

Equivalence Relation between Initial Values and Solutions for Evolution p -Laplacian Equation in Unbounded Space

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Abstract. In this paper, an equivalence relation between the ω -limit set of initial values and the ω -limit set of solutions is established for the Cauchy problem of evolution p -Laplacian equation in the unbounded space $Y_\sigma(\mathbb{R}^N)$. To overcome the difficulties caused by the nonlinearity of the equation and the unbounded solutions, we establish the propagation estimate and the growth estimate for the solutions. It will be demonstrated that the equivalence relation can be used to study the asymptotic behavior of solutions.

AMS subject classifications: 35K55, 35B40

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1 Introduction

In this paper, we consider the asymptotic behavior of solutions for the Cauchy problem of the evolution p -Laplacian equation

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$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad (1.1)$$

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N, \quad (1.2)$$

where $p > 2$ and the nonnegative initial value

$$u_0 \in Y_\sigma(\mathbb{R}^N) \equiv \left\{ \varphi \in C(\mathbb{R}^N) : \lim_{|x| \rightarrow \infty} (1 + |x|^2)^{-\frac{\sigma}{2}} \varphi(x) = 0 \right\}$$

with $0 \leq \sigma < \frac{p}{p-2}$.

Since the beginning of this century, there has been a great interest in the complicated asymptotic behavior of solutions for some evolution equations [1–8]. To do this, a successful method is to establish the relation between the initial values and the solutions for the evolution equations in some Banach spaces. In 2002, it was Vázquez and Zuazua [9] who first considered the relation between the ω -limit set of initial values and the ω -limit set of solutions to the problem (1.1)-(1.2) in the bounded space $L^\infty(\mathbb{R}^N)$. They found that the set of accumulation points of the rescaled solutions $u(t^{\frac{1}{p}}x, t)$ to the problem (1.1)-(1.2) in $L_{\text{loc}}^\infty(\mathbb{R}^N)$ as $t \rightarrow \infty$ coincides with the set of $\{S(1)(\varphi)\}$, where φ ranges over the set of the accumulation points as $\lambda \rightarrow \infty$ of the family $\{u_0(\lambda x); \lambda > 0\}$ in the weak-star topology of $L^\infty(\mathbb{R}^N)$. By using this relation, they proved that the complicated asymptotic behavior can happen in the solutions. Later Cazenave, Dickstein and Weissler [10–13] investigated the relation between the rescaled solutions $t^{\frac{\mu}{2}}u(t^\beta x, t)$ ($\mu, \beta > 0$) and the initial values for the heat equation in bounded space $C_0(\mathbb{R}^N)$. They also used these relation to investigate the complicated asymptotic behavior of solutions. They also study the complicated asymptotic behavior of solutions for the Navier-Stokes equations and the Schrödinger equation [14, 15]. In our recent papers [16, 17], we revealed that there exists an equivalence relation between the ω -limit set of initial values and the ω -limit set of rescaled solutions $t^{\frac{\mu}{2}}u(t^\beta x, t)$ ($\mu, \beta > 0$) in bounded space $C_0(\mathbb{R}^N)$, and use this relation to study the complicated asymptotic behavior of solutions for the Cauchy problem of the porous medium equation and the Cauchy problem of the evolution p -Laplacian equation respectively. The studies of other asymptotic behavior of solutions for the evolution equations can be found in [18–23].

Note that the relations in the above works are only considered in some bounded spaces. It follows from the existence theory for the evolution p -Laplacian equation that the solutions of the problem (1.1)-(1.2) are global even if the initial data belong to some unbounded spaces [24–26]. Our interest here is to study

the relation between initial values and solutions for the problem (1.1)-(1.2) in the unbounded space $Y_\sigma(\mathbb{R}^N)$. The difficulties in our studies are mainly caused by the unbounded solutions and the nonlinearity of Eq. (1.1). Fortunately, we can establish the propagation estimate and the growth estimate for these unbounded solutions to overcome these difficulties. By using the properties of solutions in the unbounded space, we obtain that if

$$u_0 \in Y_\sigma^+(\mathbb{R}^N) \equiv \{\varphi \in Y_\sigma(\mathbb{R}^N); \varphi \geq 0\} \quad \text{with} \quad 0 \leq \sigma < \frac{p}{p-2},$$

then

$$\omega^\sigma(u_0) = S(1)\Omega^\sigma(u_0), \quad (1.3)$$

where

$$\omega^\sigma(u_0) \equiv \left\{ f \in Y_\sigma(\mathbb{R}^N); \exists t_n \rightarrow \infty \text{ s.t. } t_n^{-\frac{\sigma}{p-\sigma(p-2)}} u\left(t_n^{\frac{1}{p-\sigma(p-2)}} x, t_n\right) \rightarrow f(x) \text{ in } Y_\sigma(\mathbb{R}^N) \right\},$$

$$\Omega^\sigma(u_0) \equiv \left\{ \varphi \in Y_\sigma(\mathbb{R}^N); \exists \lambda_n \rightarrow \infty \text{ s.t. } \lambda_n^{-\frac{2\sigma}{p-\sigma(p-2)}} u_0\left(\lambda_n^{\frac{2}{p-\sigma(p-2)}} x\right) \rightarrow \varphi(x) \text{ in } Y_\sigma(\mathbb{R}^N) \right\}.$$

The relation (1.3) can be used to prove complicated asymptotic behavior of solutions to the problem (1.1)-(1.2). Since there exists an initial value $u_0 \in B_M^{\sigma,+} \equiv \{\phi \in Y_\sigma^+(\mathbb{R}^N); \|\phi\|_{Y_\sigma(\mathbb{R}^N)} \leq M\}$ such that $\Omega^\sigma(u_0) = B_M^{\sigma,+}$, solutions of the problem (1.1)-(1.2) can exhibit complicate asymptotic behavior by the relation (1.3), according to Vázquez and Zuazua [9].

