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Characteristic Local Discontinuous Galerkin Methods for Incompressible Navier-Stokes Equations

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Abstract. By combining the characteristic method and the local discontinuous Galerkin method with carefully constructing numerical fluxes, variational formulations are established for time-dependent incompressible Navier-Stokes equations in $\mathbb{R}^2$. The nonlinear stability is proved for the proposed symmetric variational formulation. Moreover, for general triangulations the priori estimates for the L^2-norm of the errors in both velocity and pressure are derived. Some numerical experiments are performed to verify theoretical results.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Navier-Stokes equations, local discontinuous Galerkin method, symmetric variational formulation.

1 Introduction

Based on the assumption that the fluid, at the scale of interest, is a continuum, and the conservation of momentum (often alongside mass and energy conservation), the equation to describe the motion of fluid substances can be derived, which is named after the French engineer and physicist Claude-Louis Navier and the Ireland mathematician and physicist George Gabriel Stokes to recognize their fundamental contributions. Nowadays, it is still the central equation to fluid mechanics. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary $\partial\Omega$ and $T>0$ is a finite quantity. The

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time-dependent Navier-Stokes equation for an incompressible viscous fluid confined in Ω is [27]:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, & (\boldsymbol{x},t)\in\Omega\times[0,T], \\ \nabla\cdot\boldsymbol{u}=0, & (\boldsymbol{x},t)\in\Omega\times[0,T], \\ \boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \\ \boldsymbol{u}(\boldsymbol{x},t)=0, & (\boldsymbol{x},t)\in\partial\Omega\times[0,T]. \end{cases} \tag{1.1}$$

It is well-known that the problem has a unique solution and $\boldsymbol{u}\in L^2(0,T;H_0^1(\Omega)^2)\cap L^\infty(0,T;L^2(\Omega)^2)$, $p\in W^{-1,\infty}(0,T;L_0^2(\Omega))$ for $\boldsymbol{u}_t\in L^2(0,T;\boldsymbol{X}')$, the body force function $\boldsymbol{f}\in L^2(0,T;H^{-1}(\Omega)^2)$ and $\boldsymbol{u}_0\in H(\mathrm{div},\Omega)$ [27]. The constant ν is the fluid viscosity coefficient. Since p is uniquely defined up to an additive constant, we also assume that $\int_\Omega p\,d\boldsymbol{x}=0$. The $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is a nonlinear convective term and

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = u_1\frac{\partial\boldsymbol{u}}{\partial x} + u_2\frac{\partial\boldsymbol{u}}{\partial y}.$$

The idea of the characteristic methods dates back to the works of Douglas and Russell in 1982 [15]. Later on Süli [26] and Boukir et al. [4] extended the idea to two and three dimensional nonlinear coupled system, and performed the detailed numerical analysis for the incompressible Navier-Stokes equation. In the context of linear advection-diffusion equations, Eulerian-Lagrangian characteristic methods were proved to converge independent of the vanishing viscosity parameter [28, 29] or even in the case of degenerate diffusion coefficient [32]. The Eulerian-Lagrangian characteristic method was also used to solve the equation modelling the subsurface porous medium flow with error estimate [30]. Being different from the above ideas, here we use the characteristic method to tackle the time derivative term and the nonlinear convective term together and solve the considered equation with first order accuracy in time. It seems that the characteristic methods have many advantages compared to a high-order Runge-Kutta scheme or a high-order finite difference scheme [14], such as 1) efficient in solving the advection-dominated diffusion problems; 2) easily obtaining the existence and uniqueness of the solutions of the discretized system; 3) making the nonlinear equations linear and conveniently tackling the nonlinear obstacles; 4) easily performing the numerical stability analysis; 5) physically discretizing the material derivative [8].

Because of the inherent performances of the Navier-Stokes or Stokes equations in characterizing the turbulence (most flows occurring in nature are turbulent) in fluids or gases, from the finite element methods to discontinuous Galerkin methods a lot of research works on these topics have been done [3, 9–12, 17–19, 21, 24]. To our knowledge, there are less works on the discontinuous Galerkin method to solve the time-dependent incompressible Navier-Stokes equation, and much less on the local discontinuous Galerkin method (LDG). Recently splitting the nonlinearity and incompressibility, and using discontinuous or continuous finite element methods in space, Girault et al. solved the time-dependent incompressible Navier-Stokes equation [17] with discontinuous Galerkin methods.

In this paper, we use the local discontinuous Galerkin methods to discretize the space derivative of the considered equation. By introducing the local auxiliary variable, the order of the diffusion term can be reduced. The symmetric formulation arising from using penalty terms makes stability and error analysis possible. The introduced auxiliary variable $\bar{\boldsymbol{\sigma}}=\sqrt{\nu}\nabla\boldsymbol{u}$ lessens the challenges caused by the big Reynold number since $\sqrt{\nu}$ is not as small as ν. The lucky thing is that we still keep the general advantages of discontinuous Galerkin methods, namely, to get high order accuracy and to perform h_p-adaptivity, and the high parallelizability, etc.

The contribution of this work is an extension of local discontinuous Galerkin methods for the Stokes system [10] with the characteristic local discontinuous Galerkin (CLDG) method to the time dependent incompressible Navier-Stokes equations. We use some similar schemes in [10] to carry out the existence and error estimate of the pressure.

The outline of this paper is as follows. In Section 2 we derive the CLDG scheme and prove the existence, uniqueness of numerical solution. In Section 3 we prove the nonlinear stability. In Section 4 we carry out the priori estimates for L^2-norm of the errors in the velocity and pressure. In Section 5 some numerical experiments are given to verify theoretical results and illustrate the performance of the proposed scheme. In Section 6 some concluding remarks are given.

2 Derivation of the numerical scheme

We first introduce the notations, and then focus on deriving the full discrete numerical scheme of the time-dependent incompressible Navier-Stokes equations.

2.1 Preliminaries

For the mathematical setting of the Navier-Stokes problems, we describe some Sobolev spaces. The $L^2(\Omega)$ and $L_0^2(\Omega)$ are the classical space of square integrable functions with the inner product $(f,g)=\int_\Omega fg\,d\boldsymbol{x}$ and the subspace of functions of $L^2(\Omega)$ with zero mean value respectively,

$$L_0^2(\Omega)=\left\{v\in L^2(\Omega):\int_\Omega v\,d\boldsymbol{x}=0\right\}.$$

It is well-known that $C_0^\infty(\Omega)$ and $H_0^1(\Omega)$ are the space of infinitely differentiable functions with compact support and the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ respectively.

$$H^1(\Omega)=\left\{v\in L^2(\Omega):\nabla v\in L^2(\Omega)\right\},$$

and $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. Denote $\boldsymbol{X}$ as the space of functions of $H_0^1(\Omega)^2$ with zero divergence,

$$\boldsymbol{X}=\left\{\boldsymbol{v}\in H_0^1(\Omega)^2:\nabla\cdot\boldsymbol{v}=0\right\},$$

and $\boldsymbol{X}'$ as its dual space.

For any Banach space W, let $L^p[0,T;W], 1 \le p < \infty$ and $L^\infty[0,T;W]$ denote the spaces of p-integrable functions with norms

$$\|v\|_{L^p[0,T;W]} = \left(\int_0^T \|v(t)\|_W^p\right)^{1/p}, \quad \|v\|_{L^\infty[0,T;W]} = \mathrm{esssup}_{t\in[0,T]}\|v\|_W.$$

Let $H^1[0,T;W]$ denote the space of functions with square integral derivatives with norm

$$\|v\|_{H^1[0,T;W]} = \left(\int_0^T \|v\|_W^2\,dt + \int_0^T \|\partial_t v\|_W^2\,dt\right)^{1/2}.$$

The fundamental work spaces for solving the Navier-Stokes equations are $\boldsymbol{X}$ and $\mathbb{M} := L_0^2(\Omega)$.

The inner product and norm of vector functions $\boldsymbol{v} = (v_i)_{1\le i\le d}$ are defined by

$$(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x}, \qquad \|\boldsymbol{v}\|_0 = \left(\sum_{i=1}^d \|v_i\|_{L^2(\Omega)}^2\right)^{1/2}.$$

The gradient of a vector function $\boldsymbol{v}: \mathbb{R}^d \to \mathbb{R}^d$ and the divergence of a matrix function $\bar{\boldsymbol{\sigma}}: \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are given by

$$\nabla\boldsymbol{v} = \left(\frac{\partial v_i}{\partial x_j}\right)_{1\le i,j\le d}, \qquad \nabla\cdot\bar{\boldsymbol{\sigma}} = \left(\sum_{j=1}^d \frac{\partial\bar{\sigma}_{ij}}{\partial x_j}\right)_{1\le i,j\le d}.$$

Consequently, for a vector function $\boldsymbol{v} = (v_i)_{1\le i\le d}$, we have

$$\Delta\boldsymbol{v} = \nabla\cdot\nabla\boldsymbol{v} = (\Delta v_i)_{1\le i\le d}.$$

The L^2 inner product of two matrix functions $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\tau}}$ is defined by

$$(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\tau}}) = \int_\Omega \bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\tau}}\,d\boldsymbol{x} = \int_\Omega \sum_{1\le i,j\le d} \bar{\sigma}_{ij}\bar{\tau}_{ij}\,d\boldsymbol{x},$$

equipped with the norm

$$\|\bar{\boldsymbol{\sigma}}\|_0 = (\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\sigma}})_\Omega^{1/2} = \left(\int_\Omega \bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}}\,d\boldsymbol{x}\right)^{1/2} = \left(\int_\Omega \sum_{1\le i,j\le d} \bar{\sigma}_{ij}^2\,d\boldsymbol{x}\right)^{1/2}.$$

Let Ω be a bounded polygonal domain subdivided into elements E. Here E is a triangle or a quadrilateral in 2D. We assume that the intersection of two elements is either empty, or an edge (2D). The regular mesh is considered which means

$$\forall E \in \mathscr{E}_h, \qquad \frac{h_E}{\rho_E} \le C,$$

where $\mathscr{E}_h$ is the subdivision of Ω, C is a constant, h_E is the diameter of the element E, and ρ_E is the diameter of the inscribed circle in element E. Throughout this work $h = \max_{E\in\mathscr{E}_h} h_E$.

We introduce the Broken Sobolev space [23] for any real number s,

$$H^s(\mathscr{E}_h)^2 = \left\{ \boldsymbol{v} \in L^2(\Omega)^2 : \forall E \in \mathscr{E}_h, \boldsymbol{v}|_E \in H^s(E)^2 \right\},$$

equipped with the Broken Sobolev norm:

$$\|\boldsymbol{v}\|_s = \left(\sum_{E\in\mathscr{E}_h} \sum_{i=1}^{d} \|v_i\|^2_{H^s(E)} \right)^{1/2}.$$

We denote by $\mathscr{E}_h^B$ the set of edges of the subdivision $\mathscr{E}_h$. Let $\mathscr{E}_h^i$ denote the set of interior edges; and $\mathscr{E}_h^b = \mathscr{E}_h^B \setminus \mathscr{E}_h^i$ the set of edges on $\partial\Omega$. With each edge e, we have a unit normal vector $\boldsymbol{n}_e$. If e is on the boundary $\partial\Omega$, then $\boldsymbol{n}_e$ is taken to be the unit outward vector normal to $\partial\Omega$ [23].

