

## Optimal Defined Contribution Pension Management with Salary and Risky Assets Following Jump Diffusion Processes

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**Abstract.** The paper considers an optimal asset allocation problem for a defined contribution pension plan during the accumulation phase. The salary follows a stochastic process, which combines a compound Poisson jump with Brownian uncertainty. The plan aims to minimise the quadratic loss function over finite time horizon by investing in the market of risky assets and bank account. The risky assets are subjected to Poisson jump and Brownian motion. The closed-form optimal investment decision is derived from the corresponding Hamilton-Jacobi-Bellman equation.

**AMS subject classifications:** 97M30, 93E20

**Key words:** Compound Poisson process, defined contribution pension plan, stochastic optimal control, dynamic programming approach, Hamilton-Jacobi-Bellman equation.

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### 1. Introduction

Pension is an important financial instrument for individuals to reallocate their wealth from working life to retirement. Generally, there are two kinds of pension arrangements — viz. defined contribution (DC) and defined benefit (DB) plans. In the former case, the contributions are fixed in advance by the pension sponsor and the benefits depend on the investment earnings, so that the majority of risks are borne by the individual itself. In the other pension plan, the benefits are fixed in the contract, while the contributions are designed by the sponsor to keep the fund in balance. Thus the sponsor bears the majority of risks and the individual does not experience any losses.

Generally, a pension plan consists of accumulation and decumulation phases. In the first phase, also called the contribution phase, the pensioner pays contributions to the pension

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trustee during the employment period. In the other one, a pension annuity or a lump sum is received and can be converted to a whole life assurance with a death benefit.

There are a variety of works concentrating on the optimal investment and the management of DC pension scheme. For instance, Haberman and Vigna [12] used a dynamic programming approach to derive a formula for the optimal investment allocation in the DC pension scheme whose funds are invested in an  $n$  asset market. For general multi-asset financial markets with stochastic investment opportunities and stochastic contributions, Menoncin and Vigna [15] solved a mean-variance optimisation problem in the accumulation phase of DC pension schemes. Under the requirement of inflation protected guarantee, Tang *et al.* [19] obtained an optimal asset allocation decision for economic environment with risks arising from real and nominal interest rates. Other optimal management and investment problems for DC pension funds have been recently studied by Wang *et al.* [21] and Josa-Fombellida *et al.* [8].

In order to model the dynamics of risky assets, a geometric Brownian motion is widely used. However, it does not properly match the market prices, so that the jump diffusion model seems to be more appropriate. Considering the optimal consumption and portfolio rules, Merton [16] studied the Poisson jumps in a dynamic portfolio problem. Ngwira and Gerrard [17] investigated DB pension management problems and developed an optimal contribution and investment strategy incorporating jumps into the risky asset price process. Delong *et al.* [5] and Liang *et al.* [14] considered mean-variance problems under the assumption that the dynamics of the stock price is driven by a Lévy noise. On the other hand, the jump diffusion stock price model for the DC pension management remains little studied. In particular, Sun *et al.* [18] analysed the precommitment and equilibrium investment strategies under the assumption that the stock prices follow a jump diffusion process.

However, using the approach of Sun *et al.* [18], one can show that the investment strategy is only optimal at time 0, and the pension trustee is assumed to be precommitted to the target determined at the initial time. This problem motivated us to use the dynamic programming approach to consider a closed-form investment strategy for DC type pension plans. The stock price is stochastic and driven by a Brownian motion and a compound Poisson jump. In contrast to [18], investment strategy in our work is optimal not only at the initial time but also in what follows.

It is worth well to use the jump diffusion processes to study the salary dynamics, since salary may not raise gradually and continuously. It can have a positive jump when the employee is promoted from a lower to a higher position or if another company proposes a higher salary. In fact, such models are already used in the salary studies — e.g. assuming that salary follows a jump diffusion process, Bian *et al.* [1] developed an optimal retirement strategy of a DB type pension plan. The model is also employed by Calvo-Garrido and Vázquez [4] in pricing pension plans.

We also note that in DB pension scheme, the benefits are partially or totally determined by salaries. In particular, a shock in salary can produce shocks in the evolution of benefits. Therefore, Josa-Fombellida and Rincón-Zapatero [9] consider jumps in the evolution of benefits in a DB pension plan. This is another reason for using the jump diffusion processes

in the description of salary evolution. These examples motivate us to use the jump diffusion salary model in optimisation of the DC pension scheme. Moreover, since the salary has a close connection to contribution rate, we consider contribution rate as a jump diffusion process in a DC pension optimal management problem.

The rest of the paper is structured as follows. Section 2 describes the financial market with stochastic interest rate and two tradeable assets, which are of interest to our pension management. Section 3 introduces stochastic salary processes and considers a stochastic optimal control problem in DC pension scheme. This allows to minimise the quadratic loss function over a finite horizon. A closed form solution of this stochastic control problem is obtained from a related HJB equation. Section 4 is devoted to sensitivity analysis of the previous results. Finally, our conclusions are Section 5.

## 2. Notations and Assumptions

We consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a probability measure  $\mathbb{P}$  on  $\Omega$  and  $\mathcal{F} = \mathcal{F}^B \vee \mathcal{F}^N$ . The filtration  $\mathcal{F}^B = \{\mathcal{F}_t^B\}_{t \geq 0}$  is generated by a three dimensional Brownian motion  $(B_r, B_S, B_L)$  — i.e.

$$\mathcal{F}_t^B = \sigma \{(B_r(s), B_S(s), B_L(s)); 0 \leq s \leq t\}.$$

Suppose that there is an instantaneous correlation  $\rho(t) \in [-1, 1]$  between  $B_r$  and  $B_S$ . For simplicity, we assume that  $B_S$  and  $B_L$  are independent and fluctuations in financial market and the future salary have little influence on each other. The filtration  $\mathcal{F}^N = \{\mathcal{F}_t^N\}_{t \geq 0}$  is generated by a two-dimensional Poisson process  $(N_S, N_L)$  with constant intensity  $(\lambda_S, \lambda_L)$ ,  $\lambda_S, \lambda_L \in \mathbb{R}_+$  — i.e.

$$\mathcal{F}_t^N = \sigma \{(N_S(s), N_L(s)); 0 \leq s \leq t\}.$$

Assuming that  $N_S$  and  $N_L$  are independent, we get that the corresponding Brownian motions are independent of Poisson processes on this space.

Let  $r(t)$  denote the instantaneous interest rate. We assume that it is an Ornstein-Uhlenbeck process — i.e. it satisfies the equation

$$dr(t) = (a - b r(t))dt + \sigma_r dB_r(t), \quad r(0) = r_0. \quad (2.1)$$

Such mean-reverting type processes have been introduced by Vasicek [20] in order to describe the dynamics of the interest rates. We note that  $a$  is the long-term mean of the interest rate,  $b$  the mean reversion coefficient and  $\sigma_r$  the volatility associated with the diffusion component. All parameters in the Eq. (2.1) are positive constants.

