Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 4, pp. 483-495

DOI: 10.4208/aamm.11-m11175
August 2012

Influence of Gravity and Taper on the Vibration of a Standing Column

C. Y. Wang*

Departments of Mathematics and Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA

Received 16 November 2011; Accepted (in revised version) 9 May 2012

Available online 10 July 2012

Abstract. The stability and natural vibration of a standing tapered vertical column under its own weight are studied. Exact stability criteria are found for the pointy column and numerical stability boundaries are determined for the blunt tipped column. For vibrations we use an accurate, efficient initial value numerical method for the first three frequencies. Four kinds of columns with linear taper are considered. Both the taper and the cross section shape of the column have large influences on the vibration frequencies. It is found that gravity decreases the frequency while the degree of taper may increase or decrease frequency. Vibrations may occur in two different planes.

AMS subject classifications: 74K10, 74H45, 74H55

Key words: Vibration, column, taper, weight.

1 Introduction

The standing column under the influence of gravity models towers, tall buildings, free-standing poles and antennas. The stability of a uniform standing column was solved in the nineteenth century by Greenhill [1] using what is now known as Bessel functions. See Wang et al. [2] for a review on column stability. The vibration of a uniform standing column was recently studied by Virgin et al. [3], whose experimental results confirm numerical predictions superbly.

For strength reasons the standing column is usually not uniform but tapered, wide at base and narrow at the top. Dinnik [4] studied analytically the stability of a power-law tapered standing column, whose tip must decrease into a sharp point. For other cases numerical or semi-numerical methods, such as the Ritz method [5,6], finite elements [7], series expansions [8], integral equations [9] must be used.

*Corresponding author.
Email: cywang@math.msu.edu (C. Y. Wang)

There have been many papers on the vibration of a tapered beam without a compressive axial force. See e.g., [10]. However, to the author's knowledge, there are no reports on the important problem of the vibration of a standing tapered column which is affected by gravity. Since no analytic solutions exist when gravity is present, we shall use a highly efficient initial value method adapted from Barasch and Chen [11] and Wang [12].

2 Formulation

The equation for small vibrations of a non-uniform Euler-Bernoulli column subjected to an axial force can be derived by considering an elemental segment or from energy considerations, e.g., [13]

$$\frac{\partial^2}{\partial x'^2}\Big(EI(x')\frac{\partial^2 y'}{\partial x'^2}\Big)+\frac{\partial}{\partial x'}\Big(F(x')\frac{\partial y'}{\partial x'}\Big)+\rho(x')\frac{\partial^2 y'}{\partial t'^2}=0. \tag{2.1}$$

Here (x', y') are the longitudinal and transverse coordinates of the column (origin at the base), EI is the flexural rigidity, F is the axial force, ρ is the mass per length and t' is the time. Now for a free standing column of height L

$$F=g\int_{x'}^{L}\rho(x')dx', \tag{2.2}$$

where g is the gravitational acceleration. Let

$$EI(x')=EI_0 l(x'),\qquad \rho(x')=\rho_0 r(x'), \tag{2.3}$$

where EI_0 is the maximum of EI and ρ_0 is the maximum of ρ, both occurring at the base at $x'=0$. Consider a harmonic vibration with frequency ω'

$$y'=w'(x')e^{i\omega' t'}. \tag{2.4}$$

Normalize all lengths by the column length L, the time by $L^2\sqrt{\rho_0/EI_0}$ and drop primes. Eq. (2.1) becomes

$$\frac{d^2}{dx^2}\Big[l(x)\frac{d^2 w}{dx^2}\Big]+\beta\frac{d}{dx}\Big[\int_x^1 r(x)dx\frac{dw}{dx}\Big]-\omega^2 r(x)w=0. \tag{2.5}$$

Here

$$\beta=\frac{g\rho_0 L^3}{EI_0},\qquad \omega=\omega' L^2\sqrt{\rho_0/EI_0} \tag{2.6}$$

are non-dimensional parameters representing gravity force and frequency respectively. At the base of the beam, the column is clamped

$$w(0)=0,\qquad \frac{dw}{dx}(0)=0. \tag{2.7}$$

At the top, the column is free (moment and shear vanish)

$$l(1)\frac{d^2w}{dx^2}\Big|_{x=1}=0,\qquad \frac{d}{dx}\Big[l(x)\frac{d^2w}{dx^2}\Big]\Big|_{x=1}=0. \tag{2.8}$$

Eqs. (2.5), (2.7), (2.8) are to be solved for the eigenvalues or frequencies ω.