If $\boldsymbol{v}$ belongs to $H^1(\mathscr{E}_h)^2$, the trace of $\boldsymbol{v}$ along any side of one element E is well defined. If two elements E_1^e and E_2^e are neighbors and share one common side e, there are two traces of $\boldsymbol{v}$ belonging to e. We assume that the normal vector $\boldsymbol{n}_e$ is oriented from E_1^e to E_2^e, the average and jump are defined by respectively

$$\{\boldsymbol{v}\} = \frac{1}{2}\left(\boldsymbol{v}|_{\partial E_1^e} + \boldsymbol{v}|_{\partial E_2^e}\right) \quad \text{and} \quad [\boldsymbol{v}] = (\boldsymbol{v}|_{\partial E_1^e} - \boldsymbol{v}|_{\partial E_2^e}), \;\; \forall e \in \partial E_1^e \bigcap \partial E_2^e.$$

If e is on $\partial\Omega$, we have the definition:

$$\{\boldsymbol{v}\} = [\boldsymbol{v}] = \boldsymbol{v}|_{\partial E}, \quad \forall e \in \partial E \bigcap \partial\Omega.$$

2.2 CLDG scheme

By introducing an auxiliary variable $\bar{\boldsymbol{\sigma}} = \sqrt{\nu}\nabla\boldsymbol{u}$ [3,13], we rewrite (1.1) as a mixed form:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \sqrt{\nu}\nabla\cdot\bar{\boldsymbol{\sigma}} + \nabla p = \boldsymbol{f}, & (\boldsymbol{x},t) \in \Omega\times[0,T], \\ \bar{\boldsymbol{\sigma}} = \sqrt{\nu}\nabla\boldsymbol{u}, & (\boldsymbol{x},t) \in \Omega\times[0,T], \\ \nabla\cdot\boldsymbol{u} = 0, & (\boldsymbol{x},t) \in \Omega\times[0,T], \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \boldsymbol{u}(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \partial\Omega\times[0,T], \end{cases} \tag{2.1}$$

where $\nu = 1/Re$ is the viscosity coefficient. Obviously, $\sqrt{\nu}$ is small enough we have $\sqrt{\nu} > \nu$.

Before presenting the variational form, we clarify the notation: $\boldsymbol{v}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n} := \sum_{i,j=1}^{2} v_i \bar{\sigma}_{ij} n_j := \bar{\boldsymbol{\sigma}} : (\boldsymbol{v}\otimes\boldsymbol{n})$. Multiplying the first, second, and the third equation of (2.1) by the smooth test functions $(\boldsymbol{v}, \bar{\boldsymbol{\tau}}, q)$ respectively, and integrating by parts over an arbitrary subset $E \in \mathscr{E}_h$, we get the following weak variational formulation, namely to find the solution $(\boldsymbol{u},\bar{\boldsymbol{\sigma}},\boldsymbol{p})\in \mathbb{V}\times\mathbb{V}^2\times\mathbb{Q}$ such that for any functions $(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\in\mathbb{V}\times\mathbb{V}^2\times\mathbb{Q}$,

$$\left\{\begin{array}{l} \int_E(\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v}\,d\boldsymbol{x}+\int_E\sqrt{\nu}\bar{\boldsymbol{\sigma}}:\nabla\boldsymbol{v}\,d\boldsymbol{x}-\int_{\partial E}\sqrt{\nu}\boldsymbol{v}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}_E\,ds \\ \quad -\int_E p\nabla\cdot\boldsymbol{v}\,d\boldsymbol{x}+\int_{\partial E}p\boldsymbol{v}\cdot\boldsymbol{n}_E\,ds=\int_E\boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x}, \\ \int_E\bar{\boldsymbol{\sigma}}:\bar{\boldsymbol{\tau}}\,d\boldsymbol{x}-\int_E\sqrt{\nu}\nabla\boldsymbol{u}:\bar{\boldsymbol{\tau}}\,d\boldsymbol{x}=0, \\ \int_E\nabla\cdot\boldsymbol{u}q\,d\boldsymbol{x}=0, \end{array}\right. \tag{2.2}$$

where $\boldsymbol{n}_E$ is the outward unit normal to ∂E, and

$$\begin{aligned} \mathbb{V}&=\left\{\boldsymbol{v}\in L^2(\Omega)^2:\boldsymbol{v}|_E\in H^1(E)^2,\forall E\in\mathscr{E}_h\right\}, \\ \mathbb{V}^2&=\left\{\bar{\boldsymbol{\sigma}}\in(L^2(\Omega)^2)^2:\bar{\boldsymbol{\sigma}}|_E\in(H^1(E)^2)^2,\forall E\in\mathscr{E}_h\right\}, \\ \mathbb{Q}&=\left\{q\in\mathbb{M}:q|_E\in H^1(E),\forall E\in\mathscr{E}_h\right\}. \end{aligned}$$

The exact solution $(\boldsymbol{u},\bar{\boldsymbol{\sigma}},p)$ will be approximated by the functions $(\boldsymbol{u}_h,\bar{\boldsymbol{\sigma}}_h,p_h)$ belonging to the finite element spaces $\mathbb{V}_h\times\mathbb{V}_h^2\times\mathbb{Q}_h$

$$\begin{aligned} \mathbb{V}_h&=\left\{\boldsymbol{v}\in L^2(\Omega)^2:\boldsymbol{v}|_E\in\mathbb{P}^k(E)^2,\forall E\in\mathscr{E}_h\right\}, \\ \mathbb{V}_h^2&=\left\{\bar{\boldsymbol{\sigma}}\in(L^2(\Omega)^2)^2:\bar{\boldsymbol{\sigma}}|_E\in(\mathbb{P}^k(E)^2)^2,\forall E\in\mathscr{E}_h\right\}, \\ \mathbb{Q}_h&=\left\{q\in\mathbb{M}:q|_E\in\mathbb{P}^k(E),\forall E\in\mathscr{E}_h\right\}, \end{aligned}$$

where $\mathbb{P}^k(E)$ denotes the set of all polynomials of degree at most $k\geq 1$ on E.

To find $(\boldsymbol{u}_h,\bar{\boldsymbol{\sigma}}_h,p_h)\in\mathbb{V}_h\times\mathbb{V}_h^2\times\mathbb{Q}_h$ for any functions $(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\in\mathbb{V}_h\times\mathbb{V}_h^2\times\mathbb{Q}_h$ and $E\in\mathscr{E}_h$, the following holds

$$\left\{\begin{array}{l} \int_E(\partial_t\boldsymbol{u}_h+(\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}_h)\cdot\boldsymbol{v}\,d\boldsymbol{x}+\int_E\sqrt{\nu}\bar{\boldsymbol{\sigma}}_h:\nabla\boldsymbol{v}\,d\boldsymbol{x}-\int_{\partial E}\sqrt{\nu}\boldsymbol{v}\cdot\bar{\boldsymbol{\sigma}}_h^*\cdot\boldsymbol{n}_E\,ds \\ \quad -\int_E p_h\nabla\cdot\boldsymbol{v}\,d\boldsymbol{x}+\int_{\partial E}p_h^*\boldsymbol{v}\cdot\boldsymbol{n}_E\,ds=\int_E\boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x}, \\ \int_E\bar{\boldsymbol{\sigma}}_h:\bar{\boldsymbol{\tau}}\,d\boldsymbol{x}-\int_E\sqrt{\nu}\nabla\boldsymbol{u}_h:\bar{\boldsymbol{\tau}}\,d\boldsymbol{x}=0, \\ \int_E\nabla\cdot\boldsymbol{u}_h q\,d\boldsymbol{x}=0, \end{array}\right. \tag{2.3}$$

where $\bar{\boldsymbol{\sigma}}_h^*$ and p_h^* are to be determined by numerical fluxes. By carefully adding penalty terms and choosing the numerical fluxes

$$\bar{\boldsymbol{\sigma}}_h^*=\{\bar{\boldsymbol{\sigma}}_h\},\quad p_h^*=\{p_h\}, \tag{2.4}$$

we develop the following numerical scheme

$$\begin{cases} \big(\partial_t \boldsymbol{u}_h+(\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}_h,\boldsymbol{v}\big)+(\bar{\boldsymbol{\sigma}}_h,\sqrt{\nu}\nabla\boldsymbol{v})-(\{\bar{\boldsymbol{\sigma}}_h\},\sqrt{\nu}[\boldsymbol{v}]\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^B} \\ \quad -(p_h,\nabla\cdot\boldsymbol{v})+(\{p_h\},[\boldsymbol{v}]\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^B}+([\boldsymbol{u}_h],[\boldsymbol{v}])_{\mathscr{E}_h^B}=(\boldsymbol{f},\boldsymbol{v}), \\ (\bar{\boldsymbol{\sigma}}_h,\bar{\boldsymbol{\tau}})-(\sqrt{\nu}\nabla\boldsymbol{u}_h,\bar{\boldsymbol{\tau}})+(\{\bar{\boldsymbol{\tau}}\},\sqrt{\nu}[\boldsymbol{u}_h]\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^B}=0, \\ (q,\nabla\cdot\boldsymbol{u}_h)-(\{q\},[\boldsymbol{u}_h]\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^B}+([p_h],[q])_{\mathscr{E}_h^i}=0, \end{cases} \tag{2.5}$$

for any functions $(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\in\mathbb{V}_h\times\mathbb{V}_h^2\times\mathbb{Q}_h$. The exact solution $(\boldsymbol{u},p)$ of (1.1) is expected to be at least continuous with homogeneous boundary. So added penalty terms $(\{\bar{\boldsymbol{\tau}}\},\sqrt{\nu}[\boldsymbol{u}_h]\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^B}$, $([\boldsymbol{u}_h],[\boldsymbol{v}])_{\mathscr{E}_h^B}$, $(\{q\},[\boldsymbol{u}_h]\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^B}$ and $([p_h],[q])_{\mathscr{E}_h^i}$ still keep the consistency of the scheme. Moreover, the locality of the discontinuous Galerkin method still remains since the penalty term $(\{\bar{\boldsymbol{\tau}}\},\sqrt{\nu}[\boldsymbol{u}_h]\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^B}$ in the second equation is independent of $\bar{\boldsymbol{\sigma}}_h$. The most important thing is that these additions make the variational formulation symmetric, which makes the stability and error analysis possible.