We consider a market of two tradeable underlying instruments, which are traded continuously over time, are perfectly divisible and satisfy the following conditions:

1. The money market account  $S_0(t)$  follows the stochastic differential equation

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1. \quad (2.2)$$

2. The dynamics of the risky asset is described by the equation

$$dS(t) = \mu_S(r(t))S(t)dt + \sigma_S S(t)dB_S(t) + S(t-)d \sum_{i=1}^{N_S(t)} Y_i^S, \quad S(0) = S_0, \quad (2.3)$$

where  $\mu_S(r) = r + \xi$  is the mean rate of the return,  $\xi \in \mathbb{R}_+$  the expectation of the excess return from investing in the risky asset,  $\sigma_S$  a constant volatility associated with the diffusion component, and  $\{Y_i^S\}$  is a sequence of independent and identically distributed variables with finite first and second moments  $\mu_1^S$  and  $\mu_2^S$  independent of  $\{N_S\}$ . Note that  $B_S$  and  $N_S$  do not depend on stochastic processes.

Let  $\theta$  be the Sharpe ratio of risk for the portfolio — cf. [2], defined by

$$\theta = \frac{\xi + \lambda_S \mu_1^S}{\sqrt{\sigma_S^2 + \lambda_S \mu_2^S}}. \quad (2.4)$$

We also assume that  $\mu + \lambda_S \mu_1^S > r$  or, equivalently,  $\xi > -\lambda_S \mu_1^S$  so that the risky asset is attractive to the pension manager.

### 3. Optimal Portfolio

Consider the accumulation phase of a typical DC pension plan where the employee pays the contributions during the employment period [11]. According to [13], the contribution associated with the pension fund is a constant proportion  $\kappa$  of the participant's salary. However, because of possible abruptness in the salary, it is more realistic to use the jump diffusion model in order to describe the evolution of salary. Let us suppose that the instantaneous salary  $L(t)$  at time  $t$  is described by the following stochastic differential equation:

$$dL(t) = \mu_L L(t)dt + \sigma_L L(t)dB_L(t) + L(t-)d \sum_{i=1}^{N_L(t)} Y_i^L, \quad L(0) = L_0, \quad (3.1)$$

where  $\mu_L$  and  $\sigma_L$  are, respectively, the constant drift and volatility of the salary, and  $\{Y_i^L\}$  is the sequence of independent and identically distributed random variables with finite first and second moments  $\mu_1^L$  and  $\mu_2^L$ , which do not depend on  $\{N_L\}$ . Note that  $B_L$  and  $N_L$  are independent stochastic processes.

The stochastic control problem consists in finding an optimal design for the assets in the accumulation phase of a DC pension plan from the hire date  $t = 0$  to the retirement  $t = T$  of the pension member. Note that for a chosen investment policy, the fund evolution is described by the equation

$$dX(t) = (X(t) - \pi(t)) \frac{dS_0(t)}{S_0(t)} + \pi(t) \frac{dS(t)}{S(t-)} + \kappa L(t)dt, \quad X(0) = X_0 > 0, \quad (3.2)$$

where  $\pi(t)$  is the capital invested in stocks at time  $t$  and  $X(t) - \pi(t)$  the investment in the bank account. Borrowing and short-selling are also possible and negative value of  $\pi(t)$  means that the manager takes a short position of risky assets, while negative  $X(t) - \pi(t)$  means the borrowing from the bank at the interest rate  $r(t)$  to purchase risky assets.

Substituting (2.2) and (2.3) into the Eq. (3.2) yields

$$dX(t) = [r(t)X(t) + (\mu_S - r(t))\pi(t) + \kappa L(t)] dt + \pi(t)\sigma_S dB_S(t) + \pi(t-)d \sum_{i=1}^{N_S(t)} Y_i^S. \quad (3.3)$$

Suppose that the investment strategy  $\pi$  is an  $\mathcal{F}_t$ -measurable function such that

$$\mathbb{E} \left\{ \int_0^\infty \pi^2(t) dt \right\} < \infty, \quad (3.4)$$

and the sponsor preference is modeled by quadratic, penalising deviations from prescribed targets with the quadratic loss function

$$Q(t, X(t)) = (X(t) - F(t))^2. \quad (3.5)$$

The target of the pension manager is described by the function  $F(t)$ . According to [10],  $F$  is the price of the desired annuity in a DC pension plan. If the pension plan is of the DB type, then  $F$  can be connected to the benefits provided by the pension trustee.

Any deviation from the target  $F$  are penalised so that if wealth  $X$  is different from the target, a penalty measured by the loss function must be paid. For pension plans, there are various applications of the quadratic loss function — cf. [3, 10]. We note that the deviation above the target is also penalised by the quadratic loss function, which is a drawback of the model. Nevertheless, according to [10], the target is a natural limitation to the overall risk of the portfolio and as soon as it is reached, there is no need to take a risk so that the surplus becomes undesirable. On account of these considerations, we adopt the modified quadratic loss function from [6], which has the form

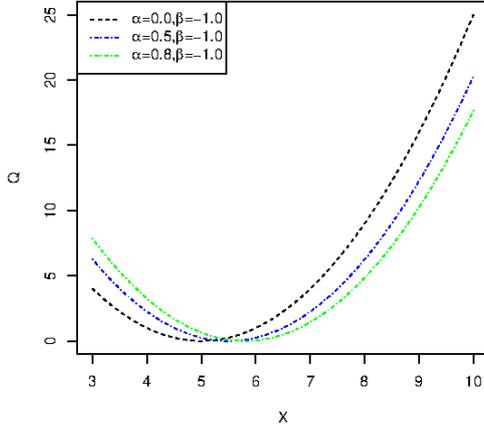
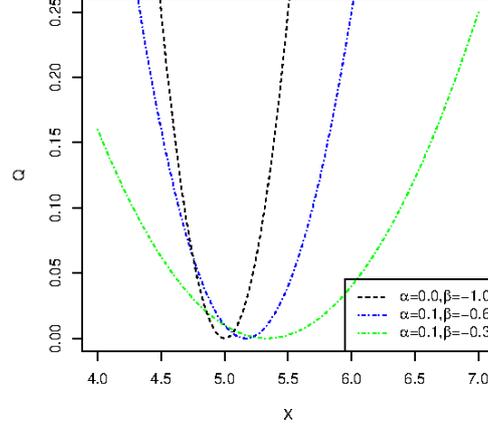
$$Q(t, X(t)) = [\alpha + \beta(X(t) - F(t))]^2, \quad (3.6)$$

where  $\alpha > 0$  and  $\beta < 0$ , so that under-funding is more penalised than over-funding. An additional advantage of the quadratic loss function is that it allows to establish an explicit solution of the stochastic optimal control problem and a closed-form investment decision.

Figs. 1 and 2 demonstrate the role of the parameters  $\alpha$  and  $\beta$  in the modified loss function (3.6) with the target  $F \equiv 5$ . If  $\alpha = 0$  and  $\beta = -1$ , function (3.6) becomes the initial loss function (3.5). However, the pension manager may choose different  $\alpha$  and  $\beta$  to avoid the deficiencies of our model.