We are interested in the important cases where the column has linear taper. In general the rigidity and density vary as follows

$$l=(1-cx)^m,\qquad r=(1-cx)^n. \tag{2.9}$$

Here $0\le c\le 1$ represents the degree of taper and m, n are positive constants. Thus

$$\int_x^1 r dx=\begin{cases}1-x, & c=0,\\ \dfrac{(1-cx)^{n+1}-(1-c)^{n+1}}{c(n+1)}, & c\neq 0.\end{cases} \tag{2.10}$$

If $c=0$, the column is uniform. If $c=1$, the column has a pointy tip. For $c\neq 1$, Eqs. (2.8) reduce to

$$\frac{d^2w}{dx^2}\Big|_{x=1}=0,\qquad \frac{d^3w}{dx^3}\Big|_{x=1}=0. \tag{2.11}$$

Although Eqs. (2.5), (2.7), (2.9), (2.11) can be solved numerically for general values of m and n, we shall consider only the four most important cases. Fig. 1(a) shows a solid tapered column of circular cross section. In this case the density is proportional to the radius squared and $n=2$. The rigidity is proportional to the radius to the fourth power and $m=4$. The same exponents also apply to any regular polygonal cross section, including the square and the equilateral triangle. Fig. 1(b) shows a tapered composite column composed of N inclined uniform legs, connected to each other by webs or trusses of negligible mass (not shown) compared to that of the legs. These are called "tower" by Gere and Carter [14]. In general if in any cross section the legs are at the vertex of a regular polygon, then $m=2$ and $n=0$. Fig. 1(c) shows a solid column of constant thickness and tapered sides. The vibration properties are different in the two principle directions. If the column vibrates about the axis A-A which is perpendicular to the thickness direction, then $m=1$ and $n=1$. If the column vibrates about the axis B-B which is parallel to the thickness direction, then $m=3$ and $n=1$. Note that EI_0 will be different for each direction. Fig. 1(d) shows a composite column composed of two inclined plates strengthened by webs or trusses, called "open web" [14]. Let the plates have constant width. For vibrations about A-A axis, which is perpendicular to the width direction, the exponents are $m=0$ and $n=0$. For vibrations about B-B axis, which is parallel to the width direction, the exponents are $m=2$ and $n=0$.

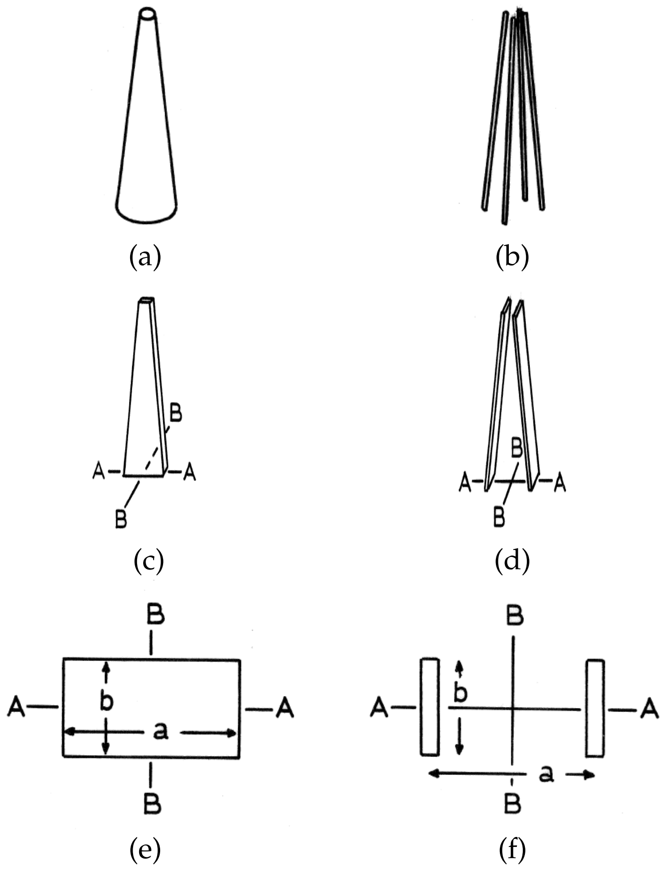

Figure 1: (a) Taper in both transverse directions ($m = 4$, $n = 2$); (b) "Tower" construction ($m = 2$, $n = 0$); (c) Constant thickness and tapered sides ($m = 3$, $n = 1$ and $m = n = 1$); (d) "Open web" construction ($m = 2$, $n = 0$ and $m = n = 0$); (e) Cross section at the base for Fig. 1(c); (f) Cross section at the base for Fig. 1(d).