The rest of this paper is organized as follows. In the next section, we give some definitions and properties. Section 3 is devoted to some estimates for the solutions of the problem (1.1)-(1.2) with the initial values $u_0 \in Y_\sigma(\mathbb{R}^N)$. The equivalence relation is studied in Section 4. Using the equivalence relation to investigate the asymptotic behavior is given in the last section.

2 Preliminaries

In this section we first introduce some concepts and give some propositions about the solutions to the problem (1.1)-(1.2).

Definition 2.1. ([24,25]) For $r > 0$, $f \in L_{\text{loc}}^1(\mathbb{R}^N)$, let

$$\|\|\varphi\|\|_r = \sup_{R \geq r} R^{-\frac{N(p-2)+p}{p-2}} \int_{\{|x| \leq R\}} |\varphi(x)| dx \quad \text{and} \quad \ell(\varphi) = \lim_{r \rightarrow \infty} \|\|\varphi\|\|_r.$$

The space X_0 is defined to be

$$X_0 \equiv \{\varphi \in X; \ell(\varphi) = 0\}$$

with the norm $\|\cdot\|_1$. Hence it is a Banach space.

If the initial value $u_0 \in X_0$, the existence and uniqueness of global weak solution of the problem (1.1)-(1.2) had been proved in [24–26], and this solution satisfies the following proposition.

Proposition 2.1. ([25, 26]) *The Cauchy problem of the evolution p -Laplacian equation (1.1)-(1.2) generates a continuous bounded semigroup in X_0 given by*

$$S(t): u_0 \rightarrow u(x, t). \quad (2.1)$$

In other words, $u(x, t) = S(t)u_0 \in C([0, \infty); X_0)$. Moreover, if $u_0 \in L^q(\mathbb{R}^N)$ with $1 \leq q \leq \infty$, then $S(t)$ is a contraction bounded semigroup in $L^q(\mathbb{R}^N)$.

The unbounded spaces $L^\infty(\rho_\sigma)$ and $Y_\sigma(\mathbb{R}^N)$ is defined as follows.

Definition 2.2. Let $0 \leq \sigma < \infty$, $\rho_\sigma(x) = (1 + |x|^2)^{-\frac{\sigma}{2}}$. The weighted space $L^\infty(\rho_\sigma)$ is defined to be

$$L^p(\rho_\sigma) \equiv \{\varphi \in L^1_{\text{loc}}(\mathbb{R}^N): \varphi \rho_\sigma \in L^\infty(\mathbb{R}^N)\}$$

with $\|\varphi\|_{L^\infty(\rho_\sigma)} = \|\varphi \rho_\sigma\|_{L^\infty(\mathbb{R}^N)}$. The space $Y_\sigma(\mathbb{R}^N)$ is defined by

$$Y_\sigma(\mathbb{R}^N) \equiv \left\{ \varphi(x) \in C(\mathbb{R}^N): \lim_{|x| \rightarrow \infty} \varphi(x) \rho_\sigma(x) = 0 \right\}$$

with the norm $\|\varphi\|_{Y_\sigma(\mathbb{R}^N)} = \|\varphi \rho_\sigma\|_{L^\infty(\mathbb{R}^N)}$.

It is easy to prove that $Y_\sigma(\mathbb{R}^N)$ and $L^\infty(\rho_\sigma)$ are Banach spaces.

Definition 2.3. Suppose that $u_0 \in Y_\sigma(\mathbb{R}^N)$ with $0 \leq \sigma < \frac{p}{p-2}$. For $\varphi(x) \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\lambda > 0$, let

$$D_\lambda^\sigma \varphi(x) \equiv \lambda^{-\frac{2\sigma}{p-\sigma(p-2)}} \varphi(\lambda^{\frac{2}{p-\sigma(p-2)}} x).$$

Then we define the limit set $\Omega^\sigma(u_0)$ by

$$\Omega^\sigma(u_0) \equiv \left\{ \varphi \in Y_\sigma(\mathbb{R}^N); \exists \lambda_n \rightarrow \infty \text{ s.t. } D_{\lambda_n}^\sigma[u_0] \xrightarrow{w^*} \varphi \text{ in } Y_\sigma(\mathbb{R}^N) \text{ as } n \rightarrow \infty \right\}.$$

The ω -limit set $\omega^\sigma(u_0)$ is defined to be

$$\omega^\sigma(u_0) \equiv \left\{ f \in Y_\sigma(\mathbb{R}^N); \exists t_n \rightarrow \infty \text{ s.t. } D_{\sqrt{t_n}}^\sigma[S(t_n)u_0] \xrightarrow{n \rightarrow \infty} f \text{ in } Y_\sigma(\mathbb{R}^N) \right\},$$

where $S(t)$ is the semigroup given in (2.1).