Let us assume that the sponsor preference is to minimise the expected value of the terminal quadratic loss. We consider objective function  $J(t, X, L, r)$  defined by

$$J(t, X, L, r) = \mathbb{E}_t[Q(T, X(T))],$$

Figure 1: Impact of  $\alpha$ .Figure 2: Impact of  $\beta$ .

where  $\mathbb{E}_t$  is the conditional expectation given all the information up to time  $t$ . In order to solve the stochastic optimisation problem, we use a dynamic programming approach. Consider the value function

$$V(t, X, L, r) = \min_{\{\pi \in \mathcal{A}_{X,L,r}\}} \{J(t, X, L, r) : \text{subject to (3.3), (3.1), (2.1)}\},$$

and let  $\mathcal{A}_{X,L,r}$  denote the class of all admissible controls — i.e. it is the set of all measurable processes  $\{\pi(t)\}_{t \geq 0}$  such that  $\pi$  satisfies the Eq. (3.4) and  $X$ ,  $L$  and  $r$  satisfy the Eqs. (3.3), (3.1) and (2.1), respectively.

In stochastic optimal control theory, the HJB equation provides the connection between the value function and optimal control [7, 23]. We consider the function  $v(t, X, L, r) = [\alpha + \beta(X(T) - F(T))]^2$  and let  $v_t$ ,  $v_X$ ,  $v_L$ ,  $v_r$ ,  $v_{XX}$ ,  $v_{LL}$ ,  $v_{rr}$ , and  $v_{X,r}$  be the partial derivatives of first and second order of the function  $v$  with respect to  $t$ ,  $X$ ,  $L$  and  $r$ .

**Theorem 3.1.** *Assume that  $v(t, X, L, r) \in C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R} \times (0, +\infty)) \cap C([0, T] \times \mathbb{R} \times \mathbb{R} \times (0, +\infty))$  and satisfies the equation*

$$\begin{aligned} & v_t + [rX + (\mu_S - r)\pi + \kappa L]v_X + \mu_L L v_L + (a - b r)v_r \\ & + \frac{1}{2}\pi^2 \sigma_S^2 v_{XX} + \frac{1}{2}\sigma_L^2 L^2 v_{LL} + \frac{1}{2}\sigma_r^2 v_{rr} + \pi \sigma_S \sigma_r \rho v_{X,r} \\ & + \lambda_S [\mathbb{E}v(t, X + \pi Y_1^S, L, r) - v(t, X, L, r)] \\ & + \lambda_L [\mathbb{E}v(t, X, L(1 + Y_1^L), r) - v(t, X, L, r)] \\ & \equiv G(v; \pi) \geq 0 \end{aligned}$$

for any  $\pi \in \mathcal{A}_{X,L,r}$  and any  $(t, X, L, r) \in ([0, T] \times \mathbb{R} \times \mathbb{R} \times (0, +\infty))$ . Then

$$v(t, X, L, r) \geq V(t, X, L, r)$$

for any  $\pi \in \mathcal{A}_{X,L,r}$ . Moreover, if there exists an admissible strategy  $\pi^* \in \mathcal{A}_{X,L,r}$  such that  $G(v; \pi) = 0$  for all  $(t, X, L, r) \in ([0, T] \times \mathbb{R} \times \mathbb{R} \times (0, +\infty))$ , then

$$v(t, X, L, r) = V(t, X, L, r)$$

and this strategy is optimal.

The proof of this result is standard and omitted here.

We set

$$\varpi := \theta^2, \quad \vartheta(t) := \frac{\sigma_S \sigma_r \rho(t)(\xi + \lambda_S \mu_1^S)}{\sigma_S^2 + \lambda_S \mu_2^S}, \quad \zeta(t) := \frac{\sigma_S^2 \sigma_r^2 \rho^2(t)}{\sigma_S^2 + \lambda_S \mu_2^S},$$

where  $\theta$  is the Sharpe ratio (2.4). An explicit optimal investment strategy is given by the following theorem.

**Theorem 3.2.** Choose  $\tau \leq T$  and let

$$\begin{aligned} \gamma(t) &= \frac{2}{b}(1 - e^{-b(T-t)}), \\ \delta(t) &= \beta^2 e^{\int_t^T h(s) ds}, \\ h(t) &= (a - 2\vartheta)\gamma(t) + \left(\frac{1}{2}\sigma_r^2 - \zeta\right)\gamma^2(t) - \varpi, \\ g(t) &= \frac{1}{b}(1 - e^{-b(T-t)}), \\ f(t) &= 2(\alpha\beta - \beta^2 F(T)) e^{\int_t^T l(s) ds}, \\ l(t) &= (a - \vartheta - \zeta\gamma(t))g(t) + \frac{1}{2}\sigma_r^2 g^2(t) - \varpi - \vartheta\gamma(t), \\ \omega(t; \tau) &= \gamma(\tau)e^{-b(\tau-t)} + \frac{1}{b}(1 - e^{-b(\tau-t)}), \\ \varepsilon(t; \tau) &= 2\kappa\delta(\tau)e^{\int_t^\tau y(s) ds}, \\ y(t) &= \mu_L + \lambda_L \mu_1^L - \varpi - \vartheta\gamma(t) + (a - \vartheta - \zeta\gamma(t))\omega(t; \tau) + \frac{1}{2}\sigma_r^2 \omega^2(t; \tau). \end{aligned} \tag{3.7}$$

Then the investment strategy

$$\begin{aligned} \pi^*(t) &= -\frac{\xi + \lambda_S \mu_1^S}{\sigma_S^2 + \lambda_S \mu_2^S} \left[ X + \left( \frac{1}{2\delta(t)} \int_t^T \varepsilon(t; \tau) e^{(\omega(t; \tau) - \gamma(t))r} d\tau \right) L + \frac{f(t)}{2\delta(t)} e^{(g(t) - \gamma(t))r} \right] \\ &\quad - \frac{\sigma_S \sigma_r \rho}{\sigma_S^2 + \lambda_S \mu_2^S} \left[ \gamma(t)X + \left( \frac{1}{2\delta(t)} \int_t^T \varepsilon(t; \tau) \omega(t; \tau) e^{(\omega(t; \tau) - \gamma(t))r} d\tau \right) L \right. \\ &\quad \left. + \frac{f(t)g(t)}{2\delta(t)} e^{(g(t) - \gamma(t))r} \right] \end{aligned} \tag{3.8}$$

is optimal and the optimal quadratic loss function has the form

$$\begin{aligned}
V(t, X, L, r) = & \delta(t)e^{\gamma(t)r} X^2 + f(t)e^{g(t)r} X \\
& + \left[ \int_t^T \int_{-\infty}^{+\infty} f_C(\tau, \zeta) \frac{e^{-((n-\zeta)^2)/4m}}{2\sqrt{\pi m}} e^{(2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L))(\tau-t)} d\zeta d\tau \right] L^2 \\
& + \left[ \int_t^T \int_{-\infty}^{+\infty} f_D(\tau, \zeta) \frac{e^{-((n-\zeta)^2)/4m}}{2\sqrt{\pi m}} e^{(\mu_L + \lambda_L \mu_1^L)(\tau-t)} d\zeta d\tau \right] L \\
& + \left[ \int_t^T \varepsilon(t; \tau) e^{\omega(t; \tau)r} d\tau \right] XL \\
& + (\alpha - \beta F)^2 - \int_t^T \frac{1}{2} \varpi \frac{f^2(t)}{2\delta(t)} + \vartheta \frac{f^2(t)g(t)}{2\delta(t)} + \frac{1}{2} \varsigma \frac{f^2(t)g^2(t)}{2\delta(t)} ds,
\end{aligned}$$