3 The initial value method

The boundary value problem is difficult since the four boundary conditions are evenly divided at both ends of the column. We shall use a simple initial value method briefly described as follows. Let

$$w = C_1 w_1(x) + C_2 w_2(x), \tag{3.1}$$

where w_1 and w_2 are any independent functions that satisfy the initial conditions Eq. (2.7). Prescribe initial conditions such that

$$w_1(0) = 0, \qquad \frac{dw_1}{dx}(0) = 0, \qquad \frac{d^2w_1}{dx^2}(0) = 1, \qquad \frac{d^3w_1}{dx^3}(0) = 0, \tag{3.2a}$$

$$w_2(0) = 0, \qquad \frac{dw_2}{dx}(0) = 0, \qquad \frac{d^2w_2}{dx^2}(0) = 0, \qquad \frac{d^3w_2}{dx^3}(0) = 1. \tag{3.2b}$$

Then Eq. (2.5) is integrated by the Runge-Kutta method for both w_1 and w_2. Eq. (2.11) gives, for non-trivial solutions, the condition

$$\begin{vmatrix} \dfrac{d^2w_1}{dx^2}(1) & \dfrac{d^2w_2}{dx^2}(1) \\ \dfrac{d^3w_1}{dx^3}(1) & \dfrac{d^3w_2}{dx^3}(1) \end{vmatrix} = 0. \tag{3.3}$$

The frequencies are obtained by bisection to satisfy Eq. (3.3). The errors of both Runge-Kutta and bisection can be prescribed to any accuracy. Comparisons of this method with other numerical methods are given in the next section.

4 Stability

Vibration is only viable if the standing column is statically stable. We first present some exact stability solutions. By setting frequency to zero and integrating once, Eq. (2.5) gives

$$\frac{d}{dx}\Big[l(x)\frac{d^2w}{dx^2}\Big] + \beta\Big[\int_x^1 r(x)dx\frac{dw}{dx}\Big] = \text{constant}. \tag{4.1}$$

The boundary condition Eq. (2.8) shows the constant is zero. We set

$$z = 1 - x, \qquad \theta(z) = \frac{dw}{dx}. \tag{4.2}$$

For the uniform column where $l = 1$ and $r = 1$, Eq. (4.1) becomes

$$\frac{d^2\theta}{dz^2} + \beta z\theta = 0. \tag{4.3}$$

The boundary conditions are

$$\frac{d\theta}{dz}(0) = 0, \tag{4.4a}$$
$$\theta(1) = 0. \tag{4.4b}$$

The solution to Eq. (4.3) up to a multiplying constant and satisfying Eq. (4.4a) is

$$\theta = \sqrt{z}J_{-\frac{1}{3}}\Big(\frac{2\sqrt{\beta}}{3}z^{\frac{3}{2}}\Big), \tag{4.5}$$

where J is the Bessel function of the first kind. Eq. (4.4b) then gives the exact stability equation

$$J_{-\frac{1}{3}}\Big(\frac{2\sqrt{\beta}}{3}\Big) = 0. \tag{4.6}$$

The roots are $\beta = 7.83735, 55.977, 148.508$ but only the lowest (buckling) load is significant. Greenhill [1] obtained 7.833.