Throughout this work, we use the notations

$$(\boldsymbol{w},\boldsymbol{v})=\sum_{E\in\mathscr{E}_h}(\boldsymbol{w},\boldsymbol{v})_E,\quad (\boldsymbol{w},\boldsymbol{v})_{\mathscr{E}_h^i}=\sum_{e\in\mathscr{E}_h^i}(\boldsymbol{w},\boldsymbol{v})_e,\quad (\boldsymbol{w},\boldsymbol{v})_{\mathscr{E}_h^B}=\sum_{e\in\mathscr{E}_h^B}(\boldsymbol{w},\boldsymbol{v})_e.$$

Definitions of the bilinear forms:

$$\begin{aligned} &\mathbb{a}(\bar{\boldsymbol{\sigma}}_h,\boldsymbol{v})=(\bar{\boldsymbol{\sigma}}_h,\sqrt{\nu}\nabla\boldsymbol{v})-(\{\bar{\boldsymbol{\sigma}}_h\},\sqrt{\nu}[\boldsymbol{v}]\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^B}, \\ &\mathbb{b}(p_h,\boldsymbol{v})=-(p_h,\nabla\cdot\boldsymbol{v})+(\{p_h\},[\boldsymbol{v}]\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^B}, \\ &\mathbb{c}(\boldsymbol{u}_h,\boldsymbol{v})=([\boldsymbol{u}_h],[\boldsymbol{v}])_{\mathscr{E}_h^B}, \\ &\mathbb{d}(p_h,q)=([p_h],[q])_{\mathscr{E}_h^i}. \end{aligned}$$

By integration by parts, the forms $\mathbb{a}(\bar{\boldsymbol{\sigma}}_h,\boldsymbol{v})$ and $\mathbb{b}(p_h,\boldsymbol{v})$ also can be rewritten as

$$\begin{aligned} &\mathbb{a}(\bar{\boldsymbol{\sigma}}_h,\boldsymbol{v})=-(\nabla\cdot\bar{\boldsymbol{\sigma}}_h,\sqrt{\nu}\boldsymbol{v})+([\bar{\boldsymbol{\sigma}}_h],\sqrt{\nu}\{\boldsymbol{v}\}\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^i}, \\ &\mathbb{b}(p_h,\boldsymbol{v})=(\nabla p_h,\boldsymbol{v})-([p_h],\{\boldsymbol{v}\}\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^i}. \end{aligned} \tag{2.6}$$

2.3 Characteristic method

For each positive integer N, let $0=t^0<t^1<\cdots<t^N=T$ be a partition of T into subintervals $J^n=(t^{n-1},t^n]$, with uniform mesh and the interval length $\Delta t=t^n-t^{n-1},1\le n\le N$ and $\boldsymbol{u}^n=\boldsymbol{u}(\boldsymbol{x},t^n)$. The characteristic tracing back along the field $\boldsymbol{u}^{n-1}$ of a point $\boldsymbol{x}\in\Omega$ at time t^n to t^{n-1} is approximated by [1,6,25]:

$$\dot{\mathbf{x}}(\boldsymbol{x},t^{n-1}):=\boldsymbol{x}-\boldsymbol{u}^{n-1}\Delta t.$$

We assume that the characteristics do not cross each other, which can be satisfied by choosing sufficiently small Δt, since physically there are no crossings for the characteristics. Consequently, the approximation for the hyperbolic part of (1.1) at time t^n can be derived by

$$\partial_t \boldsymbol{u}^n + \boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n \approx \frac{\boldsymbol{u}^n - \dot{\boldsymbol{u}}^{n-1}}{\Delta t},$$

where $\dot{\boldsymbol{u}}^{n-1} = \boldsymbol{u}(\dot{\mathbf{x}}(\boldsymbol{x},t^{n-1}))$.

Lemma 2.1 (Time truncation error [25])**.** *Let* $E(\boldsymbol{x},n) = \frac{\boldsymbol{u}^n - \dot{\boldsymbol{u}}^{n-1}}{\Delta t} - (\partial_t \boldsymbol{u}^n + \boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n)$. *If* $\boldsymbol{u} \in C^4([\Delta t, T]; H^3(\Omega)^2)$ *and* $t^n > \Delta t$, *then*

$$E(\boldsymbol{x},n) = -\Delta t \left(\frac{1}{2} \frac{d^2 \boldsymbol{g}_x^n}{dt^2} + \frac{\partial \boldsymbol{u}}{\partial t} \cdot \nabla \boldsymbol{u}(\boldsymbol{x},t^n) \right) + \mathcal{O}(\Delta t^2), \tag{2.7}$$

where $\boldsymbol{g}_x^n(t) = \boldsymbol{u}(\boldsymbol{x} - (t^n - t)\boldsymbol{u}(\boldsymbol{x},t),t)$.

Hence the fully discretized scheme, the characteristic local discontinuous Galerkin (CLDG) scheme, corresponding to the variational formulation (2.5) is to find $(\boldsymbol{u}_h^n, \bar{\boldsymbol{\sigma}}_h^n, p_h^n) \in \mathbb{V}_h \times \mathbb{V}_h^2 \times \mathbb{Q}_h$ for any functions $(\boldsymbol{v}, \bar{\boldsymbol{\tau}}, q) \in \mathbb{V}_h \times \mathbb{V}_h^2 \times \mathbb{Q}_h$ such that

$$\left\{ \begin{array}{l} \left(\frac{\boldsymbol{u}_h^n - \check{\boldsymbol{u}}_h^{n-1}}{\Delta t}, \boldsymbol{v} \right) + (\bar{\boldsymbol{\sigma}}_h^n, \sqrt{\nu} \nabla \boldsymbol{v}) - (\{\bar{\boldsymbol{\sigma}}_h^n\}, \sqrt{\nu}[\boldsymbol{v}] \otimes \boldsymbol{n}_e)_{\mathscr{E}_h^B} \\ \quad -(p_h^n, \nabla \cdot \boldsymbol{v}) + (\{p_h^n\}, [\boldsymbol{v}] \cdot \boldsymbol{n}_e)_{\mathscr{E}_h^B} + ([\boldsymbol{u}_h^n], [\boldsymbol{v}])_{\mathscr{E}_h^B} = (\boldsymbol{f}^n, \boldsymbol{v}), \\ (\bar{\boldsymbol{\sigma}}_h^n, \bar{\boldsymbol{\tau}}) - (\sqrt{\nu} \nabla \boldsymbol{u}_h^n, \bar{\boldsymbol{\tau}}) + (\{\bar{\boldsymbol{\tau}}\}, \sqrt{\nu}[\boldsymbol{u}_h^n] \otimes \boldsymbol{n}_e)_{\mathscr{E}_h^B} = 0, \\ (q, \nabla \cdot \boldsymbol{u}_h^n) - (\{q\}, [\boldsymbol{u}_h^n] \cdot \boldsymbol{n}_e)_{\mathscr{E}_h^B} + ([p_h^n], [q])_{\mathscr{E}_h^i} = 0, \end{array} \right. \tag{2.8}$$

where $\check{\boldsymbol{u}}_h^{n-1} = \boldsymbol{u}_h(\check{\mathbf{x}}(\boldsymbol{x},t^{n-1})) = \boldsymbol{u}_h(\boldsymbol{x} - \boldsymbol{u}_h^{n-1}\Delta t, t^{n-1})$, and $\check{\boldsymbol{u}}_h^0 = \boldsymbol{u}^0$.

We rewrite (2.8) as a compact formulation: Find $(\boldsymbol{u}_h^n, \bar{\boldsymbol{\sigma}}_h^n, p_h^n) \in \mathbb{V}_h \times \mathbb{V}_h^2 \times \mathbb{Q}_h$ for any functions $(\boldsymbol{v}, \bar{\boldsymbol{\tau}}, q) \in \mathbb{V}_h \times \mathbb{V}_h^2 \times \mathbb{Q}_h$ such that

$$\left\{ \begin{array}{l} \left(\frac{\boldsymbol{u}_h^n - \check{\boldsymbol{u}}_h^{n-1}}{\Delta t}, \boldsymbol{v} \right) + \mathbb{a}(\bar{\boldsymbol{\sigma}}_h^n, \boldsymbol{v}) + \mathbb{b}(p_h^n, \boldsymbol{v}) + \mathbb{c}(\boldsymbol{u}_h^n, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \\ (\bar{\boldsymbol{\sigma}}_h^n, \bar{\boldsymbol{\tau}}) - \mathbb{a}(\bar{\boldsymbol{\tau}}, \boldsymbol{u}_h^n) = 0, \\ -\mathbb{b}(q, \boldsymbol{u}_h^n) + \mathbb{d}(p_h^n, q) = 0. \end{array} \right. \tag{2.9}$$

For notational and analytic convenience, we define the following equality:

$$\begin{aligned} &\mathscr{A}(\boldsymbol{u}_h^n, \bar{\boldsymbol{\sigma}}_h^n, p_h^n; \boldsymbol{v}, \bar{\boldsymbol{\tau}}, q) \\ =&\mathbb{a}(\bar{\boldsymbol{\sigma}}_h^n, \boldsymbol{v}) + \mathbb{b}(p_h^n, \boldsymbol{v}) + \mathbb{c}(\boldsymbol{u}_h^n, \boldsymbol{v}) + (\bar{\boldsymbol{\sigma}}_h^n, \bar{\boldsymbol{\tau}}) \\ &- \mathbb{a}(\bar{\boldsymbol{\tau}}, \boldsymbol{u}_h^n) - \mathbb{b}(q, \boldsymbol{u}_h^n) + \mathbb{d}(p_h^n, q), \end{aligned} \tag{2.10}$$

and the right side hand

$$\mathscr{F}(\boldsymbol{v}) = (\boldsymbol{f}^n, \boldsymbol{v}). \tag{2.11}$$

Remark 2.1. Note that we take $(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)=(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)$ at (2.10) to obtain a semi-norm $\left|\cdot\right|_{\mathscr{A}}$, that is

$$\begin{aligned}&\left|(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)\right|_{\mathscr{A}}^2\\ =&\mathscr{A}(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n;\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)\\ =&\mathbb{c}(\boldsymbol{u}_h^n,\boldsymbol{u}_h^n)+(\bar{\boldsymbol{\sigma}}_h^n,\bar{\boldsymbol{\sigma}}_h^n)+\mathbb{d}(p_h^n,p_h^n)\\ =&\sum_{e\in\mathscr{E}_h^B}\|[\boldsymbol{u}_h^n]\|_{L^2(e)}^2+\|\bar{\boldsymbol{\sigma}}_h^n\|_0^2+\sum_{e\in\mathscr{E}_h^i}\|[p_h^n]\|_{L^2(e)}^2.\end{aligned}\tag{2.12}$$

2.4 Existence and uniqueness of CLDG solution

In order to prove the existence and uniqueness of approximation solution of the CLDG scheme of problem (1.1), we shall introduce the following mild condition on the local spaces

$$q\in\mathbb{P}^k(E):\ \int_E \boldsymbol{v}\cdot\nabla q\,d\boldsymbol{x}=0,\ \ \forall\,\boldsymbol{v}\in\mathbb{P}^k(E)^2,\ \ \text{then}\ \ \nabla q=0\ \text{ on }\ E.\tag{2.13}$$

Obviously $\nabla\mathbb{P}^k(E)\subset\mathbb{P}^k(E)^2$; Eq. (2.13) is satisfied with $k\geq 1$; see [5,10].

Lemma 2.2. *If the approximation spaces $\mathbb{V}_h\times\mathbb{V}_h^2\times\mathbb{Q}_h$ are spanned by the polynomial space $\mathbb{P}^k(E)$ with $k\geq 1$, then there exists a unique solution $(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)\in\mathbb{V}_h\times\mathbb{V}_h^2\times\mathbb{Q}_h$ satisfying (2.8).*

Proof. To ensure the computability of the CLDG scheme for problem (1.1), we begin by showing that the variational formulation of (2.8) is uniquely solvable for $(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)$ at each time step n. As (2.8) represents a finite system of linear equations, the uniqueness of $(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)$ is equivalent to the existence.

Setting $\check{\boldsymbol{u}}_h^{n-1}=\boldsymbol{f}=0$ and taking $\boldsymbol{v}=\boldsymbol{u}_h^n,\bar{\boldsymbol{\tau}}=\bar{\boldsymbol{\sigma}}_h^n,q=p_h^n$ in (2.9), we have

$$\frac{1}{\Delta t}\|\boldsymbol{u}_h^n\|_0^2+|(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)|_{\mathscr{A}}^2=0,\tag{2.14}$$

which implies $\boldsymbol{u}_h^n=\boldsymbol{0},\bar{\boldsymbol{\sigma}}_h^n=\bar{\boldsymbol{0}}$, and $[p_h^n]\big|_e=0,\forall\,e\in\mathscr{E}_h^i$. It follows from (2.9), that

$$\forall\boldsymbol{v}\in\mathbb{V}_h,\quad b(\boldsymbol{v},p_h^n)=0.$$

From identity (2.6), we get

$$b(\boldsymbol{v},p_h^n)=\sum_{E\in\mathscr{E}_h}\int_E\nabla p_h^n\cdot\boldsymbol{v}\,d\boldsymbol{x}=0,\ \ \forall\,\boldsymbol{v}\in\mathbb{V}_h.$$

We conclude from Eq. (2.13) that $\nabla p_h^n=\boldsymbol{0}$ on each $E\in\mathscr{E}_h$, and $[p_h^n]\big|_e=0,\forall\,e\in\mathscr{E}_h^i$, that p_h^n is a constant. Since $p_h^n\in\mathbb{M}$, that is, $\int_\Omega p_h^n d\boldsymbol{x}=0$, we have $p_h^n=0$. □

3 Stability analysis

In this subsection, before presenting and proving the numerical stability result, we shall give the following lemma.

