show that the operator $\mu_{\Omega,h}^{\rho}$ is a bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and from $L^{1+\epsilon}(\mathbb{R}^n)$ to $L^{1+\epsilon}(\mathbb{R}^n)$, respectively. Thus, by Theorem 2.1, (2.1) and (2.2), we have

$$\left|\mu_{\Omega,h}^{\rho}f\right|_{p}\leq C\alpha^{-\left(\frac{\epsilon-\frac{\epsilon+1}{p}}{\epsilon-1}\right)}\left|f\right|_{p}\tag{2.3}$$

for all $1+\epsilon<p<2$. Letting $\epsilon\to0^+$, we would get

$$\left|\mu_{\Omega,h}^{\rho}f\right|_{p}\leq\frac{C}{\alpha^{\frac{1}{p}}}\left|f\right|_{p}$$

for $1<p<2$. Similarly, for $M>2$, we have by Theorem 1.2 that

$$\left|\mu_{\Omega,h}^{\rho}f\right|_{M}\leq\frac{C}{\alpha}\left|f\right|_{M}.\tag{2.4}$$

Interpolating between (2.1) and (2.4) yields

$$\left|\mu_{\Omega,h}^{\rho}f\right|_{p}\leq C\alpha^{-\left(\frac{\frac{1}{M}-\frac{1}{p}}{\frac{2}{M}-1}\right)}\left|f\right|_{p}\tag{2.5}$$

for all $2<p<M$. Letting $M\to\infty$ gives

$$\left|\mu_{\Omega,h}^{\rho}f\right|_{p}\leq\frac{C}{\alpha^{\frac{1}{p}}}\left|f\right|_{p}$$

for $2<p<\infty$.

Now, we show that the power $(1/p)$ is sharp. We shall work out the case $p=2$ and $\rho=\alpha$ is a positive real number. We shall also assume $0<\alpha<1$. Set

$$\Omega(x)=(x_1)'=\frac{x_1}{|x|}.$$

Then $\Omega$ satisfies (1.1) and $\Omega\in L^2(\mathbb{S}^{n-1})$. On the other hand, let $f(x)=x_1$ if $|x|<1$ and $f(x)=0$ if $|x|\geq1$. Then $f\in L^2(\mathbb{R}^n)$. In fact,

$$|f|_2=\frac{1}{\sqrt{n+2}}\left|\Omega\right|_2.$$

Now,

$$\begin{aligned}\left|\mu_{\Omega,h}^{\rho}f\right|_2^2&\geq\int_{\mathbb{R}^n}\int_3^\infty\left|\int_{\mathbb{S}^{n-1}}\int_0^t\Omega(y')f(x-ry')\frac{dr}{r^{1-\alpha}}d\sigma(y')\right|^2\frac{dt}{t^{1+2\alpha}}dx\\&\geq\int_{|x|<1}\int_3^\infty\left|\int_{\mathbb{S}^{n-1}}\int_0^t\Omega(y')f(x-ry')\frac{dr}{r^{1-\alpha}}d\sigma(y')\right|^2\frac{dt}{t^{1+2\alpha}}dx.\end{aligned}$$

By noticing that $f(x-ry')=0$ whenever $|x|<1$ and $r>2$, it follows from the last integral that

$$\begin{aligned}
\left|\mu^{\rho}_{\Omega,h}f\right|_2^2 &\geq \int_{|x|<1}\int_3^\infty \left|\int_{\mathbb{S}^{n-1}}\int_0^2 \Omega(y')(x_1-ry'_1)\frac{dr}{r^{1-\alpha}}d\sigma(y')\right|^2 \frac{dt}{t^{1+2\alpha}}dx \\
&= \int_{|x|<1}\int_3^\infty \left|\int_{\mathbb{S}^{n-1}}\int_0^2 (\Omega(y'))^2 r^\alpha dr d\sigma(y')\right|^2 \frac{dt}{t^{1+2\alpha}}dx \\
&= |\Omega|_2^4\left(\frac{2^{\alpha+1}}{1+\alpha}\right)^2\left(\frac{1}{3^{2\alpha}}\right)\frac{1}{2\alpha}\,|B(0,1)| \\
&\geq \frac{C}{\sqrt{\alpha}}\,|f|_2, \qquad (2.6)
\end{aligned}$$

where $|B(0,1)|$ is the volume of the ball $B(0,1)=\{x\in\mathbb{R}^n : |x|<1\}$ and $C$ is a constant independent of $\alpha$. Here, (2.6) follows by (1.1). This completes the proof. $\square$