where

$$\begin{aligned}
f_C(t, r) = & -\frac{1}{4} \delta^{-1}(t) e^{-\gamma(t)r} \left[ \theta \int_t^T \varepsilon(t; \tau) e^{\omega(t; \tau)r} d\tau \right. \\
& \left. + \frac{\sigma_S \sigma_r \rho(t)}{\sqrt{\sigma_S^2 + \lambda_S \mu_2^S}} \int_t^T \varepsilon(t; \tau) \omega(t; \tau) e^{\omega(t; \tau)r} d\tau \right]^2 \\
& + \kappa \int_t^T \varepsilon(t; \tau) e^{\omega(t; \tau)r} d\tau, \\
f_D(t, r) = & -\frac{1}{2} f(t) \delta^{-1}(t) e^{-g(t)r} \left[ (\varpi + \vartheta g(t)) \int_t^T \varepsilon(t; \tau) e^{\omega(t; \tau)r} d\tau \right. \\
& \left. + (\varsigma g(t) + \vartheta) \int_t^T \varepsilon(t; \tau) \omega(t; \tau) e^{\omega(t; \tau)r} d\tau \right] + \kappa f(t) e^{g(t)r},
\end{aligned}$$

and

$$m = \frac{\sigma_r^2}{4b} (1 - e^{-2b(\tau-t)}), \quad n = r e^{-b(\tau-t)} + \frac{a}{b} (1 - e^{-b(\tau-t)}).$$

The proof of this result is given in Appendix.

Figs. 3 and 4 show possible paths for the optimal strategy  $\pi^*$  and the optimal fund  $X^*$ .

#### 4. Sensitivity Analysis

In order to investigate the influence of parameters on the optimal investment decision, we provide a sensitivity analysis. Unless otherwise stated, the relevant parameters are chosen as follows:  $\xi = 0.01$ ,  $\sigma_S = 0.5$ ,  $\mu_1^S = 0.1$ ,  $\mu_2^S = 0.8$ ,  $\lambda_S = 0.3$ ,  $\mu_L = 0.2$ ,  $\sigma_L = 0.5$ ,

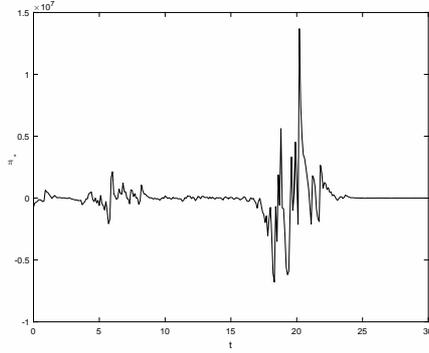


Figure 3: Possible path of optimal strategy.

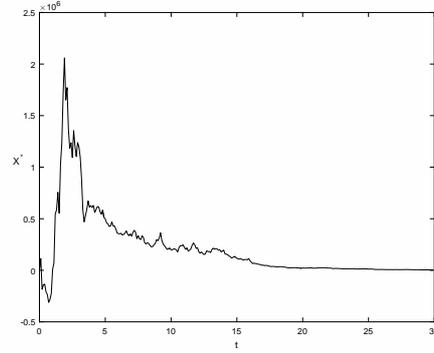
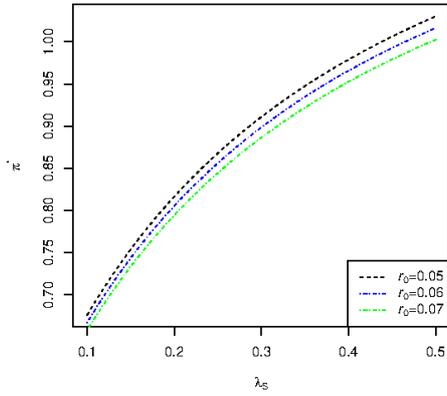
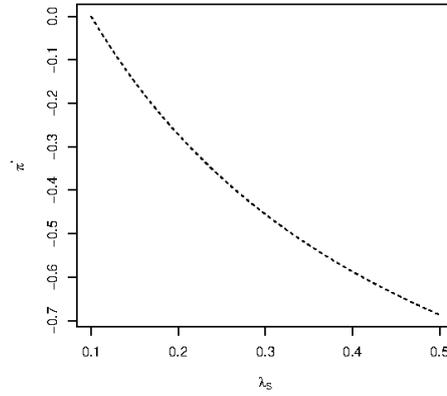


Figure 4: Possible path of optimal fund.

Figure 5: Impact of  $\lambda_S$  with positive  $\mu_1^S$ .Figure 6: Impact of  $\lambda_S$  with negative  $\mu_1^S$ .

$\mu_1^L = 0.3$ ,  $\mu_2^L = 0.8$ ,  $\lambda_L = 0.1$ ,  $\sigma_r = 0.1$ ,  $a = 0.1$ ,  $b = 1$ ,  $t = 0$ ,  $T = 30$ ,  $X_0 = 1$ ,  $L_0 = 1$ ,  $F = 5$ ,  $r_0 = 0.05$ ,  $\rho = 0$ ,  $\kappa = 0.1$ ,  $\alpha = 0.1$  and  $\beta = -0.1$ .

Fig. 5 shows the connection between the intensity of the Poisson jump of the risky asset and the optimal investment amount at the initial time for the positive mean of the jump size  $\mu_1^S$ . Note that the investment in stock grows along with the increase of  $\lambda_S$ . Moreover, the risky assets become more attractive if the probability of positive jumps is high. Fig. 6 shows the connection between  $\lambda_S$  and  $\pi^*$  for  $\mu_1^S = -0.1$ . For negative jump size, the investor holds a short position of the stock, and the absolute value of  $\pi^*$  becomes higher when the intensity of jump becomes higher. Fig. 6 also shows that for the same  $\lambda_S$ , the investment  $\pi^*$  admits lower values with higher interest rates. The manager transfers a part of the fund from the stock into the bank account, since the later has higher return and lower risk as is shown in Fig. 13.