Lemma 3.1 ([4,7,15,25])**.** *Define $\check{\mathcal{X}}_{\boldsymbol{x}}^n(t)=\boldsymbol{x}-(t^n-t)\boldsymbol{u}_h^{n-1}, \forall t\in[t_{n-2},t_n], 2\leq n\leq N$. Under the condition $\Delta t<\frac{1}{2L_n}, L_n=\max_{1\leq i\leq n}\|\boldsymbol{u}_h^i\|_{1,\infty}$ on each time step t^n, for any function $\boldsymbol{v}\in L^2(\Omega)$, there is*

$$\|\check{\boldsymbol{v}}\|_0^2-\|\boldsymbol{v}\|_0^2\leq C\Delta t\|\boldsymbol{v}\|_0^2, \tag{3.1}$$

where $\boldsymbol{u}_h^n\in\mathbb{V}_h\subset W^{1,\infty}(\Omega)$ and $\check{\boldsymbol{v}}=\boldsymbol{v}(\boldsymbol{x}-\Delta t\boldsymbol{u}_h^{n-1})$.

The proof can be found in [7,15,25].

Theorem 3.1 (Numerical stability)**.** *The CLDG scheme of* (2.8) *is nonlinear stable, for any integer $N=1,2,3,\cdots$,*

$$\begin{aligned}&\|\boldsymbol{u}_h^N\|_0^2+2\Delta t\sum_{n=1}^N\left|(\boldsymbol{u}_h^n,\bar{\sigma}_h^n,p_h^n)\right|_{\mathscr{A}}^2+\sum_{n=1}^N\|\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}\|_0^2\\ \leq& C\Delta t\sum_{n=1}^N\|\boldsymbol{f}^n\|_0^2+C\|\boldsymbol{u}^0\|_0^2,\end{aligned}$$

where $\Delta t<\frac{1}{2L_n}, L_n=\max_{1\leq i\leq n}\|\boldsymbol{u}_h^i\|_{1,\infty}$, $\boldsymbol{u}^0=\boldsymbol{u}_h^0$, $|\cdot|_{\mathscr{A}}$ is defined by (2.12)*), C is a generic constant.*

Proof. Taking $\boldsymbol{v}=2\Delta t\boldsymbol{u}_h^n$, $\bar{\boldsymbol{\tau}}=2\Delta t\bar{\sigma}_h^n$, and $q=2\Delta tp_h^n$ respectively in (2.9), we get the following equations

$$2(\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1},\boldsymbol{u}_h^n)+2\Delta t\left|(\boldsymbol{u}_h^n,\bar{\sigma}_h^n,p_h^n)\right|_{\mathscr{A}}^2=2\Delta t\mathscr{F}(\boldsymbol{u}_h^n),$$

and

$$2(\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1},\boldsymbol{u}_h^n)=\|\boldsymbol{u}_h^n\|_0^2-\|\check{\boldsymbol{u}}_h^{n-1}\|_0^2+\|\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}\|_0^2.$$

Now, we estimate the bound of $\|\check{\boldsymbol{u}}_h^{n-1}\|_0^2-\|\boldsymbol{u}_h^{n-1}\|_0^2$. Since $\mathbb{V}_h$ is a subset of $W^{1,\infty}(\Omega)$, from Lemma 3.1, it follows that

$$\|\check{\boldsymbol{u}}_h^{n-1}\|_0^2-\|\boldsymbol{u}_h^{n-1}\|_0^2\leq C\Delta t\|\boldsymbol{u}_h^{n-1}\|_0^2. \tag{3.2}$$

By the definition of $\mathscr{F}$, Hölder inequality and Young inequality, there is

$$\begin{aligned}&\|\boldsymbol{u}_h^n\|_0^2-\|\boldsymbol{u}_h^{n-1}\|_0^2+2\Delta t\left|(\boldsymbol{u}_h^n,\bar{\sigma}_h^n,p_h^n)\right|_{\mathscr{A}}^2+\|\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}\|_0^2\\ \leq& C\Delta t\|\boldsymbol{u}_h^{n-1}\|_0^2+\Delta t\|\boldsymbol{f}^n\|_0^2+\Delta t\|\boldsymbol{u}_h^n\|_0^2.\end{aligned} \tag{3.3}$$

Summing up the above equation from $n=1$ to N, we have

$$\begin{aligned}
&\|\boldsymbol{u}_h^N\|_0^2-\|\boldsymbol{u}_h^0\|_0^2+2\Delta t\sum_{n=1}^N\big|(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)\big|_{\mathscr{A}}^2+\sum_{n=1}^N\|\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}\|_0^2\\
\le& C\Delta t\sum_{n=1}^N\|\boldsymbol{u}_h^{n-1}\|_0^2+\Delta t\sum_{n=1}^N\|\boldsymbol{f}^n\|_0^2+\Delta t\sum_{n=1}^N\|\boldsymbol{u}_h^n\|_0^2.
\end{aligned}$$

Then the following holds

$$\begin{aligned}
&\|\boldsymbol{u}_h^N\|_0^2+2\Delta t\sum_{n=1}^N\big|(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)\big|_{\mathscr{A}}^2+\sum_{n=1}^N\|\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}\|_0^2\\
\le& C\Delta t\sum_{n=1}^N\|\boldsymbol{u}_h^n\|_0^2+\Delta t\sum_{n=1}^N\|\boldsymbol{f}^n\|_0^2+(C\Delta t+1)\|\boldsymbol{u}_h^0\|_0^2.
\end{aligned}$$

From the discrete Grönwall inequality, we have

$$\begin{aligned}
&\|\boldsymbol{u}_h^N\|_0^2+2\Delta t\sum_{n=1}^N\big|(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)\big|_{\mathscr{A}}^2+\sum_{n=1}^N\|\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}\|_0^2\\
\le& e^{CT}\left(\Delta t\sum_{n=1}^N\|\boldsymbol{f}^n\|_0^2+(C\Delta t+1)\|\boldsymbol{u}_h^0\|_0^2\right).
\end{aligned}$$

The proof is completed. □

4 Error analysis

In this section, we shall give error estimates for the CLDG scheme of (2.8). For the sake of simplicity, we introduce some notations as follows

$$\begin{aligned}
&\boldsymbol{\xi}_1^n=\boldsymbol{\Pi}\boldsymbol{u}^n-\boldsymbol{u}_h^n, \quad \boldsymbol{\xi}_2^n=\boldsymbol{\Pi}\boldsymbol{u}^n-\boldsymbol{u}^n, \quad \boldsymbol{e}_u^n=\boldsymbol{\xi}_1^n-\boldsymbol{\xi}_2^n=\boldsymbol{u}^n-\boldsymbol{u}_h^n,\\
&\bar{\boldsymbol{\eta}}_1^n=\bar{\Pi}\bar{\boldsymbol{\sigma}}^n-\bar{\boldsymbol{\sigma}}_h^n, \quad \bar{\boldsymbol{\eta}}_2^n=\bar{\Pi}\bar{\boldsymbol{\sigma}}^n-\bar{\boldsymbol{\sigma}}^n, \quad \bar{\boldsymbol{e}}_{\bar{\sigma}}^n=\bar{\boldsymbol{\eta}}_1^n-\bar{\boldsymbol{\eta}}_2^n=\bar{\boldsymbol{\sigma}}^n-\bar{\boldsymbol{\sigma}}_h^n,\\
&\zeta_1^n=\Pi p^n-p_h^n, \quad \zeta_2^n=\Pi p^n-p^n, \quad e_p^n=\zeta_1^n-\zeta_2^n=p^n-p_h^n,
\end{aligned}$$

where $\boldsymbol{\Pi}:\mathbb{V}\mapsto\mathbb{V}_h$,$\bar{\Pi}:\mathbb{V}^2\mapsto\mathbb{V}_h^2$ and $\Pi:\mathbb{Q}\mapsto\mathbb{Q}_h$ are linear continuous L^2-projection operators onto the corresponding finite element spaces.

Throughout this work, we assume that the solution $(\boldsymbol{u},p)$ of (1.1) satisfies the regularity

$$\begin{aligned}
&\boldsymbol{u}\in L^\infty(0,T;W^{1,\infty}(\Omega)^2)\cap L^\infty(0,T;H^{k+1}(\Omega)^2)\cap C^4([\Delta t,T];H^3(\Omega)^2),\\
&\boldsymbol{u}\in H^1(0,T;H^{-1}(\Omega)^2),\ \partial_t\boldsymbol{u}\in L^2(0,T;H^{k+1}(\Omega)^2),\ \partial_{tt}\boldsymbol{u}\in L^2(0,T;L^2(\Omega)^2),\\
&p\in L^2(0,T;H^{k+1}(\Omega))\cap L^2(0,T;L_0^2(\Omega)),
\end{aligned}\tag{4.1}$$

where k is the degree of approximation polynomials.

Next we review some lemmas for our analysis. The first one is the standard approximation result for any linear continuous projection operator Π from $H^{s+1}(E)$ onto $V_h(E)=\{v: v|_E \in \mathbb{P}^l(E),\ l\geq 0\}$ satisfying $\Pi v=v$ for any $v\in\mathbb{P}^l(E)$. The second one is the standard trace inequality. The other two are some results about the characteristic methods.

Lemma 4.1 ([5,10]). *Let $v\in H^{s+1}(E), s\geq 0$. Let Π be a linear continuous projection operator from $H^{s+1}(E)$ onto $V_h(E)$ such that $\Pi v=v$ for any $v\in\mathbb{P}^l(E)$. For $m=0,1$, there is*

$$\begin{cases} |v-\Pi v|_{H^m(E)} \leq Ch_E^{\min(s,l)+1-m}\|v\|_{H^{s+1}(E)}, \\ \|v-\Pi v\|_{L^2(\partial E)}\leq Ch_E^{\min(s,l)+1/2}\|v\|_{H^{s+1}(E)}. \end{cases} \tag{4.2}$$

Lemma 4.2 ([5,10]). *For any $v\in V_h(E)$,*

$$\|v\|_{L^2(\partial E)}\leq Ch_E^{-1/2}\|v\|_{L^2(E)}, \tag{4.3}$$

where C is a generic constant independent of h_E.

Lemma 4.3 ([7,15,25]). *If $\Delta t<\frac{1}{2L_n}, L_n=\max_{1\leq i\leq n}\|\boldsymbol{u}_h^i\|_{1,\infty}$, then for any function $v\in H^1(\Omega)$ and each time step n, there is a constant C, such that*

$$\|v(\boldsymbol{x})-v(\check{\mathbf{x}})\|_0\leq C\Delta t\|\nabla v\|_0, \tag{4.4}$$

where $\check{\mathbf{x}}=\boldsymbol{x}-\Delta t\boldsymbol{u}_h^{n-1}$.

See the proof in page 12 of [25].

Lemma 4.4 ([7,15,25]). *If $v\in L^2(\Omega)$ and $\Delta t<\frac{1}{2L_n}, L_n=\max_{1\leq i\leq n}\|\boldsymbol{u}_h^i\|_{1,\infty}$, then for any time step n, there exists a constant C such that*

$$\|v(\boldsymbol{x})-v(\check{\mathbf{x}})\|_{-1}\leq C\Delta t\|v\|_0, \tag{4.5}$$

where $\check{\mathbf{x}}=\boldsymbol{x}-\Delta t\boldsymbol{u}_h^{n-1}$.