## 3 Further study

As pointed out in the introduction section, in this section we present a subclass of the class $l^\infty(L^q)(\mathbb{R}^+)$ where the corresponding operator $\mu^{\rho}_{\Omega,h}$ is bounded on $L^2$ under the condition $\Omega\in L^1(\mathbb{S}^{n-1})$. If $q=\infty$, $l^\infty(L^\infty)(\mathbb{R}^+)=L^\infty(\mathbb{R}^+)$. For $1\leq q<\infty$, let $\mathcal{D}_q$ be the space of all measurable radial functions $h$ on $\mathbb{R}^n$ which satisfy

$$\frac{h(r)}{r^{1/q'}}\in l^\infty(L^q)(\mathbb{R}^+), \tag{3.1a}$$

$$\sum_{j=1}^{\infty}\left(\int_{2^j}^{2^{j+1}}|h(r)|^q\,\frac{dr}{r}\right)^{1/q}<\infty. \tag{3.1b}$$

It is obvious that $\mathcal{D}_q\subset l^\infty(L^q)(\mathbb{R}^+)$ and this inclusion is proper for $1\leq q<\infty$. In fact, for $j\in\mathbb{Z}^-$, we have

$$\left(\int_{2^j}^{2^{j+1}}|h(r)|^q\,\frac{dr}{r}\right)^{1/q}=\left(\int_{2^j}^{2^{j+1}}\left|\frac{h(r)}{r^{1/q'}}\right|^q r^{\frac{q}{q'}}\,\frac{dr}{r}\right)^{1/q}\leq C\left|h/r^{1/q'}\right|_{l^\infty(L^q)(\mathbb{R}^+)}.$$

On the other hand, for $j\in\mathbb{Z}^+$, by (3.1b) we have

$$\left(\int_{2^j}^{2^{j+1}}|h(r)|^q\,\frac{dr}{r}\right)^{1/q}\leq\sum_{j=1}^{\infty}\left(\int_{2^j}^{2^{j+1}}|h(r)|^q\,\frac{dr}{r}\right)^{1/q}<\infty.$$

Notice further that the constant functions are contained in $l^\infty(L^q)(\mathbb{R}^+)$ but not in $\mathcal{D}_q$.

On the other hand,

$$\mathcal{D}_q\nsubseteq L^\infty(\mathbb{R}^+). \tag{3.2}$$

To see (3.2), we construct a function $h \in \mathcal{D}_q \backslash L^\infty(\mathbb{R}^+)$. For convenience, we consider the case $q = 2$. Define $h$ on $\mathbb{R}^+$ by $h(r) = \sqrt[4]{n}\, r$, if $r \in [1 + \frac{1}{n+1}, 1 + \frac{1}{n}]$, $n \in \mathbb{N}$ and $h(r) = 0$ otherwise. It is clear that $h$ is not bounded. To see that $h \in \mathcal{D}_q$, we first observe that since $h(r) = 0$ for all $r \geq 2$, it follows that $h$ satisfies (3.1b). To see that $h$ satisfies (3.1a), notice

$$\left(\int_1^2 \left|\frac{h(r)}{r^{1/2}}\right|^2 \frac{dr}{r}\right)^{1/2} = \left(\sum_{n=1}^{\infty} \int_{1+\frac{1}{n+1}}^{1+\frac{1}{n}} \left|\frac{h(r)}{r^{1/2}}\right|^2 \frac{dr}{r}\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n(n+1)}\right)^{\frac{1}{2}} < \infty.$$

Now, we have the following result:

**Theorem 3.1.** *If $h \in \mathcal{D}_q$ for some $1 \leq q < \infty$ and $\Omega \in L^1(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying* (1.1), *then $\mu^{\rho}_{\Omega,h}$is bounded on $L^2(\mathbb{R}^n)$.*

*Proof.* By simple change of variables and Plancherel's theorem, we have

$$\left|\mu^{\rho}_{\Omega,h}\right|_2^2 \leq \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right|^2 \left[\int_0^\infty \left|t^{-\rho} \int_{|y|\leq t} e^{-2\pi i y\cdot\xi} \, |y|^{-n+\rho} \, h(|y|)\Omega(y)dy\right|^2 \frac{dt}{t}\right] d\xi. \tag{3.3}$$

On the other hand, by Minkowski's integral inequality, we have

$$\begin{aligned}
&\left(\int_0^\infty \left|t^{-\rho} \int_{|y|\leq t} e^{-2\pi i y\cdot\xi} \, |y|^{-n+\rho} \, h(|y|)\Omega(y)dy\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\
=&\left(\int_0^\infty \left|\int_0^\infty \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')\chi_{[0,t]}(r) r^{\rho-1} d\sigma(y')dr\right|^2 \frac{dt}{t^{1+2\rho}}\right)^{\frac{1}{2}} \\
\leq&\int_0^\infty \left(\int_0^\infty \left|\int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')\chi_{[0,t]}(r) d\sigma(y')\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{\frac{1}{2}} \frac{dr}{r^{1-\alpha}} \\
=&\int_0^\infty \left|\int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y')\right| \left(\int_r^\infty \frac{dt}{t^{1+2\alpha}}\right)^{\frac{1}{2}} \frac{dr}{r^{1-\alpha}} \\
=&\frac{1}{\sqrt{2\alpha}} \int_0^\infty \left|\int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y')\right| \frac{dr}{r}.
\end{aligned} \tag{3.4}$$