High volatility of jump size of the stocks leads to a higher risk. In order to avoid risk, the optimal investment in the risky asset decreases with the grows of  $\mu_2^S$  and positive mean jump size as Fig. 7 shows. The absolute value of the optimal investment also diminishes if the mean jump size  $\mu_1^S$  is negative and  $\mu_2^S$  grows — cf. Fig. 8. Similar to the role of  $\mu_2^S$ ,

a higher volatility constant  $\sigma_S$  indicates the high risk of stocks. Figs. 9 and 10 demonstrate the influence of  $\sigma_S$  on the optimal decision for  $\mu_1^S = 0.1$  and  $\mu_1^S = -0.1$ , respectively. Figs. 9 and 10 also show that for the same  $\sigma_S$ , the investment  $\pi^*$  has a lower (absolute) value with a longer terminal time  $T$ . An explanation of this phenomenon is that the investment becomes more risky over a longer time. Therefore, it got smaller in order to avoid risk, as demonstrated in Fig. 14. Fig. 11 shows that the pension trustee holds a lower amount of risky assets with a higher initial earning. Since the target  $F(T)$  remains unchanged, it might be easier to reach the target with a higher  $L_0$ , without investing a large amount of money into stocks. Fig. 12 demonstrates the impact of the target  $F(T)$ . For higher targets, the manager has to invest more into risky assets to obtain a higher return. Figs. 15 and 16 show the influence of salary parameters on the optimal investment strategy. In particular, if a higher jump scale  $\mu_1^L$  or a higher jump intensity  $\lambda_L$  is expected, pensioners take less risky assets. Similar to the discussions concerning Fig. 11, it is easier to reach the same target with a higher  $\mu_1^L$  or  $\lambda_L$ . Therefore, it is not necessary to make large investment into risky assets. Note that Figs. 17 and 18 show the impact of  $\mu_1^L$  and  $\lambda_L$  along  $[0, T]$ .

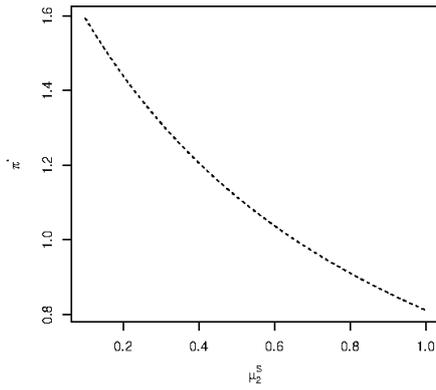


Figure 7: Impact of  $\mu_2^S$  with positive  $\mu_1^S$ .

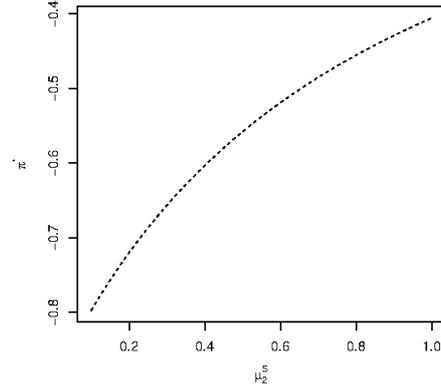


Figure 8: Impact of  $\mu_2^S$ ,  $\mu_1^S = -0.1$ .

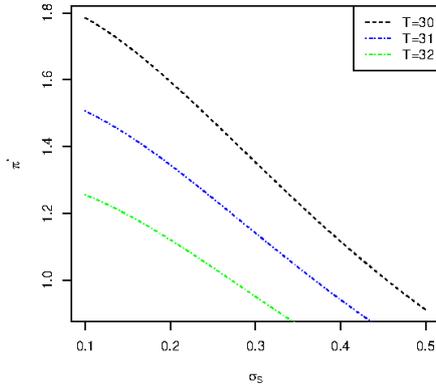


Figure 9: Impact of  $\sigma_S$  with positive  $\mu_1^S$ .

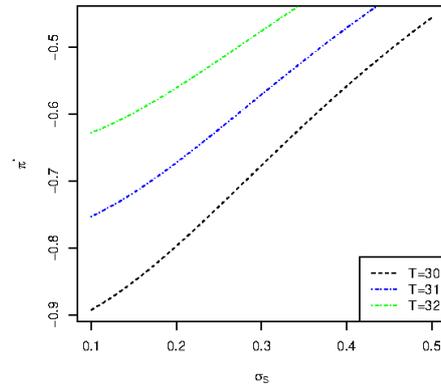


Figure 10: Impact of  $\sigma_S$  with negative  $\mu_1^S$ .

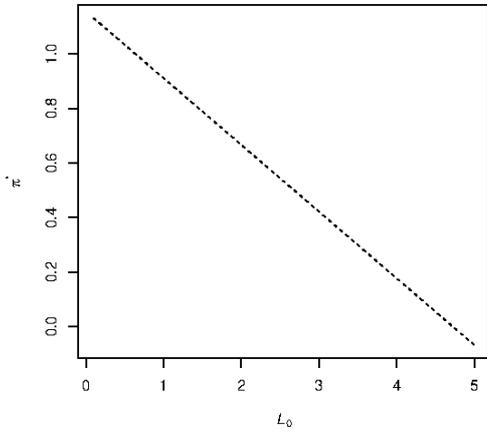


Figure 11: Impact of  $L_0$ .

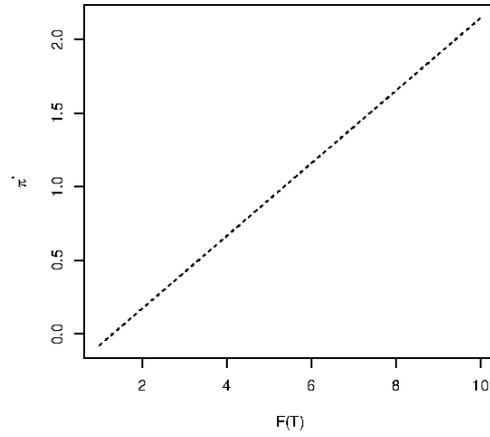


Figure 12: Impact of  $F(T)$ .

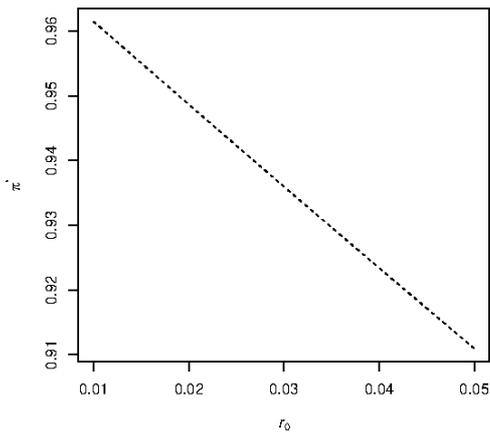


Figure 13: Impact of  $r_0$ .

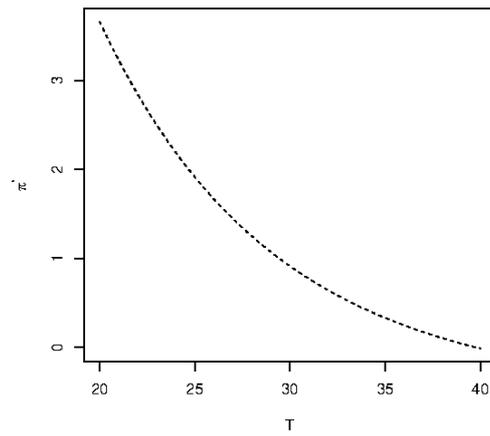


Figure 14: Impact of  $T$ .