The proof can be found in [7,15,25].

4.1 Error in velocity

Theorem 4.1 (Error estimate of the velocity). *Let $(\boldsymbol{u}^n,p^n)$ be the solution of (1.1)) at time $t=t^n$ and $\bar{\boldsymbol{\sigma}}^n\in(H^{k+1}(\Omega)^2)^2$, $(\boldsymbol{u}_h^n,\bar{\sigma}_h^n,p_h^n)$ the solution of the CLDG scheme of (2.8)). If $\Delta t<\frac{1}{2L_n}, L_n=\max_{1\leq i\leq n}\|\boldsymbol{u}_h^i\|_{1,\infty}$, and the regularity (4.1)) satisfied, then for any integer $N=1,2,\cdots$,*

$$\begin{aligned}&\|\boldsymbol{e}_u^N\|_0^2+\Delta t\sum_{n=1}^N\left|(\boldsymbol{e}_u^n,\bar{\boldsymbol{e}}_{\bar{\sigma}}^n,e_p^n)\right|_{\mathscr{A}}^2+\sum_{n=1}^N\|\boldsymbol{e}_u^n-\check{\boldsymbol{e}}_u^{n-1}\|_0^2\\ &\leq C\Delta t^2+\nu Ch^{2k}+Ch^{2k},\end{aligned} \tag{4.6}$$

where $k\geq 1$, C is a generic constant independent of the parameters $\nu,\Delta t,h$ but depends on certain Sobolev norms of the exact solution. Note that the above error estimate derived holds uniformly with respect to the small viscosity ν $(0<\nu\ll 1)$.

Proof. The weak form of the exact solution $(\boldsymbol{u}^n,\bar{\boldsymbol{\sigma}}^n,p^n)$ satisfies (2.5) since the consistency of the scheme. Taking $\boldsymbol{v}=\boldsymbol{\xi}_1^n$, $\bar{\boldsymbol{\tau}}=\bar{\boldsymbol{\eta}}_1^n$, $q=\zeta_1^n$ in (2.5) at $t=t^n$ and in (2.8), subtracting (2.8) from (2.5), we obtain

$$\begin{aligned}
&\left(\partial_t\boldsymbol{u}^n+(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n-\frac{\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)+|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)|_{\mathscr{A}}^2\\
=&\mathscr{A}(\boldsymbol{\xi}_2^n,\bar{\boldsymbol{\eta}}_2^n,\zeta_2^n;\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\\
=&\mathbb{a}(\bar{\boldsymbol{\eta}}_2^n,\boldsymbol{\xi}_1^n)+\mathbb{b}(\zeta_2^n,\boldsymbol{\xi}_1^n)+\mathbb{c}(\boldsymbol{\xi}_2^n,\boldsymbol{\xi}_1^n)+(\bar{\boldsymbol{\eta}}_2^n,\bar{\boldsymbol{\eta}}_1^n)\\
&-\mathbb{a}(\bar{\boldsymbol{\eta}}_1^n,\boldsymbol{\xi}_2^n)-\mathbb{b}(\zeta_1^n,\boldsymbol{\xi}_2^n)+\mathbb{d}(\zeta_2^n,\zeta_1^n)\\
=&\sum_{i=1}^{7}\mathcal{I}_i,
\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}
&\mathcal{I}_1=\mathbb{a}(\bar{\boldsymbol{\eta}}_2^n,\boldsymbol{\xi}_1^n),\quad \mathcal{I}_2=\mathbb{b}(\zeta_2^n,\boldsymbol{\xi}_1^n),\quad \mathcal{I}_3=\mathbb{c}(\boldsymbol{\xi}_2^n,\boldsymbol{\xi}_1^n),\quad \mathcal{I}_4=(\bar{\boldsymbol{\eta}}_2^n,\bar{\boldsymbol{\eta}}_1^n),\\
&\mathcal{I}_5=-\mathbb{a}(\bar{\boldsymbol{\eta}}_1^n,\boldsymbol{\xi}_2^n),\quad \mathcal{I}_6=-\mathbb{b}(\zeta_1^n,\boldsymbol{\xi}_2^n),\quad \mathcal{I}_7=\mathbb{d}(\zeta_2^n,\zeta_1^n).
\end{aligned}$$

Now, we estimate each term $\mathcal{I}_i$, respectively. By the property of L^2-projection operator $\bar{\Pi}$, the Hölder inequality, and Lemma 4.1, we obtain

$$\begin{aligned}
\mathcal{I}_1&=(\bar{\boldsymbol{\eta}}_2^n,\sqrt{\nu}\nabla\boldsymbol{\xi}_1^n)-(\{\bar{\boldsymbol{\eta}}_2^n\},\sqrt{\nu}[\boldsymbol{\xi}_1^n]\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^B}\\
&\leq\sum_{e\in\mathscr{E}_h^B}\sqrt{\nu}\,\|\{\bar{\boldsymbol{\eta}}_2^n\}\|_{L^2(e)}\|[\boldsymbol{\xi}_1^n]\otimes\boldsymbol{n}_e\|_{L^2(e)}\\
&\leq C\Big(\sum_{e\in\mathscr{E}_h^B}\nu\,\|\{\bar{\boldsymbol{\eta}}_2^n\}\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{e\in\mathscr{E}_h^B}\|[\boldsymbol{\xi}_1^n]\|_{L^2(e)}^2\Big)^{1/2}\\
&\leq\sqrt{\nu}Ch^{k+1/2}\big|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\big|_{\mathscr{A}}.
\end{aligned}$$

Similarly, we deduce

$$\begin{aligned}
\mathcal{I}_2&=-(\zeta_2^n,\nabla\cdot\boldsymbol{\xi}_1^n)+(\{\zeta_2^n\},[\boldsymbol{\xi}_1^n]\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^B}\\
&\leq\Big(\sum_{e\in\mathscr{E}_h^B}\|\{\zeta_2^n\}\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{e\in\mathscr{E}_h^B}\|[\boldsymbol{\xi}_1^n]\|_{L^2(e)}^2\Big)^{1/2}\\
&\leq Ch^{k+1/2}\big|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\big|_{\mathscr{A}},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_3&\leq\Big(\sum_{e\in\mathscr{E}_h^B}\|[\boldsymbol{\xi}_2^n]\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{e\in\mathscr{E}_h^B}\|[\boldsymbol{\xi}_1^n]\|_{L^2(e)}^2\Big)^{1/2}\\
&\leq Ch^{k+1/2}\big|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\big|_{\mathscr{A}}.
\end{aligned}$$

Note that $\mathcal{I}_4=0$ because the property of L^2-projection operator $\bar{\Pi}$. By the property of L^2-projection operator $\mathbf{\Pi}$, the Young inequality, and trace inequality,

$$\begin{aligned}
\mathcal{I}_5 &= (\nabla\cdot\bar{\boldsymbol{\eta}}_1^n,\sqrt{\nu}\boldsymbol{\xi}_2^n)-([\bar{\boldsymbol{\eta}}_1^n],\sqrt{\nu}\{\boldsymbol{\xi}_2^n\}\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^i}\\
&=-\Big([\bar{\boldsymbol{\eta}}_1^n],\sqrt{\nu}\{\boldsymbol{\xi}_2^n\}\otimes\boldsymbol{n}_e\Big)_{\mathscr{E}_h^i}\\
&\le \sum_{e\in\mathscr{E}_h^i}\sqrt{\nu}\,\|\{\boldsymbol{\xi}_2^n\}\otimes\boldsymbol{n}_e\|_{L^2(e)}\|[\bar{\boldsymbol{\eta}}_1^n]\|_{L^2(e)}\\
&\le \sum_{e\in\mathscr{E}_h^i}\sqrt{\nu}\|\{\boldsymbol{\xi}_2^n\}\otimes\boldsymbol{n}_e\|_{L^2(e)}\Big(Ch_{E_1^e}^{-1/2}\|\bar{\boldsymbol{\eta}}_1^n\|_{L^2(E_1^e)}+Ch_{E_2^e}^{-1/2}\|\bar{\boldsymbol{\eta}}_1^n\|_{L^2(E_2^e)}\Big)\\
&\le \sqrt{\nu}Ch^{-1/2}\Big(\sum_{e\in\mathscr{E}_h^i}\|\{\boldsymbol{\xi}_2^n\}\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{e\in\mathscr{E}_h^i}(\|\bar{\boldsymbol{\eta}}_1^n\|_{L^2(E_1^e)}+\|\bar{\boldsymbol{\eta}}_1^n\|_{L^2(E_2^e)})^2\Big)^{1/2}\\
&\le \sqrt{\nu}Ch^k\big|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\big|_{\mathscr{A}}.
\end{aligned}$$

From identity (2.6), with the same deduction, there are

$$\begin{aligned}
\mathcal{I}_6 &= -(\nabla\zeta_1^n,\boldsymbol{\xi}_2^n)+([\zeta_1^n],\{\boldsymbol{\xi}_2^n\}\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^i}\\
&=([\zeta_1^n],\{\boldsymbol{\xi}_2^n\}\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^i}\\
&\le C\Big(\sum_{e\in\mathscr{E}_h^i}\|\{\boldsymbol{\xi}_2^n\}\cdot\boldsymbol{n}_e\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{e\in\mathscr{E}_h^i}\|[\zeta_1^n]\|_{L^2(e)}^2\Big)^{1/2}\\
&\le Ch^{k+1/2}\big|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\big|_{\mathscr{A}},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_7 &\le \sum_{e\in\mathscr{E}_h^i}\|[\zeta_2^n]\|_{L^2(e)}\|[\zeta_1^n]\|_{L^2(e)}\\
&\le \Big(\sum_{e\in\mathscr{E}_h^i}\|[\zeta_2^n]\|_{L^2(e)}^2\Big)^{1/2}\Big(\sum_{e\in\mathscr{E}_h^i}\|[\zeta_1^n]\|_{L^2(e)}^2\Big)^{1/2}\\
&\le Ch^{k+1/2}\big|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\big|_{\mathscr{A}}.
\end{aligned}$$

Now let us tackle the first term of the left side of Eq. (4.7). It is easy to obtain that

$$\left(\partial_t\boldsymbol{u}^n+(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n-\frac{\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)$$

$$
\begin{aligned}
&=\left(\partial_t \boldsymbol{u}^n+(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n-\frac{\boldsymbol{u}^n-\dot{\boldsymbol{u}}^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)+\left(\frac{\check{\boldsymbol{u}}^{n-1}-\dot{\boldsymbol{u}}^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\\
&\quad+\left(\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)-\left(\frac{\boldsymbol{\xi}_2^n-\check{\boldsymbol{\xi}}_2^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\\
&=\sum_{i=1}^4\mathcal{B}_i.
\end{aligned}\tag{4.8}
$$

From Lemma 2.1 and Hölder's inequality, there is

$$
\begin{aligned}
|\mathcal{B}_1|&=\left|\left(\partial_t \boldsymbol{u}^n+(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n-\frac{\boldsymbol{u}^n-\dot{\boldsymbol{u}}^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\right|\\
&\le C\Delta t\,\|\boldsymbol{\xi}_1^n\|_0\\
&\le C\Delta t^2+C\,\|\boldsymbol{\xi}_1^n\|_0^2.
\end{aligned}
$$

By the definitions of $\check{\mathbf{x}}$ and $\dot{\mathbf{x}}$,

$$
\check{\mathbf{x}}-\dot{\mathbf{x}}=\Delta t(\boldsymbol{u}_h^{n-1}-\boldsymbol{u}^{n-1}).
$$