For $\lambda > 0$, the commutative relation between the semigroup operator $S(t)$ and the dilation operator $D_\lambda^{\mu,\beta} \varphi(x) \equiv \lambda^\mu \varphi(\lambda^{2\beta}x)$

$$D_\lambda^{\mu,\beta} [S(\lambda^2 t)u_0] = S(\lambda^{2-2p\beta-\mu(p-2)}t) [D_\lambda^{\mu,\beta} u_0] \tag{2.2}$$

had been proven in [7, 17].

Definition 2.4. Let $d(x) \equiv \sup\{R; u_0(y) = 0 \text{ a.e. in } B_R(x)\}$. The positive set of $u(x,t)$ at time t is defined to be

$$\Omega(t) \equiv \{x \in \mathbb{R}^N; u(x,t) > 0\}.$$

The ρ -neighborhood of $\Omega(t)$ is given by

$$\Omega_\rho(t) \equiv \{x \in \mathbb{R}^N; d(x, \Omega(t)) < \rho\},$$

where $d(x, \Omega(t)) \equiv \sup\{R; B_R(x) \cap \Omega(t) = \emptyset\}$.

3 Some estimates

To study the relation between initial values and solutions for the problem (1.1)-(1.2) in unbounded space $Y_\sigma(\mathbb{R}^N)$, we need to give some estimates about the solutions first. The following lemma had been proven in [24, 25].

Lemma 3.1. ([24, 25]) Let $u(x,t)$ be the nonnegative weak solutions of the problem (1.1)-(1.2). For given $x_0 \in \mathbb{R}^N$, if

$$B(x_0) = \sup_{R>0} R^{-\frac{N(p-2)+p}{p-2}} \int_{B_R(x_0)} u_0(y) dy < \infty,$$

where $B_R(x_0) = \{y; |x_0 - y| < R\}$, then

$$u(x_0, t) = 0 \quad \text{for all } 0 < t \leq CB(x_0)^{-(p-2)}.$$

The following theorem concerns the propagation estimate for the solutions of the problem (1.1)-(1.2) with $u_0 \in L^\infty(\rho_\sigma)$.

Theorem 3.1 (Propagation Estimate). Suppose $u_0 \in L^\infty(\rho_\sigma)$ with $0 \leq \sigma < \frac{p}{p-2}$, then for $0 \leq t_1 \leq t_2 < \infty$, one can get

$$\Omega(t_2) \subset \Omega_{\rho(t_2-t_1)}(t_1),$$

where

$$\rho(t) = C \max \left(\|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{p-2}{p}} t^{\frac{1}{p}}, \|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{p-2}{p-\sigma(p-2)}} t^{\frac{1}{p-\sigma(p-2)}} \right).$$

Proof. We only need to consider the case $t_1 = 0$. Suppose that $x_0 \in \mathbb{R}^N$ with $d(x_0) > 0$. If $R < d(x_0)$, then the following equality holds

$$\int_{B_R(x_0)} u_0(y) dy = 0; \quad (3.1)$$

and if $R \geq d(x_0)$, we deduce from Definition (2.1) and $0 \leq \sigma < \frac{p}{p-2}$ that

$$\begin{aligned} & R^{-\frac{N(p-2)+p}{p-2}} \int_{B_R(x_0)} u_0(y) dy \\ &= R^{-\frac{N(p-2)+p}{p-2}} \int_{B_R(x_0)} u_0(y) \rho_\sigma(x) (1+|x|^2)^{\frac{\sigma}{2}} dy \\ &\leq \|u_0\|_{L^\infty(\rho_\sigma)} R^{-\frac{N(p-2)+p}{p-2}} (1+R^2)^{\frac{\sigma}{2}} \int_{B_R(x_0)} dy \\ &\leq 2^{\frac{\sigma}{2}} \|u_0\|_{L^\infty(\rho_\sigma)} \max(R^{-\frac{p}{p-2}}, R^{-\frac{p}{p-2}+\sigma}) \\ &\leq 2^{\frac{\sigma}{2}} \|u_0\|_{L^\infty(\rho_\sigma)} \max(d(x_0)^{-\frac{p}{p-2}}, d(x_0)^{-\frac{p}{p-2}+\sigma}). \end{aligned}$$

Consequently,

$$\begin{aligned} B(x_0) &= \sup_{R \geq d(x_0)} R^{-\frac{N(p-2)+p}{p-2}} \int_{B_R(x_0)} u_0(y) dy \\ &\leq C \|u_0\|_{L^\infty(\rho_\sigma)} \max\left(d(x_0)^{-\frac{p}{p-2}}, d(x_0)^{-\frac{p}{p-2}+\sigma}\right) \end{aligned}$$

holds by (3.1). Then it follows from Lemma 3.1 that

$$u(x_0, t) = 0 \quad \text{for all } 0 \leq t \leq C \|u_0\|_{L^\infty(\rho_\sigma)}^{-(p-2)} \min(d(x_0)^p, d(x_0)^{p-\sigma(p-2)}),$$

and therefore

$$\Omega(t) \subset \Omega_{\rho(t)}(0),$$

where

$$\rho(t) = C \max\left(\|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{p-2}{p}} t^{\frac{1}{p}}, \|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{p-2}{p-\sigma(p-2)}} t^{\frac{1}{p-\sigma(p-2)}}\right),$$

and the proof is complete. \square

In the following theorem, we estimate the growth estimate of solutions $u(x, t)$ to the problem (1.1)-(1.2) with the initial value $u_0 \in L^\infty(\rho_\sigma)$.