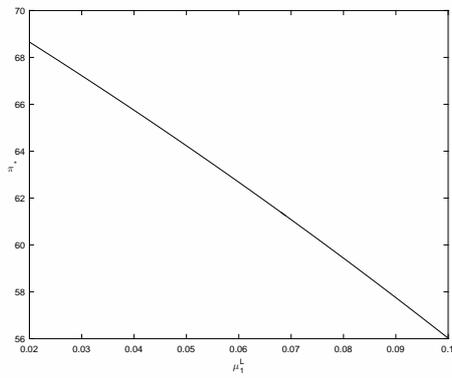


Figure 15: Impact of  $\mu_1^L$ .

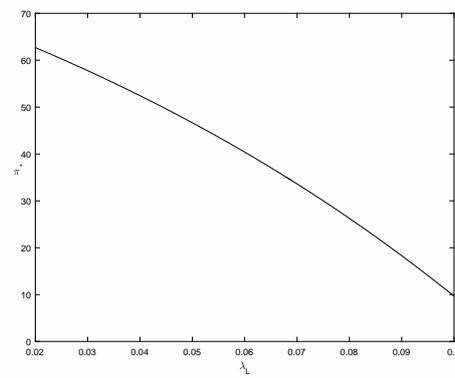
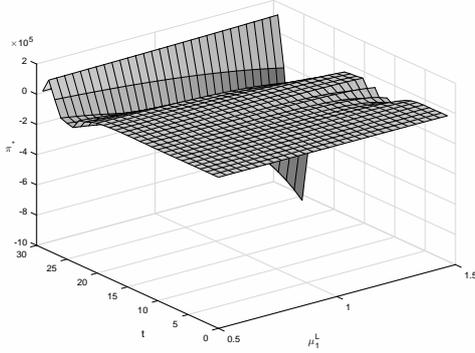
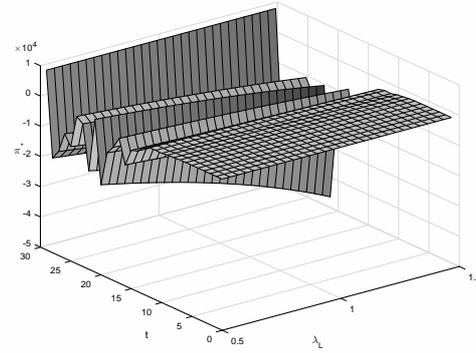


Figure 16: Impact of  $\lambda_L$ .

Figure 17: Impact of  $\mu_1^L$  and  $t$ .Figure 18: Impact of  $\lambda_L$  and  $t$ .

## 5. Conclusion

This work studies the accumulation phase of typical DC pension plans when the contributions are paid during the employment period. The contributions constitute a fixed fraction  $\kappa$  of the participant's salary. At the same time, the pension manager invests contributions into financial market of risky assets and bank accounts. The manager determines the optimal investment strategy to keep the fund to the target prescribed and minimise the quadratic loss function. In order to avoid the deficiencies of the general loss function, a modified loss function is adopted — cf. Eq.(3.6), so that the pension manager can operate with different parameters  $\alpha$  and  $\beta$ .

In the DC type pension plans, the salary and risky assets follow the jump diffusion processes. In order to determine an investment strategy, the dynamic programming technique is used and an explicit solution of the optimal problem is derived from the corresponding HJB equation. Analytic solutions of these equations are presented in Appendix.

In Section 3 we show that the first summand of the optimal investment strategy in the Eq. (3.8) is proportional to  $(\xi + \lambda_S \mu_1^S)/(\sigma_S^2 + \lambda_S \mu_2^S)$ . This corresponds to  $\xi/\sigma_S^2$ , which is called "the optimal growth portfolio strategy" in the model without jumps in the risky assets — i.e. if  $\mu_1^S = \mu_2^S \equiv 0$ .

In Section 4 we show that risky assets become more attractive in the case of the higher probability of positive jumps and lower volatility. It also indicates that stocks become less attractive for a higher initial value, expected rate and the probability of positive jumps of the salary.

## Appendix A

*Proof of Theorem 3.2.* We assume that the value function  $V$  has the form

$$V(t, X, L, r) = A(t, r)X^2 + B(t, r)X + C(t, r)L^2 + D(t, r)L + E(t, r)XL + G(t, r), \quad (\text{A.1})$$

where functions  $A, B, C, D, E$  and  $G$  satisfy the conditions

$$A(T, r) = \beta^2, \quad B(T, r) = 2(\alpha\beta - \beta^2 F(T)),$$

$$G(T, r) = (\alpha - \beta F(T))^2, \quad C(T, r) = D(T, r) = E(T, r) = 0.$$

Moreover, we also assume that  $V \in C^{1,2}$  and  $V_{XX} > 0$ .

Substituting the representation (A.1) into the HJB equation and using the first order condition, we arrive at the optimal investment decision

$$\pi^* = -\frac{\mu_s - r + \lambda_s \mu_1^S}{\sigma_s^2 + \lambda_s \mu_2^S} \left( X + \frac{E}{2A} L + \frac{B}{2A} \right) - \frac{\sigma_s \sigma_r \rho}{\sigma_s^2 + \lambda_s \mu_2^S} \left( \frac{A_r}{A} X + \frac{E_r}{2A} L + \frac{B_r}{2A} \right), \quad (\text{A.2})$$

where  $A_r$ ,  $B_r$  and  $E_r$  are respectively partial derivatives of  $A$ ,  $B$  and  $E$  with respect to  $r$ . Other partial derivatives are denoted analogously.

Substituting the Eq. (A.2) into the HJB equation and rearranging it, we obtain

$$\begin{aligned} & \left[ A_t + (2r - \varpi)A + ((a - b)r - 2\vartheta)A_r + \frac{1}{2}\sigma_r^2 A_{rr} - \varsigma \frac{A_r^2}{A} \right] X^2 \\ & + \left[ B_t + \left( r - \varpi - \vartheta \frac{A_r}{A} \right) B + \left( (a - b)r - \vartheta - \varsigma \frac{A_r}{A} \right) B_r + \frac{1}{2}\sigma_r^2 B_{rr} \right] X \\ & + \left[ C_t + (2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L))C + (a - b)rC_r + \frac{1}{2}\sigma_r^2 C_{rr} \right. \\ & \left. - \frac{1}{4}\varpi \frac{E^2}{A} - \frac{1}{2}\vartheta \frac{EE_r}{A} - \frac{1}{4}\varsigma \frac{E_r^2}{A} + \kappa E \right] L^2 \\ & + \left[ D_t + (\mu_L + \lambda_L \mu_1^L)D + (a - b)rD_r + \frac{1}{2}\sigma_r^2 D_{rr} - \frac{1}{2}\varpi \frac{BE}{A} - \frac{1}{2}\varsigma \frac{B_r E_r}{A} \right. \\ & \left. - \vartheta \frac{B_r E + BE_r}{2A} + \kappa B \right] L \\ & + \left[ E_t + \left( r + \mu_L + \lambda_L \mu_1^L - \varpi - \vartheta \frac{A_r}{A} \right) E + \left( (a - b)r - \vartheta - \varsigma \frac{A_r}{A} \right) E_r \right. \\ & \left. + \frac{1}{2}\sigma_r^2 E_{rr} + 2\kappa A \right] XL \\ & + \left[ G_t + (a - b)rG_r + \frac{1}{2}\sigma_r^2 G_{rr} - \frac{1}{2}\varpi \frac{B^2}{2A} - \vartheta \frac{B_r B}{2A} - \frac{1}{2}\varsigma \frac{B_r^2}{2A} \right] = 0. \end{aligned} \quad (\text{A.3})$$