Using the Taylor formula, we have

$$
\begin{aligned}
|\check{\boldsymbol{u}}^{n-1}-\dot{\boldsymbol{u}}^{n-1}|&=|\boldsymbol{u}^{n-1}(\check{\mathbf{x}})-\boldsymbol{u}^{n-1}(\dot{\mathbf{x}})|\\
&\le\Delta t\,\|\nabla\boldsymbol{u}^{n-1}\|_\infty\,|\boldsymbol{u}_h^{n-1}-\boldsymbol{u}^{n-1}|\\
&\le C\Delta t\,\|\nabla\boldsymbol{u}^{n-1}\|_\infty\,(|\boldsymbol{\xi}_1^{n-1}|+|\boldsymbol{\xi}_2^{n-1}|).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\check{\boldsymbol{u}}^{n-1}-\dot{\boldsymbol{u}}^{n-1}\|_0\\
&\le C\Delta t\,\|\nabla\boldsymbol{u}^{n-1}\|_\infty\,(\|\boldsymbol{\xi}_1^{n-1}\|_0+\|\boldsymbol{\xi}_2^{n-1}\|_0)\\
&\le C\Delta t(h^{k+1}+\|\boldsymbol{\xi}_1^{n-1}\|_0).
\end{aligned}\tag{4.9}
$$

From inequality (4.9), we deduce

$$
\begin{aligned}
|\mathcal{B}_2|&=\left|\left(\frac{\check{\boldsymbol{u}}^{n-1}-\dot{\boldsymbol{u}}^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\right|\\
&\le\frac{1}{\Delta t}\,\|\check{\boldsymbol{u}}^{n-1}-\dot{\boldsymbol{u}}^{n-1}\|_0\|\boldsymbol{\xi}_1^n\|_0\\
&\le Ch^{2k+2}+C\,\|\boldsymbol{\xi}_1^{n-1}\|_0^2+C\,\|\boldsymbol{\xi}_1^n\|_0^2.
\end{aligned}
$$

By Lemma 3.1, there is

$$\begin{aligned}\mathcal{B}_3&=\left(\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\\&=\frac{1}{2\Delta t}\Big(\|\boldsymbol{\xi}_1^n\|_0^2-\|\check{\boldsymbol{\xi}}_1^{n-1}\|_0^2\Big)+\frac{1}{2\Delta t}\|\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}\|_0^2\\&\geq\frac{1}{2\Delta t}\Big(\|\boldsymbol{\xi}_1^n\|_0^2-\|\boldsymbol{\xi}_1^{n-1}\|_0^2\Big)-C\|\boldsymbol{\xi}_1^{n-1}\|_0^2+\frac{1}{2\Delta t}\|\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}\|_0^2.\end{aligned}$$

From the definition, we can get

$$\begin{aligned}\mathcal{B}_4&=-\left(\frac{\boldsymbol{\xi}_2^n-\check{\boldsymbol{\xi}}_2^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\\&=-\left(\frac{\boldsymbol{\xi}_2^n-\boldsymbol{\xi}_2^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)-\left(\frac{\boldsymbol{\xi}_2^{n-1}-\check{\boldsymbol{\xi}}_2^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right).\end{aligned}$$

Consequently, from Taylor formula and Hölder's inequality, it follows that

$$\left|\left(\frac{\boldsymbol{\xi}_2^n-\boldsymbol{\xi}_2^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\right|\leq C\Big(\|\boldsymbol{\xi}_1^n\|_0^2+\frac{1}{\Delta t}\|\partial_t\boldsymbol{\xi}_2\|_{L^2(J^n;\Omega)}^2\Big).$$

Using the Hölder inequality, Young inequality and Lemma 4.3, we have

$$\left|\left(\frac{\boldsymbol{\xi}_2^{n-1}-\check{\boldsymbol{\xi}}_2^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\right|\leq C\Big(\|\boldsymbol{\xi}_1^n\|_0^2+\|\nabla\boldsymbol{\xi}_2^{n-1}\|_0^2\Big).$$

Combining $\mathcal{B}_i$, $i=1,\cdots,4$, there is

$$\begin{aligned}&\left(\partial_t\boldsymbol{u}^n+(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n-\frac{\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}}{\Delta t},\boldsymbol{\xi}_1^n\right)\\\geq&\frac{1}{2\Delta t}\Big(\|\boldsymbol{\xi}_1^n\|_0^2-\|\boldsymbol{\xi}_1^{n-1}\|_0^2\Big)-C\|\boldsymbol{\xi}_1^{n-1}\|_0^2\\&+\frac{1}{2\Delta t}\|\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}\|_0^2-C\|\boldsymbol{\xi}_1^n\|_0^2-\frac{C}{\Delta t}\|\partial_t\boldsymbol{\xi}_2\|_{L^2(J^n;\Omega)}^2\\&-C\|\nabla\boldsymbol{\xi}_2^{n-1}\|_0^2-Ch^{2k+2}-C\Delta t^2.\end{aligned}\tag{4.10}$$

Substituting $\mathcal{I}_i$, $i=1,\cdots,7$ and (4.10) into (4.7) and rearranging the terms in above inequality, we can obtain

$$\begin{aligned}&\frac{1}{2\Delta t}\Big(\|\boldsymbol{\xi}_1^n\|_0^2-\|\boldsymbol{\xi}_1^{n-1}\|_0^2\Big)+\frac{1}{2}\big|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\big|_{\mathscr{A}}^2+\frac{1}{2\Delta t}\|\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}\|_0^2\\\leq&C\|\boldsymbol{\xi}_1^{n-1}\|_0^2+C\|\boldsymbol{\xi}_1^n\|_0^2+\frac{C}{\Delta t}\|\partial_t\boldsymbol{\xi}_2\|_{L^2(J^n;\Omega)}^2\\&+C\|\nabla\boldsymbol{\xi}_2^{n-1}\|_0^2+C\Delta t^2+Ch^{2k+1}+\nu Ch^{2k}.\end{aligned}\tag{4.11}$$

Summing over n from 1 to N and multiplying $2\Delta t$ from the both sides of (4.11), and using discrete Grönwall inequality, we finally obtain

$$\begin{aligned}&\|\boldsymbol{\xi}_1^N\|_0^2+\Delta t\sum_{n=1}^{N}\left|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\right|_{\mathscr{A}}^2+\sum_{n=1}^{N}\|\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}\|_0^2\\ \le& C\sum_{n=1}^{N}\|\partial_t\boldsymbol{\xi}_2\|_{L^2(J^n;\Omega)}^2+C\Delta t\sum_{n=1}^{N}\|\nabla\boldsymbol{\xi}_2^{n-1}\|_0^2\\ &+C\Delta t^2+Ch^{2k+1}+\nu Ch^{2k}.\end{aligned}\tag{4.12}$$

By the triangular inequality, the desired error bound of (4.6) is obtained. □

Remark 4.1. Note that from (4.12) for any integer $N=1,2,\cdots$, there are

$$\|\boldsymbol{\xi}_1^N\|_0^2\le C\left(\Delta t^2+h^{2k}\right),\quad \sum_{n=1}^{N}\|\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}\|_0^2\le C\left(\Delta t^2+h^{2k}\right).$$

4.2 Error in pressure

Lemma 4.5 (Div-grad relation [22]). *If $\boldsymbol{v}\in H_0^1(\Omega)^2$, then*

$$\|\nabla\cdot\boldsymbol{v}\|_0\le\|\nabla\boldsymbol{v}\|_0.\tag{4.13}$$

Lemma 4.6 (Inverse inequality). *For any polynomial function v of degree i defined on E. There exists a constant C independent of h_E such that*

$$\forall 0\le j\le i,\quad \|\nabla^j v\|_{L^2(E)}\le Ch_E^{-j}\|v\|_{L^2(E)}.\tag{4.14}$$

To obtain the error estimate in the pressure, we shall recall the continuous inf-sup condition for the spaces $H_0^1(\Omega)^2$ and $L_0^2(\Omega)$.

Lemma 4.7 ([10,16,23]). *There exists a positive constant β, such that*

$$\inf_{q\in L_0^2(\Omega)}\sup_{\boldsymbol{v}\in H_0^1(\Omega)^2}\frac{(\nabla\cdot\boldsymbol{v},q)}{\|q\|_0\|\nabla\boldsymbol{v}\|_0}\ge\beta.\tag{4.15}$$

Equivalently for any $q\in L_0^2(\Omega)$, there is a function $\tilde{\boldsymbol{v}}\in H_0^1(\Omega)^2$ such that

$$-\int_\Omega q\nabla\cdot\tilde{\boldsymbol{v}}\,d\boldsymbol{x}\ge\beta_1\|q\|_0^2,\quad \|\tilde{\boldsymbol{v}}\|_1\le\beta_2\|q\|_0,\tag{4.16}$$

where $\beta_1>0$, $\beta_2>0$ are positive constants independent of $h,\Delta t,q$ and $\tilde{\boldsymbol{v}}$.

Lemma 4.8. *For any functions* $(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\in\mathbb{V}_h\times\mathbb{V}_h^2\times\mathbb{Q}_h$*, there exists a function* $\tilde{\boldsymbol{v}}\in H_0^1(\Omega)^2$ *such that for two positive constants* K_1 *and* K_2 *independent of* $h,\Delta t$ *and* q*, there are*

$$K_1\|q\|_0^2\le\mathscr{A}(\boldsymbol{v},\bar{\boldsymbol{\tau}},q;\boldsymbol{\Pi}\tilde{\boldsymbol{v}},\bar{\boldsymbol{0}},0)+K_2\big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|^2_{\mathscr{A}},\quad \|\boldsymbol{\Pi}\tilde{\boldsymbol{v}}\|_1\le C\|q\|_0, \tag{4.17}$$

where $\boldsymbol{\Pi}\tilde{\boldsymbol{v}}$ *is the* L^2*-projection of* $\tilde{\boldsymbol{v}}$ *onto the finite element space* $\mathbb{V}_h$*,* C *is a generic constant.*

Proof. With similar deduction as [10], we fix $q\in\mathbb{Q}_h\subset L_0^2(\Omega)$. From Lemma 4.7 for $(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\in\mathbb{V}_h\times\mathbb{V}_h^2\times\mathbb{Q}_h$, there is a function $\tilde{\boldsymbol{v}}\in H_0^1(\Omega)^2$ satisfying (4.16). Then by equality (2.10), we have

$$\begin{aligned}&\mathscr{A}(\boldsymbol{v},\bar{\boldsymbol{\tau}},q;\boldsymbol{\Pi}\tilde{\boldsymbol{v}},\bar{\boldsymbol{0}},0)\\&=\mathbb{a}(\bar{\boldsymbol{\tau}},\boldsymbol{\Pi}\tilde{\boldsymbol{v}})+\mathbb{b}(q,\boldsymbol{\Pi}\tilde{\boldsymbol{v}})+\mathbb{c}(\boldsymbol{v},\boldsymbol{\Pi}\tilde{\boldsymbol{v}})\\&=\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3.\end{aligned}\tag{4.18}$$