Theorem 3.2 (Growth Estimate). *Let $0 \leq \sigma < \frac{p}{p-2}$. If $0 \leq u_0 \in L^\infty(\rho_\sigma)$, then there exists a constant C such that*

$$0 \leq S(t)u_0(x) \leq C \left((1+t)^{\frac{2}{p-\sigma(p-2)}} + |x|^2 \right)^{\frac{\sigma}{2}} \quad \text{for } t > 0.$$

That is,

$$\|S(t)u_0\|_{L^\infty(\rho_\sigma)} \leq C(1+t)^{\frac{\sigma}{p-\sigma(p-2)}}.$$

Moreover, if $0 \leq u_0 \in Y_\sigma(\mathbb{R}^N)$, then

$$S(t)u_0 \in Y_\sigma(\mathbb{R}^N) \quad \text{for } t \geq 0.$$

Proof. Consider the following problem

$$\frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \tag{3.2}$$

$$u(x, 0) = v_0(x) = M|x|^\sigma \quad \text{in } \mathbb{R}^N. \tag{3.3}$$

For $\lambda > 0$, let

$$\lambda_1 = \lambda^{\frac{p-\sigma(p-2)}{2p}}, \quad \mu = -\frac{2\sigma}{p-\sigma(p-2)}, \quad \beta = \frac{1}{p-\sigma(p-2)},$$

then

$$2 - \mu(p-2) - 2p\beta = 0.$$

It follows from the commutative relation (2.2) that

$$\begin{aligned} \lambda^{-\frac{\sigma}{p}} [S(\lambda^{1-\frac{\sigma(p-2)}{p}} t)v_0](\lambda^{\frac{1}{p}} x) &= \lambda_1^\mu [S(\lambda_1^2 t)v_0](\lambda_1^{2\beta} x) \\ &= S(t)[\lambda_1^\mu v_0(\lambda_1^{2\beta} \cdot)](x) = S(t)v_0(x). \end{aligned} \tag{3.4}$$

Letting $t = 1$, $s = \lambda^{\frac{p-\sigma(p-2)}{p}}$ and $g(x) = S(1)v_0(x)$ in (3.4), one can get

$$S(s)v_0(x) = s^{\frac{\sigma}{p-\sigma(p-2)}} g(s^{-\frac{1}{p-\sigma(p-2)}} x). \tag{3.5}$$

Since $v_0 \in C(\mathbb{R}^N)$, we obtain from the regularity theory of the solutions that for $t > 0$,

$$0 \leq S(t)v_0 \in C([0, \infty) \times \mathbb{R}^N),$$

see [24,26]. Therefore (3.5) implies that that for $|x|=1$, the following limit holds:

$$s^{\frac{\sigma}{p-\sigma(p-2)}} g(s^{-\frac{1}{p-\sigma(p-2)}} x) = S(s)v_0(x) \rightarrow v_0(x) = M|x|^\sigma = M \quad (3.6)$$

as $s \rightarrow 0$. Put

$$y = s^{-\frac{1}{p-\sigma(p-2)}} x.$$

Observe that $|y| \rightarrow \infty$ as $s \rightarrow 0$. It follows from (3.6) that

$$|y|^{-\sigma} g(y) - M \rightarrow 0 \quad \text{as } |y| \rightarrow \infty. \quad (3.7)$$

Hence, there exists a nonnegative constant C such that

$$g(x) \leq C(1+|x|^2)^{\frac{\sigma}{2}},$$

then we deduce from (3.5) that

$$0 \leq S(s)v_0(x) \leq C \left(s^{\frac{2}{p-\sigma(p-2)}} + |x|^2 \right)^{\frac{\sigma}{2}},$$

therefore

$$S(t)g(x) = S(t)[S(1)v_0](x) = S(t+1)v_0(x) \leq C \left((1+t)^{\frac{2}{p-\sigma(p-2)}} + |x|^2 \right)^{\frac{\sigma}{2}}.$$

Taking $\varphi(x) = M(1+|x|^2)^{\frac{\sigma}{2}}$, we thus obtain

$$S(t)\varphi(x) \leq C \left((1+t)^{\frac{2}{p-\sigma(p-2)}} + |x|^2 \right)^{\frac{\sigma}{2}}.$$

Using Comparison Principle [24–26], we can get

$$S(t)u_0(x) \leq C \left((1+t)^{\frac{2}{p-\sigma(p-2)}} + |x|^2 \right)^{\frac{\sigma}{2}}$$

if we take $M = \|u_0\|_{L^\infty(\rho_\sigma)}$ in (3.3).