The Eq. (A.3) is a bivariate polynomial of  $X$  and  $L$ . It is valid for all  $X$  and  $L$ . Therefore, this equation is equivalent to the following six PDEs:

$$A_t + (2r - \varpi)A + ((a - b)r - 2\vartheta)A_r + \frac{1}{2}\sigma_r^2 A_{rr} - \varsigma \frac{A_r^2}{A} = 0, \quad (\text{A.4})$$

$$A(T, r) = \beta^2,$$

$$B_t + \left( r - \varpi - \vartheta \frac{A_r}{A} \right) B + \left( (a - b)r - \vartheta - \varsigma \frac{A_r}{A} \right) B_r + \frac{1}{2}\sigma_r^2 B_{rr} = 0, \quad (\text{A.5})$$

$$B(T, r) = 2(\alpha\beta - \beta^2 F),$$

$$G_t + (a - b r)G_r + \frac{1}{2}\sigma_r^2 G_{rr} - \frac{1}{2}\varpi \frac{B^2}{2A} - \vartheta \frac{B_r B}{2A} - \frac{1}{2}\varsigma \frac{B_r^2}{2A} = 0, \quad (\text{A.6})$$

$$\begin{aligned} G(T) &= (\alpha - \beta F)^2, \\ E_t + \left( r + \mu_L + \lambda_L \mu_1^L - \varpi - \vartheta \frac{A_r}{A} \right) E + \left( (a - b r) - \vartheta - \varsigma \frac{A_r}{A} \right) E_r \\ &\quad + \frac{1}{2}\sigma_r^2 E_{rr} + 2\kappa A = 0, \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} E(T, r) &= 0, \\ C_t + (2\mu_L + \sigma_L^2 + \lambda_L (\mu_2^L + 2\mu_1^L)) C + (a - b r)C_r + \frac{1}{2}\sigma_r^2 C_{rr} - \frac{1}{4}\varpi \frac{E^2}{A} \\ &\quad - \frac{1}{2}\vartheta \frac{EE_r}{A} - \frac{1}{4}\varsigma \frac{E_r^2}{A} + \kappa E = 0, \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} C(T, r) &= 0, \\ D_t + (\mu_L + \lambda_L \mu_1^L) D + (a - b r)D_r + \frac{1}{2}\sigma_r^2 D_{rr} - \frac{1}{2}\varpi \frac{BE}{A} - \frac{1}{2}\varsigma \frac{B_r E_r}{A} \\ &\quad - \vartheta \frac{B_r E + BE_r}{2A} + \kappa B = 0, \end{aligned} \quad (\text{A.9})$$

$$D(T, r) = 0,$$

where  $F = F(T)$ ,  $A = A(t, r)$ ,  $B = B(t, r)$ ,  $C = C(t, r)$ ,  $D = D(t, r)$ ,  $E = E(t, r)$  and  $G = G(t, r)$ .

The solution of the Eq. (A.4) is sought the form  $A(t, r) = \delta(t)e^{\gamma(t)r}$  with  $\delta$  and  $\gamma$  satisfying the condition  $\delta(T) = \beta^2$  and  $\gamma(T) = 0$ . Simplifying the corresponding expressions, we obtain

$$\frac{\delta'(t)}{\delta(t)} + \gamma'(t)r + 2r - \varpi + ((a - b r) - 2\vartheta)\gamma(t) + \left( \frac{1}{2}\sigma_r^2 - \varsigma \right) \gamma^2(t) = 0.$$

Functions  $\gamma$  and  $\delta$  can be now chosen to satisfy the equations

$$\gamma'(t) - b\gamma(t) + 2 = 0, \quad \delta'(t) + h(t)\delta(t) = 0. \quad (\text{A.10})$$

The solutions of (A.10) are easily found, namely,

$$\gamma(t) = \frac{2}{b} (1 - e^{-b(T-t)}), \quad \delta(t) = \beta^2 e^{\int_t^T h(s) ds},$$

where  $h(t)$  is defined in (3.7).

The solution of the Eq. (A.5) is sought the form  $B(t, r) = f(t)e^{g(t)r}$  under the conditions  $f(T) = 2(\alpha\beta - \beta^2 F)$  and  $g(T) = 0$ . Proceeding analogously to the previous considerations, we determine the functions  $f(t)$  and  $g(t)$  in (3.7).

Consider now the Eq. (A.6). Substituting  $A(t, r)$  and  $B(t, r)$  by they representations obtained from the Eqs. (A.4) and (A.5) yields

$$\begin{aligned} G_t + (a - b r)G_r + \frac{1}{2}\sigma_r^2 G_{rr} - \frac{1}{2}\varpi \frac{f^2(t)}{2\delta(t)} - \vartheta \frac{f^2(t)g(t)}{2\delta(t)} - \frac{1}{2}\varsigma \frac{f^2(t)g^2(t)}{2\delta(t)} &= 0, \\ G(T) &= (\alpha - \beta F)^2. \end{aligned} \quad (\text{A.11})$$

Since the nonhomogeneous term in (A.11) is independent of  $r$ , we assume that  $G(t, r)$  does not depend on  $r$ . It follows that  $G_r = G_{rr} = 0$  and writing  $G(t)$  for  $G(t, r)$ , we have

$$G(t) = (\alpha - \beta F)^2 - \int_t^T \frac{1}{2} \varpi \frac{f^2(t)}{2\delta(t)} + \vartheta \frac{f^2(t)g(t)}{2\delta(t)} + \frac{1}{2} \varsigma \frac{f^2(t)g^2(t)}{2\delta(t)} ds.$$

Considering the Eq. (A.7), we replace  $A(t, r)$  by its expression obtained from the Eq. (A.4), so that

$$\begin{aligned} E_t + (r + \mu_L + \lambda_L \mu_1^L - \varpi - \vartheta \gamma(t)) E + ((a - b r) - \vartheta - \varsigma \gamma(t)) E_r \\ + \frac{1}{2} \sigma_r^2 E_{rr} + 2\kappa \delta(t) e^{\gamma(t)r} = 0, \\ E(T, r) = 0. \end{aligned} \quad (\text{A.12})$$