Next we shall estimate $\mathcal{T}_i$ respectively. By the definition of $\mathcal{T}_1$

$$\begin{aligned}\big|\mathcal{T}_1\big|&=\big|\mathbb{a}(\bar{\boldsymbol{\tau}},\boldsymbol{\Pi}\tilde{\boldsymbol{v}})\big|\\&\le\big|\mathbb{a}(\bar{\boldsymbol{\tau}},\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}})\big|+\big|\mathbb{a}(\bar{\boldsymbol{\tau}},\tilde{\boldsymbol{v}})\big|\\&=\big|-(\nabla\cdot\bar{\boldsymbol{\tau}},\sqrt{\nu}(\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}))+([\bar{\boldsymbol{\tau}}],\sqrt{\nu}\{\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}\}\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^i}\big|+\big|(\bar{\boldsymbol{\tau}},\sqrt{\nu}\nabla\tilde{\boldsymbol{v}})\big|\\&=\big|([\bar{\boldsymbol{\tau}}],\sqrt{\nu}\{\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}\}\otimes\boldsymbol{n}_e)_{\mathscr{E}_h^i}\big|+\big|(\bar{\boldsymbol{\tau}},\sqrt{\nu}\nabla\tilde{\boldsymbol{v}})\big|\\&\le C\sqrt{\nu}\Big(\sum_{e\in\mathscr{E}_h^i}\|[\bar{\boldsymbol{\tau}}]\|^2_{L^2(e)}\Big)^{1/2}\Big(\sum_{e\in\mathscr{E}_h^i}\|\{\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}\}\otimes\boldsymbol{n}_e\|^2_{L^2(e)}\Big)^{1/2}+C\sqrt{\nu}\|\bar{\boldsymbol{\tau}}\|_0\|\tilde{\boldsymbol{v}}\|_1\\&\le C\sqrt{\nu}h^{1/2}\Big(\sum_{e\in\mathscr{E}_h^i}\|[\bar{\boldsymbol{\tau}}]\|^2_{L^2(e)}\Big)^{1/2}\|\tilde{\boldsymbol{v}}\|_1+C\sqrt{\nu}\|\bar{\boldsymbol{\tau}}\|_0\|\tilde{\boldsymbol{v}}\|_1\\&\le C\sqrt{\nu}\|\bar{\boldsymbol{\tau}}\|_0\|\tilde{\boldsymbol{v}}\|_1\\&\le C\sqrt{\nu}\big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|_{\mathscr{A}}\|q\|_0.\end{aligned}$$

Hence

$$\mathcal{T}_1\ge-\nu C\epsilon_1\|q\|_0^2-C\epsilon_1^{-1}\big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|^2_{\mathscr{A}}.\tag{4.19}$$

By the definition of $\mathcal{T}_2$, we obtain

$$\mathcal{T}_2=\mathbb{b}(q,\boldsymbol{\Pi}\tilde{\boldsymbol{v}})=\mathbb{b}(q,\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}})+\mathbb{b}(q,\tilde{\boldsymbol{v}}).$$

Since

$$\begin{aligned}\big|\mathbb{b}(q,\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}})\big|&=\big|([q],\{\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}\}\cdot\boldsymbol{n}_e)_{\mathscr{E}_h^i}\big|\\&\le\Big(\sum_{e\in\mathscr{E}_h^i}\|[q]\|^2_{L^2(e)}\Big)^{1/2}\Big(\sum_{e\in\mathscr{E}_h^i}\|\{\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}\}\cdot\boldsymbol{n}_e\|^2_{L^2(e)}\Big)^{1/2}\\&\le Ch^{1/2}\big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|_{\mathscr{A}}\|\tilde{\boldsymbol{v}}\|_1\\&\le Ch^{1/2}\big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|_{\mathscr{A}}\|q\|_0,\end{aligned}\tag{4.20}$$

and

$$\mathbb{b}(q,\tilde{\boldsymbol{v}}) = -(q,\nabla\cdot\tilde{\boldsymbol{v}}) \geq \beta_1 \| q \|_0^2. \tag{4.21}$$

Combining (4.20) and (4.21), there is

$$\mathcal{T}_2 \geq \beta_1 \| q \|_0^2 - Ch\epsilon_2 \| q \|_0^2 - C\epsilon_2^{-1} \big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|_{\mathscr{A}}^2. \tag{4.22}$$

Observe that

$$\begin{aligned}
\mathcal{T}_3 &= \mathbb{c}(\boldsymbol{v},\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}) \\
&\leq \Big(\sum_{e\in\mathscr{E}_h^B} \| [\boldsymbol{v}] \|_{L^2(e)}^2\Big)^{1/2} \Big(\sum_{e\in\mathscr{E}_h^B} \| [\boldsymbol{\Pi}\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}] \|_{L^2(e)}^2\Big)^{1/2} \\
&\leq Ch^{1/2} \big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|_{\mathscr{A}} \| \tilde{\boldsymbol{v}} \|_1 \\
&\leq Ch^{1/2} \big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|_{\mathscr{A}} \| q \|_0.
\end{aligned}$$

Then

$$\mathcal{T}_3 \geq -Ch\epsilon_3 \| q \|_0^2 - C\epsilon_3^{-1} \big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|_{\mathscr{A}}^2. \tag{4.23}$$

Substituting $\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3$ into (4.18), we deduce

$$\begin{aligned}
&\mathscr{A}(\boldsymbol{v},\bar{\boldsymbol{\tau}},q;\boldsymbol{\Pi}\tilde{\boldsymbol{v}},\bar{\boldsymbol{0}},0) \\
&\geq (\beta_1 - \nu C\epsilon_1 - Ch\epsilon_2 - Ch\epsilon_3) \| q \|_0^2 - (C\epsilon_1^{-1} + C\epsilon_2^{-1} + C\epsilon_3^{-1}) \big|(\boldsymbol{v},\bar{\boldsymbol{\tau}},q)\big|_{\mathscr{A}}^2,
\end{aligned} \tag{4.24}$$

where $\epsilon_1,\epsilon_2,\epsilon_3$ are chosen such that $K_1 = \beta_1 - \nu C\epsilon_1 - Ch\epsilon_2 - Ch\epsilon_3 > 0$ and $K_2 = C\epsilon_1^{-1} + C\epsilon_2^{-1} + C\epsilon_3^{-1} > 0$, and K_1,K_2 are positive constants independent of h.

Furthermore, from Lemma 4.1 we have

$$\| \boldsymbol{\Pi}\tilde{\boldsymbol{v}} \|_1 \leq \| \boldsymbol{\Pi}\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}} \|_1 + \| \tilde{\boldsymbol{v}} \|_1 \leq C \| \tilde{\boldsymbol{v}} \|_1 \leq C \| q \|_0. \tag{4.25}$$

The proof is completed. $\square$

Theorem 4.2 (Error estimate of the pressure)**.** *Let* $(\boldsymbol{u}^n,p^n)$ *be the solution of* (1.1) *at time* $t=t^n$ *and* $\bar{\boldsymbol{\sigma}}^n \in (H^{k+1}(\Omega)^2)^2$, $(\boldsymbol{u}_h^n,\bar{\boldsymbol{\sigma}}_h^n,p_h^n)$ *the solution of the CLDG scheme of* (2.8)*. If* $\Delta t < \frac{1}{2L_n}$, $L_n = \max_{1\leq i\leq n} \| \boldsymbol{u}_h^i \|_{1,\infty}$, *and the regularity* (4.1) *satisfied, then for any integer* $N = 1,2,\cdots$,

$$\Delta t \sum_{n=1}^{N} \| e_p^n \|_0^2 \leq C(\Delta t + h^{2k}/\Delta t). \tag{4.26}$$

Proof. From Lemma 4.7 and Lemma 4.8, for $\zeta_1^n \in \mathbb{Q}_h$, there exists a function $\boldsymbol{w} \in H_0^1(\Omega)^2$ and its L^2-projection $\boldsymbol{\Pi w}$ satisfying Eq. (4.17), namely

$$K_1 \| \zeta_1^n \|_0^2 \leq \mathscr{A}(\boldsymbol{\zeta}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n;\boldsymbol{\Pi w},\bar{\boldsymbol{0}},0) + K_2 \big|(\boldsymbol{\zeta}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\big|_{\mathscr{A}}^2, \quad \| \boldsymbol{\Pi w} \|_1 \leq C \| \zeta_1^n \|_0.$$

From the first equation of (2.5) and the first equation of (2.9), it follows that

$$\begin{aligned}&\left(\partial_t \boldsymbol{u}^n+(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n-\frac{\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}}{\Delta t},\boldsymbol{\Pi w}\right)+\mathscr{A}(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n;\boldsymbol{\Pi w},\bar{\boldsymbol{0}},0)\\ =&\mathscr{A}(\boldsymbol{\xi}_2^n,\bar{\boldsymbol{\eta}}_2^n,\zeta_2^n;\boldsymbol{\Pi w},\bar{\boldsymbol{0}},0).\end{aligned}\tag{4.27}$$

By Lemma 4.8 and rearranging identity (4.27),

$$\begin{aligned}K_1\|\zeta_1^n\|_0^2\le&\mathscr{A}(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n;\boldsymbol{\Pi w},\bar{\boldsymbol{0}},0)+K_2\left|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\right|_{\mathscr{A}}^2\\ \le&\left|\left(\partial_t \boldsymbol{u}^n+(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n-\frac{\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}}{\Delta t},\boldsymbol{\Pi w}\right)\right|\\ &+\left|\mathscr{A}(\boldsymbol{\xi}_2^n,\bar{\boldsymbol{\eta}}_2^n,\zeta_2^n;\boldsymbol{\Pi w},\bar{\boldsymbol{0}},0)\right|+K_2\left|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\right|_{\mathscr{A}}^2.\end{aligned}\tag{4.28}$$

From the same deduction of the characteristic term, and Lemma 4.4, we have

$$\begin{aligned}&\left|\left(\partial_t \boldsymbol{u}^n+(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n-\frac{\boldsymbol{u}_h^n-\check{\boldsymbol{u}}_h^{n-1}}{\Delta t},\boldsymbol{\Pi w}\right)\right|\\ \le&C\Delta t\|\boldsymbol{\Pi w}\|_0+\frac{1}{\Delta t}\|\check{\boldsymbol{u}}^{n-1}-\dot{\boldsymbol{u}}^{n-1}\|_0\|\boldsymbol{\Pi w}\|_0\\ &+\left|\left(\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t},\boldsymbol{\Pi w}\right)\right|+\left|\left(\frac{\boldsymbol{\xi}_2^n-\check{\boldsymbol{\xi}}_2^{n-1}}{\Delta t},\boldsymbol{\Pi w}\right)\right|\\ \le&C\left(\Delta t+h^{k+1}+\|\boldsymbol{\xi}_1^{n-1}\|_0\right)\|\boldsymbol{\Pi w}\|_1+\left\|\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t}\right\|_0\|\boldsymbol{\Pi w}\|_0\\ &+\left\|\frac{\boldsymbol{\xi}_2^n-\boldsymbol{\xi}_2^{n-1}}{\Delta t}\right\|_0\|\boldsymbol{\Pi w}\|_0+\left\|\frac{\boldsymbol{\xi}_2^{n-1}-\check{\boldsymbol{\xi}}_2^{n-1}}{\Delta t}\right\|_{-1}\|\boldsymbol{\Pi w}\|_1\\ \le&C\left(\Delta t+h^k+\left\|\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t}\right\|_0\right)\|\boldsymbol{\Pi w}\|_1\\ &+\left(\frac{1}{\sqrt{\Delta t}}\|\partial_t\boldsymbol{\xi}_2\|_{L^2(\Omega\times J^n)}+\|\boldsymbol{\xi}_2^{n-1}\|_0\right)\|\boldsymbol{\Pi w}\|_1\\ \le&C\left(\Delta t+h^k+\left\|\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t}\right\|_0\right)\|\zeta_1^n\|_0.\end{aligned}$$