We verify the second part of this theorem below. Note first that

$$0 \leq u_0 \in Y_\sigma(\mathbb{R}^N) \subset L^\infty(\rho_\sigma).$$

For given $t > 0$, $R > 0$, letting

$$R(t) = R + 1 + C \max \left(\|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{p-2}{p}} t^{\frac{1}{p}}, \|u_0\|_{L^\infty(\rho_\sigma)}^{\frac{p-2}{p-\sigma(p-2)}} t^{\frac{1}{p-\sigma(p-2)}} \right)$$

and then taking $\chi_{R+1}(x)$ be the cut-off function defined on B_{R+1} relative to B_R , we deduce from Theorem 3.1 that

$$\text{supp}[S(t)(\chi_{R+1}u_0)] \subset \{x \in \mathbb{R}^N; |x| \leq R(t)\}.$$

This means that for $t, R > 0$, the value of $S(t)u_0$ in $\mathbb{R}^N \setminus B_{R(t)}$ is only depended on the initial value u_0 in $\mathbb{R}^N \setminus B_{R+1}$, that is, if $|x| > R(t)$, then

$$S(t)[(1 - \chi_{R+1})u_0](x) = S(t)u_0(x). \quad (3.8)$$

For every $\varepsilon > 0$, it follows from the hypothesis $0 \leq u_0 \in Y_\sigma(\mathbb{R}^N)$ that there exists a constant $R_1 > 1 > 0$ such that for $|x| \geq R_1$,

$$(1 + |x|^2)^{-\frac{\sigma}{2}} u_0(x) < \frac{\varepsilon}{2}.$$

Letting $R = R_1$, we get

$$(1 - \chi_{R_1+1})u_0(x) < \varepsilon|x|^\sigma.$$

Putting $M = \varepsilon$ in (3.3), one can get from (3.7) and Comparison Principle that there exists a constant R_2 such that if $|x| > R_2$, then

$$S(t)[(1 - \chi_{R_1+1})u_0](x) \leq g(x) \leq 2\varepsilon|x|^\sigma. \quad (3.9)$$

So (3.5) and (3.9) imply that if $|x|t^{-\frac{1}{p-\sigma(p-2)}} > R_2$, then

$$S(t)[(1 - \chi_{R_1+1})u_0](x) \leq t^{\frac{\sigma}{p-\sigma(p-2)}} g(t^{-\frac{1}{p-\sigma(p-2)}}x) \leq 2\varepsilon|x|^\sigma. \quad (3.10)$$

Combing (3.8) and (3.10), we get that if

$$|x| > \max(R_1(t), t^{\frac{1}{p-\sigma(p-2)}} R_2),$$

then

$$S(t)u_0(x) = S(t)[(1 - \chi_{R_1+1})u_0](x) \leq 2\varepsilon|x|^\sigma. \quad (3.11)$$

Observe that for $t > 0$,

$$S(t)u_0(x) \in L^\infty(\rho_\sigma) \quad \text{and} \quad S(t)u_0(x) \in C(\mathbb{R}^N),$$

then it follows from (3.11) that for $t > 0$,

$$S(t)u_0 \in Y_\sigma(\mathbb{R}^N),$$

and the proof is complete. □

4 Equivalence relation

In this section, we study the relation between solutions and initial values of the problem (1.1)-(1.2) with initial value $u_0 \in Y_\sigma^+(\mathbb{R}^N) \equiv \{\varphi \in Y_\sigma(\mathbb{R}^N); \varphi \geq 0\}$.

Theorem 4.1. *Suppose $0 \leq \sigma < \frac{p}{p-2}$. If $u_0 \in Y_\sigma^+(\mathbb{R}^N)$, then*

$$\omega^\sigma(u_0) = S(1)\Omega^\sigma(u_0) \equiv \{f : f = S(1)\varphi, \varphi \in \Omega^\sigma(u_0)\}. \quad (4.1)$$

Proof. If $f \in \omega^\sigma(u_0)$, it follows from the definition of $\omega^\sigma(u_0)$ that there exists a sequence $t_n \xrightarrow{n \rightarrow \infty} \infty$ such that

$$\Gamma_{\sqrt{t_n}}^\sigma[S(1)u_0] = D_{\sqrt{t_n}}^\sigma[S(t_n)u_0] = S(1)[D_{\sqrt{t_n}}^\sigma u_0] \rightarrow f \quad \text{in } Y_\sigma(\mathbb{R}^N). \quad (4.2)$$

Note that if $\lambda > 1$ and $\phi \in Y_\sigma(\mathbb{R}^N)$, then

$$\begin{aligned} \|D_\lambda^\sigma \phi\|_{L^\infty(\rho_\sigma)} &= \sup_{x \in \mathbb{R}^N} (1 + |x|^2)^{-\frac{\sigma}{2}} \lambda^{-\frac{2\sigma}{\sigma(p-2)+p}} \phi(\lambda^{\frac{2}{\sigma(p-2)+p}} x) \\ &= \sup_{x \in \mathbb{R}^N} (1 + |\lambda^{\frac{2}{\sigma(p-2)+p}} x|^2)^{-\frac{\sigma}{2}} \phi(\lambda^{\frac{2}{\sigma(p-2)+p}} x) \left(\frac{1 + |\lambda^{\frac{2}{\sigma(p-2)+p}} x|^2}{\lambda^{\frac{2}{\sigma(p-2)+p}} (1 + |x|^2)} \right)^{\frac{\sigma}{2}} \\ &\leq \sup_{x \in \mathbb{R}^N} (1 + |\lambda^{\frac{2}{\sigma(p-2)+p}} x|^2)^{-\frac{\sigma}{2}} \phi(\lambda^{\frac{2}{\sigma(p-2)+p}} x) = \|\phi\|_{Y_\sigma(\mathbb{R}^N)}. \end{aligned} \quad (4.3)$$