For $c = 1$ an exact solution is again possible. For general m, n, Eq. (4.1) reduces to

$$\frac{d}{dz}\Big(z^m \frac{d\theta}{dz}\Big) + \frac{\beta}{n+1} z^{n+1}\theta = 0, \tag{4.7}$$

where the integration constant is set to zero due to zero shear at the top. Eq. (4.7) is rewritten as

$$z^2 \frac{d^2\theta}{dz^2} + mz\frac{d\theta}{dz} + \frac{\beta}{n+1} z^{3+n-m}\theta = 0. \tag{4.8}$$

The bounded solution is (e.g., Murphy [15])

$$\theta = z^{\frac{(1-m)}{2}} J_{\pm\frac{(1-m)}{(3+n-m)}}\Big[\frac{2}{(3+n-m)}\sqrt{\frac{\beta}{n+1}} z^{\frac{(3+n-m)}{2}}\Big], \tag{4.9}$$

where the plus sign is appropriate for all $m \geq 1$. The boundary condition at $z = 1$ gives

$$J_{\pm\frac{(1-m)}{(3+n-m)}}\Big[\frac{2}{(3+n-m)}\sqrt{\frac{\beta}{n+1}}\Big] = 0. \tag{4.10}$$

This exact solution is new. The buckling loads of interest are given in Table 1. The higher modes occur only in physically constrained columns.

For tapered columns where c is not zero or one, numerical integration is necessary. Since the stability problem is only second order, one can use a method described as follows. Let

$$\bar{z} = 1 - cx, \qquad \theta(\bar{z}) = \frac{dw}{dx}. \tag{4.11}$$

Eq. (4.1) gives

$$c^2 \frac{d}{d\bar{z}}\Big(\bar{z}^m \frac{d\theta}{d\bar{z}}\Big) + \frac{\beta}{c(n+1)}\big[\bar{z}^{n+1} - (1-c)^{n+1}\big]\theta = 0, \qquad c \neq 0, 1. \tag{4.12}$$

The boundary conditions are

$$\frac{d\theta}{d\bar{z}}(1-c) = 0, \tag{4.13a}$$

$$\theta(1) = 0. \tag{4.13b}$$

We guess β and without loss, set

$$\theta(1-c) = 1. \tag{4.14}$$

Eq. (4.12), together with Eqs. (4.13a), (4.14) is then integrated as an initial value problem until $\bar{z} = 1$. The buckling load β is found if Eq. (4.13b) is satisfied. If not, β is adjusted. Table 2 shows the primary buckling loads, where the exact values for $c = 0$ and $c = 1$ from Table 1 are included. We note the buckling load increases (more stable) with taper in all cases except $m = 2$ and $n = 0$ which is the tower construction.

Table 1: Exact lowest three buckling loads for tapered columns when $c = 1$. Also listed are values from Greenhill [1] and Dinnik [4].

m	n	β	
0	0	7.83735 55.977 148.508	7.833 [1]
2	0	3.6705 12.3046 25.8749	3.67 [4]
3	1	13.1873 35.425 25.8749	13.1 [4]
1	1	26.0243 137.121 336.992	26.0 [4]
4	2	30.5298 71.4582 127.047	30.6 [4]

Table 2: Buckling load β for tapered standing columns.

c	0	0.1	0.3	0.5	0.7	0.9	1
$m=2, n=0$	7.8374	7.5035	6.8105	6.0718	5.2606	4.3039	3.6705
$m=3, n=1$	7.8374	7.9477	8.2281	8.6391	9.3286	10.897	13.187
$m=1, n=1$	7.8374	8.3047	9.5069	11.289	14.236	20.133	26.024
$m=4, n=2$	7.8374	8.4144	9.8887	12.054	15.627	23.028	30.530

5 Vibrations

I) When gravity is absent

The vibrations of tapered beams without gravity will serve as limiting cases for our problem. If the beam is uniform and gravity is absent, $c = 0$ and $\beta = 0$. Eq. (2.5) yields the solution

$$w = C_1 \cosh(\sqrt{\omega}x) + C_2 \sinh(\sqrt{\omega}x) + C_3 \cos(\sqrt{\omega}x) + C_4 \sin(\sqrt{\omega}x). \tag{5.1}$$

The characteristic equation for the clamped-free beam is well known

$$1 + \cosh(\sqrt{\omega}) \cos(\sqrt{\omega}) = 0 \tag{5.2}$$

giving $\omega = 3.5160, 22.035, 61.697, \cdots$.