With the same deduction of $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3$, there is

$$\begin{aligned}&\left|\mathscr{A}(\boldsymbol{\xi}_2^n,\bar{\boldsymbol{\eta}}_2^n,\zeta_2^n;\boldsymbol{\Pi w},\bar{\boldsymbol{0}},0)\right|\\ \le&Ch^{k+\frac{1}{2}}\left|(\Pi\boldsymbol{w},\bar{\boldsymbol{0}},0)\right|_{\mathscr{A}}\\ \le&Ch^k\|\Pi\boldsymbol{w}\|_0\\ \le&Ch^k\|\zeta_1^n\|_0.\end{aligned}\tag{4.29}$$

From (4.28) and the Young inequality, it follows that

$$\begin{aligned}
K_1\|\zeta_1^n\|_0^2 \le & C\left(\Delta t + h^k + \left\|\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t}\right\|_0\right)\|\zeta_1^n\|_0 \\
& + Ch^k\|\zeta_1^n\|_0 + K_2\left|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\right|_{\mathscr{A}}^2 \\
\le & \frac{C}{K_1}\left(\Delta t^2 + h^{2k} + \left\|\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t}\right\|_0^2\right) \\
& + \frac{K_1}{2}\|\zeta_1^n\|_0^2 + K_2\left|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\right|_{\mathscr{A}}^2.
\end{aligned} \tag{4.30}$$

Rearranging above inequality, multiplying $2\Delta t$ for both sides, and summing n from 1 to N, from Remark 4.1 we have

$$\begin{aligned}
\Delta t\sum_{n=1}^N\|\zeta_1^n\|_0^2 \le & C((\Delta t)^2 + h^{2k}) + C\Delta t\sum_{n=1}^N\left|(\boldsymbol{\xi}_1^n,\bar{\boldsymbol{\eta}}_1^n,\zeta_1^n)\right|_{\mathscr{A}}^2 + \Delta t\sum_{n=1}^N\left\|\frac{\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}}{\Delta t}\right\|_0^2 \\
\le & C\Big(\Delta t^2 + h^{2k}\Big) + \frac{C}{\Delta t}\sum_{n=1}^N\|\boldsymbol{\xi}_1^n-\check{\boldsymbol{\xi}}_1^{n-1}\|_0^2 \\
\le & C\Big(\Delta t + h^{2k}/\Delta t\Big).
\end{aligned}$$

Using triangular inequality, we complete the proof. □

5 Numerical experiment

In this section, we give two examples to verify our theoretical error estimates. In all our experiments the uniform triangulations of squares are used. For numerical computation, the characteristic part is calculated by the higher-order accurate Gaussian quadrature rule [20] and the CLDG scheme is performed with $(\mathbb{P}^k,\mathbb{P}^k,\mathbb{P}^k)$ finite element pair $(k\ge 1)$. The time stepsize is taken as $\Delta t = \mathcal{O}(h)$ for the local $\mathbb{P}^1$-DG scheme and $\Delta t = \mathcal{O}(h^2)$ for the local $\mathbb{P}^2$-DG scheme. In Tables 1-6, K and k denote the number of elements in triangulation and the degree of approximation polynomials, respectively. Comparing the numerical solutions with the constructed analytical ones, the suboptimal convergence orders are displayed for the presented numerical schemes with a wide range of Reynolds numbers, such as $Re = 10^2, 10^3, 10^6, 10^8, 10^{12}, 10^{15}$.

Example 5.1. Consider the time-dependent incompressible Navier-Stokes equation in a square domain $\Omega = [-1,1]^2$. The exact solution is specified as

$$\begin{cases}
u_1(x,y,t) = \frac{1}{4}e^{\nu t}y(y^2-1)(x^2-1)^2, \\
u_2(x,y,t) = -\frac{1}{4}e^{\nu t}x(x^2-1)(y^2-1)^2, \\
p(x,y,t) = e^{\nu t}(x^2-1)(y^2-1).
\end{cases} \tag{5.1}$$

Then the exact solution has homogenous boundary value and the forcing term $\boldsymbol{f}$ can be determined by (1.1) for any given ν.

Tables 1-3 display the L^2-norm errors and convergence orders of velocity and pressure for Example 5.1 at time $T=0.25$ with different choices of Reynolds numbers, such as $Re=10^3,10^6,10^{12}$. Note that the numerical convergence are greater than the analytical ones. Furthermore, the errors and convergence orders for both velocity and pressure are still uniform with analytical ones when $Re=10^{12}$ is chosen.

Table 1: The L^2-norm errors and convergence orders of velocity and pressure for Example 5.1 with $T=0.25$, $Re=10^3$.

K	$k=1$		$k=2$		$k=1$		$k=2$	
	$\lVert e_{\boldsymbol{u}} \rVert_0$	*order*	$\lVert e_{\boldsymbol{u}} \rVert_0$	*order*	$\lVert e_p \rVert_0$	*order*	$\lVert e_p \rVert_0$	*order*
8	7.3307e-02	–	2.1805e-02	–	7.4411e-01	–	9.2279e-02	–
32	3.0062e-02	1.29	6.1735e-03	1.82	1.7126e-01	2.12	7.6043e-03	3.60
128	1.1684e-02	1.36	8.0651e-04	2.94	3.9616e-02	2.11	9.1111e-04	3.06
512	3.2724e-03	1.84	1.1109e-04	2.86	9.0959e-03	2.12	1.1663e-04	2.97
2048	8.4909e-04	1.95	1.4178e-05	2.97	2.1376e-03	2.09	1.4947e-05	2.96

Table 2: The L^2-norm errors and convergence orders of velocity and pressure for Example 5.1 with $T=0.25$, $Re=10^6$.

K	$k=1$		$k=2$		$k=1$		$k=2$	
	$\lVert e_{\boldsymbol{u}} \rVert_0$	*order*	$\lVert e_{\boldsymbol{u}} \rVert_0$	*order*	$\lVert e_p \rVert_0$	*order*	$\lVert e_p \rVert_0$	*order*
8	7.3405e-02	–	2.1811e-02	–	7.4408e-01	–	9.2308e-02	–
32	3.0176e-02	1.28	6.2362e-03	1.81	1.7144e-01	2.12	7.5987e-03	3.60
128	1.1803e-02	1.35	8.3770e-04	2.90	3.9716e-02	2.11	9.1076e-04	3.06
512	3.3761e-03	1.81	1.2512e-04	2.74	9.1327e-03	2.12	1.1639e-04	2.97
2048	9.0460e-04	1.90	2.0576e-05	2.60	2.1469e-03	2.09	1.4813e-05	2.97

Table 3: The L^2-norm errors and convergence orders of velocity and pressure for Example 5.1 with $T=0.25$, $Re=10^{12}$.

K	$k=1$		$k=2$		$k=1$		$k=2$	
	$\lVert e_{\boldsymbol{u}} \rVert_0$	*order*	$\lVert e_{\boldsymbol{u}} \rVert_0$	*order*	$\lVert e_p \rVert_0$	*order*	$\lVert e_p \rVert_0$	*order*
8	7.3405e-02	–	2.1811e-02	–	7.4408e-01	–	9.2308e-02	–
32	3.0177e-02	1.28	6.2363e-03	1.81	1.7144e-01	2.12	7.5986e-03	3.60
128	1.1803e-02	1.35	8.3773e-04	2.90	3.9716e-02	2.11	9.1076e-04	3.06
512	3.3762e-03	1.81	1.2514e-04	2.74	9.1327e-03	2.12	1.1639e-04	2.97
2048	9.0467e-04	1.90	2.0588e-05	2.60	2.1469e-03	2.09	1.4813e-05	2.97

Example 5.2. We further verify theoretical results of the CLDG scheme (2.8) in the domain $\Omega=[0,1]^2$. For the exact solution defined by

$$\begin{cases} u_1(x,y,t)=\cos(\nu t)\sin^2(\pi x)\sin(2\pi y), \\ u_2(x,y,t)=-\cos(\nu t)\sin(2\pi x)\sin^2(\pi y), \\ p(x,y,t)=\cos(\nu t)\sin(2\pi x)\sin(2\pi y). \end{cases} \tag{5.2}$$

The forcing term f can be determined by (1.1) for any given ν. In Tables 4-6, we choose big Reynolds numbers to demonstrate the efficiency of the presented scheme, such as $Re=10^3,10^8,10^{15}$. Note that errors and convergence orders for both velocity and pressure almost do not change with $Re=10^8$, $Re=10^{15}$, since $\cos(\nu t)$ does not change

Table 4: The L^2-norm errors and convergence orders of velocity and pressure for Example 5.2 with $T=0.5$, $Re=10^2$.

K	$k=1$		$k=2$		$k=1$		$k=2$	
	$\Vert e_u \Vert_0$	*order*	$\Vert e_u \Vert_0$	*order*	$\Vert e_p \Vert_0$	*order*	$\Vert e_p \Vert_0$	*order*
32	8.1469e-02	–	4.6263e-02	–	2.1081e-01	–	3.2568e-02	–
128	3.8797e-02	1.07	4.5086e-03	3.36	4.1743e-02	2.34	6.1042e-03	2.42
512	1.3465e-02	1.53	5.6697e-04	2.99	9.8853e-02	2.08	1.0498e-03	2.54
2048	5.4458e-03	1.31	1.0132e-04	2.48	4.3669e-03	1.18	2.3419e-04	2.16

Table 5: The L^2-norm errors and convergence orders of velocity and pressure for Example 5.2 with $T=0.5$, $Re=10^8$.

K	$k=1$		$k=2$		$k=1$		$k=2$	
	$\Vert e_u \Vert_0$	*order*	$\Vert e_u \Vert_0$	*order*	$\Vert e_p \Vert_0$	*order*	$\Vert e_p \Vert_0$	*order*
32	1.0755e-01	–	1.1246e-01	–	2.1007e-01	–	4.9090e-02	–
128	7.6459e-02	0.49	1.7180e-02	2.71	4.0161e-02	2.39	1.2860e-02	1.93
512	2.9048e-02	1.40	2.9317e-03	2.55	1.7924e-02	1.16	3.1689e-03	2.02
2048	1.3325e-02	1.12	1.0135e-03	1.53	1.0965e-02	0.71	7.3794e-04	2.10

Table 6: The L^2-norm errors and convergence orders of velocity and pressure for Example 5.2 with $T=0.5$, $Re=10^{15}$.

K	$k=1$		$k=2$		$k=1$		$k=2$	
	$\Vert e_u \Vert_0$	*order*	$\Vert e_u \Vert_0$	*order*	$\Vert e_p \Vert_0$	*order*	$\Vert e_p \Vert_0$	*order*
32	1.0755e-01	–	1.1246e-01	–	2.1007e-01	–	4.9090e-02	–
128	7.6459e-02	0.49	1.7180e-02	2.71	4.0161e-02	2.39	1.2860e-02	1.93
512	2.9048e-02	1.40	2.9318e-03	2.55	1.7924e-02	1.16	3.1689e-03	2.02
2048	1.3325e-02	1.12	1.0136e-03	1.53	1.0965e-02	0.71	7.3795e-04	2.10

when ν is small enough (corresponding Re big enough). Comparing with the errors and convergence orders for Example 5.1, even the errors and the convergence orders are not good as Example 5.1, they still coincide with theoretical results.

6 Conclusions

By carefully constructing the numerical fluxes, adding the penalty terms, and using the characteristic method to discretize the time derivative and nonlinear convective term, we design the effective LDG scheme to solve the time-dependent incompressible Navier-Stokes equations in $\mathbb{R}^2$. Besides the general advantages of the LDG scheme, the proposed scheme is theoretically proved or numerically verified to have the following benefits: 1) it is symmetric, so easy to do theoretical analysis and numerical computation; 2) theoretically proved to be nonlinear stable; 3) numerically verified to have the suboptimal convergence orders; 4) the scheme is efficient for a wide range of Reynolds numbers.

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