This means that

$$\|D_{\sqrt{t_n}}^\sigma u_0\|_{Y_\sigma(\mathbb{R}^N)} \leq \|u_0\|_{Y_\sigma(\mathbb{R}^N)} \leq M \quad (4.4)$$

for all $n \geq 1$. Hence there exists a subsequence $\{t_{n_k}\}$, which we still write as $\{t_n\}$, and a function $\varphi \in Y_\sigma(\mathbb{R}^N)$ such that

$$D_{\sqrt{t_n}}^\sigma u_0 \xrightarrow{w^*} \varphi \quad \text{in } Y_\sigma(\mathbb{R}^N). \quad (4.5)$$

Consequently,

$$\varphi \in \Omega^\sigma(u_0) \quad \text{and} \quad \|\varphi\|_{Y_\sigma(\mathbb{R}^N)} \leq \|u_0\|_{Y_\sigma(\mathbb{R}^N)}. \quad (4.6)$$

For every $\epsilon > 0$, applying Theorem 3.2 to u_0 and φ , we see that there exists a $R > 0$ such that if $|x| \geq R$, then

$$|S(1)u_0(x)| < \frac{\epsilon}{3} \quad \text{and} \quad |S(1)\varphi(x)| < \frac{\epsilon}{3}. \quad (4.7)$$

Comparison Principle, (4.4) and (4.7) imply that if $x \geq R$, then

$$|S(1)[D_{\sqrt{t_n}}^\sigma u_0](x)| < \frac{\epsilon}{3} \tag{4.8}$$

for all $n \geq 1$. Taking

$$R(u_0) = R + 1 + C \max \left(\|u_0\|_{Y_\sigma(\mathbb{R}^N)}^{\frac{p-2}{p}}, \|u_0\|_{Y_\sigma(\mathbb{R}^N)}^{\frac{p-2}{p-\sigma(p-2)}} \right)$$

and letting $\chi_{R(u_0)}(x)$ be the cut-off function defined on $B_{R(u_0)+1}$ relative to $B_{R(u_0)}$, we see that

$$\begin{aligned} & \text{supp}[(1 - \chi_{R(u_0)})u_0] \\ & \subset \left\{ x \in \mathbb{R}^N; |x| \geq R + 1 + C \max \left(\|u_0\|_{Y_\sigma(\mathbb{R}^N)}^{\frac{p-2}{p}}, \|u_0\|_{Y_\sigma(\mathbb{R}^N)}^{\frac{p-2}{p-\sigma(p-2)}} \right) \right\}. \end{aligned}$$

Then applying Theorem 3.1 to $(1 - \chi_{R(u_0)})u_0$, we get

$$\text{supp}[S(1)(1 - \chi_{R(u_0)})u_0] \subset \{x \in \mathbb{R}^N; |x| \geq R + 1\},$$

hence

$$\text{supp}S(1)[(1 - \chi_{R(u_0)})u_0] \cap B_R = \emptyset.$$

So apply Comparison Principle to get

$$\text{supp}S(1)[(1 - \chi_{R(u_0)})D_{\sqrt{t_n}}^\sigma u_0] \cap B_R = \emptyset.$$

That is, for $x \in B_R$,

$$S(1)(D_{\sqrt{t_n}}^\sigma u_0)(x) = S(1)[\chi_{R(u_0)}(D_{\sqrt{t_n}}^\sigma u_0)](x). \tag{4.9}$$

The same result holds for $S(1)\varphi$. That is, for $x \in B_R$,

$$S(1)\varphi(x) = S(1)[\chi_{R(u_0)}\varphi](x). \tag{4.10}$$

Since $Y_\sigma(\mathbb{R}^N) \hookrightarrow \mathcal{S}'(\mathbb{R}^N) \hookrightarrow \mathcal{D}'(\mathbb{R}^N)$, (4.5) and (4.6) show that

$$D_{\sqrt{t_n}}^\sigma u_0 \rightarrow \varphi \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{4.11}$$

as $t_n \rightarrow \infty$. Observe that

$$\chi_{R(u_0)}D_{\sqrt{t_n}}^\sigma u_0, \quad \chi_{R(u_0)}\varphi \in L^1(\mathbb{R}^N).$$