In view of (3.4), we need only to show that

$$\sup_{\xi\in\mathbb{R}^n-\{0\}} \int_0^\infty \left|\int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y')\right| \frac{dr}{r} < \infty. \tag{3.5}$$

We consider two cases:

Case 1. If $|\xi| > 2$, then

$$\begin{aligned}
&\int_0^\infty \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
=&\int_0^{2/|\xi|} \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
&+ \int_{2/|\xi|}^{1} \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
&+ \int_1^\infty \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
=:&I + II + III. 
\end{aligned} \tag{3.6}$$

By the cancellation property (1.1), we get

$$\begin{aligned}
I =& \int_0^{2/|\xi|} \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
=& \int_0^{2/|\xi|} \left| \int_{\mathbb{S}^{n-1}} \left(e^{-2\pi i r y'\cdot\xi} - 1\right) h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
=& \sum_{-\infty}^{1} \int_{2^{j-1}/|\xi|}^{2^j/|\xi|} \left| \int_{\mathbb{S}^{n-1}} \left(e^{-2\pi i r y'\cdot\xi} - 1\right) h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
\leq& C\,|\Omega|_{L^1(\mathbb{S}^{n-1})}\,|h|_{l^\infty(L^q)(\mathbb{R}_+)}\,,
\end{aligned} \tag{3.7}$$

where the last inequality was obtained using (3.1b). Next, choose $j_\xi \in \mathbb{Z}$ such that $2^{j_\xi} \leq 2/\,|\xi|$. Then

$$\begin{aligned}
II =& \int_{2/|\xi|}^{1} \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
\leq& \int_{2^{j_\xi}}^{1} \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
\leq& |\Omega|_{L^1(\mathbb{S}^{n-1})} \left|h/r^{1/q'}\right|_{l^\infty(L^q)(\mathbb{R}_+)} \sum_{j=j_\xi+1}^{0} \left(2^{j-1}\right)^{1/q'} \\
\leq& C\,|\Omega|_{L^1(\mathbb{S}^{n-1})} \left|h/r^{1/q'}\right|_{l^\infty(L^q)(\mathbb{R}^+)}\,,
\end{aligned} \tag{3.8}$$

where $C$ does not depend on the choice of $j_\xi$. Finally, notice that

$$\begin{aligned}
III =& \int_1^\infty \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
\leq& C\,|\Omega|_{L^1(\mathbb{S}^{n-1})}\,,
\end{aligned} \tag{3.9}$$

where the last inequality was obtained using (3.1b). This proves (3.5) for all $\xi \in \mathbb{R}^n$ with $|\xi| > 2$.

Case 2. If $|\xi| \leq 2$, then

$$\begin{aligned}
&\int_0^\infty \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
=&\int_0^2 \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r} \\
&+\int_2^\infty \left| \int_{\mathbb{S}^{n-1}} e^{-2\pi i r y'\cdot\xi} h(r)\Omega(y')d\sigma(y') \right| \frac{dr}{r}.
\end{aligned} \tag{3.10}$$

To estimate (3.10), we follow similar argument as in Case 1. This completes the proof. □


## Acknowledgements

The work is a part of the first author's PhD thesis, which had been done under the supervision of the second and third authors.


### References


[1] A. Al-Salman and H. Al-Qassem, On the $L^p$ boundedness of rough parametric Marcinkiewicz functions, J. Inequal. Pure Appl. Math., 8(4) (2007), 108.

[2] A. Al-Salman, H. Al-Qassem, L. Cheng and Y. Pan, $L^p$ bounds for the function of Marcinkiewicz, Math. Res. Lett., 9 (2002), 697–700.

[3] A. Al-Salman, Marcinkiewicz functions along flat surfaces with hardy space kernels, J. Integral Equations Appl., 17(4) (2005), 357–373.

[4] A. Benedek, A. Calderón and R. Panzone, Convolution operators on Banach space valued functions, Proc. Nat. Acad. Sci., 48 (1962), 356–365.

[5] Y. Ding, S. Lu and K. Yabuta, A problem on rough parametric Marcinkiewicz functions, J. Austral. Math. Soc., 72 (2002), 13–21.

[6] Hörmander, Translation invariant operators, Acta Math., 104 (1960), 93–139.

[7] E. M. Stein, On the function of Littlewood-Paley, Lusin and Marcinkiewicz, Trans. Amer. Math. Soc., 88 (1958), 430–466.

[8] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton NJ, 1970.

[9] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals, Princeton University Press, Princeton, NJ, 1993.