This is a nonhomogeneous equation because of the term  $2\kappa \delta(t) e^{\gamma(t)r}$ , which also depends on the stochastic interest rate. According to [22, Proposition 2], the solution of Eq. (A.12) can be determined by solving the following homogeneous PDE with  $\tau (\tau \leq T)$ :

$$\begin{aligned} v_t(t, r; \tau) + (r + \mu_L + \lambda_L \mu_1^L - \varpi - \vartheta \gamma(t)) v(t, r; \tau) \\ + ((a - b r) - \vartheta - \varsigma \gamma(t)) v_r(t, r; \tau) + \frac{1}{2} \sigma_r^2 v_{rr}(t, r; \tau) = 0, \\ v(\tau, r; \tau) = 2\kappa \delta(\tau) e^{\gamma(\tau)r}. \end{aligned} \quad (\text{A.13})$$

The solution of the Eq. (A.12) is given by the formula

$$E(t, r) = \int_t^T v(t, r; \tau) d\tau. \quad (\text{A.14})$$

The solution of the Eq. (A.13) is sought in the form  $v(t, r; \tau) = \varepsilon(t; \tau) e^{\omega(t; \tau)r}$  under the conditions  $\varepsilon(\tau; \tau) = 2\kappa \delta(\tau)$  and  $\omega(\tau; \tau) = \gamma(\tau)$ . Simple calculations show that  $\varepsilon$  and  $\omega$  have the form (3.7).

Considering the Eqs. (A.8) and (A.9), we set

$$\begin{aligned} f_C(t, r) &= -\frac{1}{4} \varpi \frac{E^2}{A} - \frac{1}{2} \vartheta \frac{EE_r}{A} - \frac{1}{4} \varsigma \frac{E_r^2}{A} + \kappa E, \\ f_D(t, r) &= -\frac{1}{2} \varpi \frac{BE}{A} - \frac{1}{2} \varsigma \frac{B_r E_r}{A} - \vartheta \frac{B_r E + BE_r}{2A} + \kappa B, \end{aligned} \quad (\text{A.15})$$

and rewrite them as

$$\begin{aligned} C_t + (2\mu_L + \sigma_L^2 + \lambda_L (\mu_2^L + 2\mu_1^L)) C + (a - b r) C_r + \frac{1}{2} \sigma_r^2 C_{rr} + f_C(t, r) = 0, \\ C(T, r) = 0, \end{aligned} \quad (\text{A.16})$$

and

$$\begin{aligned} D_t + (\mu_L + \lambda_L \mu_1^L) D + (a - b r) D_r + \frac{1}{2} \sigma_r^2 D_{rr} + f_D(t, r) = 0, \\ D(T, r) = 0. \end{aligned} \quad (\text{A.17})$$

The unknown function  $C(t, r)$  in (A.16) can be found analogously to  $E(t, r)$  in (A.14). Thus

$$C(t, r) = \int_t^T \varphi(t, r; \tau) d\tau, \quad (\text{A.18})$$

where  $\varphi$  is the solution of the homogeneous parabolic equation

$$\begin{aligned} \varphi_t(t, r; \tau) + (2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L))\varphi(t, r; \tau) + (a - br)\varphi_r(t, r; \tau) \\ + \frac{1}{2}\sigma_r^2\varphi_{rr}(t, r; \tau) = 0, \\ \varphi(\tau, r; \tau) = f_C(\tau, r), \quad \tau \leq T. \end{aligned} \quad (\text{A.19})$$

The function  $f_C(\tau, r)$  in the boundary condition of (A.19) cannot be expressed in exponential form and it is difficult to use the previous approach to the solution of PDEs. Instead, we apply the variable transformation method. Set

$$\begin{aligned} q(m, n; \tau) &:= \varphi(t, r; \tau)e^{-(2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L))(\tau-t)}, \\ m &:= \frac{\sigma_r^2}{4b}(1 - e^{-2b(\tau-t)}), \\ n &:= re^{-b(\tau-t)} + \frac{a}{b}(1 - e^{-b(\tau-t)}), \end{aligned} \quad (\text{A.20})$$

and choose  $t = \tau$ . Then  $m = 0$ ,  $n = r$  and the boundary condition in (A.19) yields

$$q(0, n; \tau) = \varphi(\tau, r; \tau) = f_C(\tau, r) = f_C(\tau, n).$$

On the other hand, calculating the partial derivatives of  $\varphi$  in (A.20), we obtain

$$\begin{aligned} \varphi_t &= \left[ \left( -\frac{\sigma_r^2}{2}e^{-2b(\tau-t)}q_m - (a - br)e^{-b(\tau-t)}q_n \right) - q(m, n; \tau)(2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L)) \right] \\ &\quad \times \exp((2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L))(\tau - t)), \\ \varphi_r &= \exp((2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L) - b)(\tau - t))q_n, \\ \varphi_{rr} &= \exp((2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L) - 2b)(\tau - t))q_{nn}, \end{aligned} \quad (\text{A.21})$$

where  $q_m$ ,  $q_n$  and  $q_{nn}$  are the partial derivatives of  $q$  with respect to  $m$  and  $n$ . Substituting (A.21) into (A.19) and using the initial value of  $q(m, n; \tau)$  leads to the heat equation

$$\begin{aligned} q_m(m, n; \tau) - q_{nn}(m, n; \tau) &= 0, \\ q(0, n; \tau) &= f_C(\tau, n). \end{aligned} \quad (\text{A.22})$$

The solution of (A.22) is well-known — viz.

$$q(m, n; \tau) = \int_{-\infty}^{+\infty} f_C(\tau, \zeta) \frac{e^{-(n-\zeta)^2/4m}}{2\sqrt{\pi m}} d\zeta.$$

Substituting the above  $q(m, n; \tau)$  in the first equation of (A.20), we obtain correlation between  $\varphi$  and  $q$ . Taking into account the Eq. (A.18), we write the solution of the Eqs. (A.16) in the form

$$\begin{aligned} C(t, r) &= \int_t^T \varphi(t, r; \tau) d\tau \\ &= \int_t^T \int_{-\infty}^{+\infty} f_C(\tau, \zeta) \frac{e^{-((n-\zeta)^2/4m)}}{2\sqrt{\pi m}} e^{(2\mu_L + \sigma_L^2 + \lambda_L(\mu_2^L + 2\mu_1^L))(\tau-t)} d\zeta d\tau \end{aligned}$$

with  $f_C$  defined in (A.15).

The solutions of (A.16) and (A.17) can be found analogously — e.g. (A.17) has the solution

$$D(t, r) = \int_t^T \int_{-\infty}^{+\infty} f_D(\tau, \zeta) \frac{e^{-((n-\zeta)^2/4m)}}{2\sqrt{\pi m}} e^{(\mu_L + \lambda_L \mu_1^L)(\tau-t)} d\zeta d\tau,$$

where  $f_D$ ,  $m$  and  $n$  are defined by (A.15) and Eqs. (A.20).

It is easily seen that

$$V_{XX} = 2A(t, r) = 2\delta(t)e^{\gamma(t)r} > 0.$$

Replacing  $A$ ,  $B$  and  $E$  in (A.2) by the corresponding expressions, we arrive at the optimal investment strategy (3.8).  $\square$

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