It now follows from (4.11) that

$$\chi_{R(u_0)} D_{\sqrt{t_n}}^\sigma u_0 \rightharpoonup \chi_{R(u_0)} \varphi \quad \text{in } L^1(\mathbb{R}^N)$$

as $t_n \rightarrow \infty$. So for $0 < \tau < 1$, Proposition 2.1 implies that

$$S(\tau)[\chi_{R(u_0)} D_{\sqrt{t_n}}^\sigma u_0] \rightharpoonup S(\tau)[\chi_{R(u_0)} \varphi] \quad \text{in } L^1(\mathbb{R}^N)$$

as $t_n \rightarrow \infty$. This means that

$$S(\tau)[\chi_{R(t)} D_{\sqrt{t_n}}^\sigma u_0] \rightharpoonup S(\tau)[\chi_{R(t)} \varphi] \quad \text{in } \mathcal{D}'(\mathbb{R})$$

as $t_n \rightarrow \infty$. By Theorem 3.2, (4.4) and (4.6) show that there exists a constant C such that for all $n \geq 1$,

$$\|S(\tau)[\chi_{R(t)} D_{\sqrt{t_n}}^\sigma u_0]\|_{Y_\sigma(\mathbb{R}^N)} \leq C,$$

hence

$$S(\tau)[\chi_{R(t)} D_{\sqrt{t_n}}^\sigma u_0] \xrightarrow{w^*} S(\tau)[\chi_{R(t)} \varphi] \quad \text{in } Y_\sigma(\mathbb{R}^N)$$

as $t_n \rightarrow \infty$. By the regularity of the semigroup operators $S(t)$ and $0 < \tau < 1$, we obtain

$$\|S(1)[\chi_{R(t)} D_{\sqrt{t_n}}^\sigma u_0] - S(1)[\chi_{R(t)} \varphi]\|_{Y_\sigma(\mathbb{R}^N)} \rightarrow 0$$

as $t_n \rightarrow \infty$. So (4.9) and (4.10) imply that for all $x \in B_R$,

$$\begin{aligned} & (1 + |x|^2)^{-\frac{\sigma}{2}} (S(1)[D_{\sqrt{t_n}}^\sigma u_0] - S(1)[\varphi]) \\ &= (1 + |x|^2)^{-\frac{\sigma}{2}} (S(1)[\chi_{R(t)} D_{\sqrt{t_n}}^\sigma u_0] - S(1)[\chi_{R(t)} \varphi]) \rightarrow 0 \end{aligned}$$

as $t_n \rightarrow \infty$. Then it follows from (4.7) and (4.8) that

$$\|S(1)[D_{\sqrt{t_n}}^\sigma u_0] - f\|_{Y_\sigma(\mathbb{R}^N)} = \|S(1)[D_{\sqrt{t_n}}^\sigma u_0] - S(1)[\varphi]\|_{Y_\sigma(\mathbb{R}^N)} \rightarrow 0. \quad (4.12)$$

So we deduce from (4.2), (4.5) and (4.12) that

$$f = S(1)\varphi \in S(1)\Omega^\sigma(u_0).$$

This means that

$$\omega^\sigma(u_0) \subset S(1)\Omega^\sigma(u_0). \quad (4.13)$$

On the other hand, suppose

$$f \in S(1)\Omega^\sigma(u_0). \quad (4.14)$$

Then by the definition of $S(1)\Omega^\sigma(u_0)$, there exists a function $\varphi \in \Omega^\sigma(u_0)$ such that

$$f = S(1)\varphi.$$

Hence there exists a sequence $\lambda_n \rightarrow \infty$ such that

$$D_{\lambda_n}^\sigma u_0 \rightarrow \varphi \quad \text{in } Y_\sigma(\mathbb{R}^N)$$

as $\lambda_n \rightarrow \infty$. Using a same proof of (4.12), we can get

$$\|S(1)[D_{\lambda_n}^\sigma u_0] - f\|_{Y_\sigma(\mathbb{R}^N)} = \|S(1)[D_{\lambda_n}^\sigma u_0] - S(1)[\varphi]\|_{Y_\sigma(\mathbb{R}^N)} \rightarrow 0 \quad (4.15)$$

as $\lambda_n \rightarrow \infty$. Applying commutative relation (2.2) to $S(1)[D_{\lambda_n}^\sigma u_0]$, we have

$$D_{\lambda_n}^\sigma [S(1)u_0] = S(1)[D_{\lambda_n}^\sigma u_0]. \quad (4.16)$$

Then taking $t_n = \lambda_n^2$ in (4.15), and using (4.16), we have

$$D_{\lambda_n}^\sigma [S(1)u_0] \xrightarrow{t_n \rightarrow \infty} f \quad \text{in } Y_\sigma(\mathbb{R}^N).$$

This means

$$f \in \omega^\sigma(u_0),$$

so (4.14) shows that

$$S(1)\Omega^\sigma(u_0) \subset \omega^\sigma(u_0), \quad (4.17)$$

therefore

$$\omega^\sigma(u_0) = S(1)\Omega^\sigma(u_0)$$

by (4.13), and the proof is complete. \square

5 Complicated asymptotic behavior

As an application of the relation (4.1), we follow the argument in [17] to prove that $\omega^\sigma(u_0)$ contain infinite functions. This result means that these solutions possess complicated asymptotic behavior, according to Vázquez and Zuazua [9].

Theorem 5.1. *For $M > 0$, let*

$$B_M^{\sigma,+} \equiv \{\varphi \in Y_\sigma(\mathbb{R}^N); \|\varphi\|_{Y_\sigma(\mathbb{R}^N)} \leq M \text{ and } \varphi \geq 0\}.$$

Then there exists $u_0(x) \in Y_\sigma(\mathbb{R}^N)$ such that

$$\omega^\sigma(u_0) = S(1)B_M^{\sigma,+} \equiv \{f: f = S(1)\varphi, \varphi \in B_M^{\sigma,+}\}. \quad (5.1)$$

Proof. Note that $B_M^{\sigma,+}$ with the weak-star topology is compact and separable. Let $\{\phi_i\}_{i=1}^\infty$ be a dense subset of $B_M^{\sigma,+}$. Given $\{\psi_i\} \subset B_M^{\sigma,+}$ such that for every ϕ_i , there exists a subsequence $\{\psi_{i_n}\}$ of $\{\psi_i\}$ satisfying that

$$\psi_{i_n} = \phi_i, \quad \forall i_n \geq 1.$$

Suppose

$$\lambda_i = \begin{cases} a, & \text{if } i=1, \\ a^{i(p-\sigma(p-2))} \lambda_{i-1}, & \text{if } i>1, \end{cases} \quad (5.2)$$

where $a > 2$. Let $\chi_i(x)$ be the cut-off function defined on $E_i \equiv \{x \in \mathbb{R}^N; a^{-i} < |x| < a^i\}$ relative to $E_{i-1} \equiv \{x \in \mathbb{R}^N; a^{-i+1} < |x| < a^{i-1}\}$ and assume that

$$u_0(x) = \sum_{i=1}^{\infty} D_{\lambda_i}^{-\sigma}(\chi_i(x)\psi_i(x)). \quad (5.3)$$

For $i > 1$, it follows from (5.2) that

$$\lambda_i^{\frac{2}{p-\sigma(p-2)}} a^{-i} = a^i \lambda_{i-1}^{\frac{2}{p-\sigma(p-2)}} > a^{i-1} \lambda_{i-1}^{\frac{2}{p-\sigma(p-2)}}.$$

Since

$$\text{supp}(D_{\lambda_i}^{-\sigma}(\chi_i(x)\psi_i(x))) \subset \left\{x \in \mathbb{R}^N; \lambda_i^{\frac{2}{p-\sigma(p-2)}} a^{-i} < |x| < \lambda_i^{\frac{2}{p-\sigma(p-2)}} a^i\right\},$$

we get that for $i \neq j$,

$$\text{supp} D_{\lambda_i}^{-\sigma}[\chi_i(x)\psi_i(x)] \cap \text{supp} D_{\lambda_j}^{-\sigma}[\chi_j(x)\psi_j(x)] = \emptyset, \quad (5.4)$$

hence the definition of u_0 implies that

$$u_0 \in B_M^{\sigma,+}.$$

For every $\phi \in B_M^{\sigma,+}$, there exists a subsequence $\{\phi_{i_n}\}$ of $\{\phi_i\}$, which we also write as $\{\phi_i\}$, such that

$$\phi_i \xrightarrow{w*} \phi \quad \text{in } Y_\sigma(\mathbb{R}^N) \quad (5.5)$$

as $i \rightarrow \infty$. For every ϕ_i , $i \geq 1$, it follows from (5.3) that there exists a subsequence $\{\lambda_{i_n}\}$ of $\{\lambda_i\}$ such that

$$D_{\lambda_{i_n}}^\sigma u_0 = \phi_i \quad \text{in } A_{i_n-1}$$

for all $i_n \geq 1$. Since $A_{i_n-1} \rightarrow \mathbb{R}^N \setminus \{0\}$ as $i_n \rightarrow \infty$, one can get that

$$D_{\lambda_{i_n}}^\sigma u_0 \xrightarrow{w^*} \phi_i \quad \text{in } Y_\sigma(\mathbb{R}^N)$$

$i_n \rightarrow \infty$. By diagonal method, it follows from (5.5) that

$$D_{\lambda_{i_i}}^\sigma u_0 \xrightarrow{w^*} \phi \quad \text{in } Y_\sigma(\mathbb{R}^N)$$

as $\lambda_{i_i} \rightarrow \infty$. This means that $\phi \in \Omega^\sigma(u_0)$, so

$$B_M^{\sigma,+} \subset \Omega^\sigma(u_0). \quad (5.6)$$

On the other hand, if $\phi \in \Omega^\sigma(u_0)$, it follows from the definition of $\Omega^\sigma(u_0)$ that there is a sequence $\lambda_i \rightarrow \infty$ such that

$$D_{\lambda_i}^\sigma u_0 \xrightarrow{w^*} \phi \quad \text{in } Y_\sigma(\mathbb{R}^N). \quad (5.7)$$

Since $\|D_{\lambda_i}^\sigma u_0\|_{Y_\sigma(\mathbb{R}^N)} \leq \|u_0\|_{Y_\sigma(\mathbb{R}^N)} \leq M$, $\forall i \geq 1$ by (4.3), it follows from (5.7) that $\phi \geq 0$ and $\|\phi\|_{Y_\sigma(\mathbb{R}^N)} \leq M$. So $\phi \in B_M^{\sigma,+}$, this means that

$$\Omega^\sigma(u_0) \subset B_M^{\sigma,+},$$

hence

$$\Omega^\sigma(u_0) = B_M^{\sigma,+} \quad (5.8)$$

holds by (5.6). It follows from Theorem 4.1 and (5.8) that

$$\omega^\sigma(u_0) = S(1)B_M^{\sigma,+},$$

and the proof is complete. \square

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