The first solution of non-uniform vibrating beams was probably due to Kirchhoff [16]. He studied the vibration of two-dimensional and cylindrical beams with linear taper and expressed the solution in Bessel functions. There are many papers extending Kirchhoff's exact solution, in particular Cranch and Adler [17] and Sanger [18]. Briefly, Eq. (2.5) with $\beta = 0$, $0 < c \leq 1$ can be written as

$$c^4 \frac{d^2}{d\bar{z}^2}\Big(\bar{z}^m \frac{d^2 w}{d\bar{z}^2}\Big) - \omega^2 \bar{z}^n w = 0. \tag{5.3}$$

If $m = n+2$, Eq. (5.3) can be factored into

$$\Big[\bar{z}^{-n}\frac{d}{d\bar{z}}\Big(\bar{z}^{n+1}\frac{d}{d\bar{z}}\Big)+\frac{\omega}{c^2}\Big]\Big[\bar{z}^{-n}\frac{d}{d\bar{z}}\Big(\bar{z}^{n+1}\frac{d}{d\bar{z}}\Big)-\frac{\omega}{c^2}\Big]w=0. \tag{5.4}$$

Each one of the brackets in Eq. (5.4) is a Bessel operator. When n is an integer, the solution is

$$w=\bar{z}^{-\frac{n}{2}}[C_1J_n(u)+C_2Y_n(u)+C_3I_n(u)+C_4K_n(u)], \tag{5.5}$$

where J, Y are Bessel functions and I, K are modified Bessel functions and

$$u=\frac{2\sqrt{\omega\bar{z}}}{c}. \tag{5.6}$$

The boundary conditions Eqs. (2.7), (2.8) can be simplified to the following exact characteristic equation

$$\begin{vmatrix} J_n(u_0) & Y_n(u_0) & I_n(u_0) & K_n(u_0) \\ J_{n+1}(u_0) & Y_{n+1}(u_0) & -I_{n+1}(u_0) & K_{n+1}(u_0) \\ J_{n+1}(u_1) & Y_{n+1}(u_1) & I_{n+1}(u_1) & -K_{n+1}(u_1) \\ J_{n+2}(u_1) & Y_{n+2}(u_1) & I_{n+2}(u_1) & K_{n+2}(u_1) \end{vmatrix}=0, \tag{5.7}$$

where

$$u_0=\frac{2\sqrt{\omega}}{c}, \qquad u_1=\frac{2\sqrt{\omega(1-c)}}{c}. \tag{5.8}$$

If gravity is absent, all of our relevant cases satisfy $m = n+2$ and can be expressed in term of Bessel functions above except for the $m = n = 1$ case. The latter is an important case where the thickness of the beam is constant, the width tapers linearly (Fig. 1(c)) and vibrations are perpendicular to the thickness. Wang [19] found exact solutions in terms of hypergeometric functions for general m and n. However, hypergeometric functions are seldom included as computer library functions. Thus their evaluation requires infinite series representation which, when truncated, involves an uncertain amount of error. Naguleswaran [20] used Frobenius series, but the results do not converge well for $c < 0.4$. There exist also discretization methods such as finite differences and finite elements, including a dynamic method by Downs [21]. We shall compare our initial value method described in Section 3 with these published reports. Since the $m = n = 1$ case is of some importance and has no exact solution, the results are tabulated in Table 3. The $c = 0$ case is the uniform beam from Eq. (5.2), while the $c = 1$ case is approximated by $c = 0.999$ in our numerical computation. We see that all results agree for $0.5 \le c \le 0.9$. However, for $0.1 \le c \le 0.4$ the values from the "exact" hypergeometric series and the Frobenius series fail. The method of dynamic discretization seems to be accurate but tedious to implement. For $c = 1$, Eq. (2.5) is singular at $x = 1$, where all methods encounter some difficulty. Our values for $c = 0.999$

Table 3: Comparison of our initial value method with existing numerical methods for a beam with constant thickness and linearly tapered width $(m = n = 1)$ in the absence of gravity. Parentheses from [19], square brackets from [20] and flower brackets from [21].

c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.999
ω_1	3.5160	3.6310	3.7629	3.9160 {3.9160}	4.0970 (4.5957) [4.0970]	4.3152 [4.3152]	4.5853 [4.5853] {4.5853}	4.9317 [4.9316]	5.3976 (5.3969) [5.3976] {5.3976}	6.0704 [6.0704] {6.0704}	7.1422 {7.1565}
ω_2	22.035	22.254	22.502	22.786 {22.786}	24.021 (24.021) [23.119]	23.519 [23.519]	[24.021] {24.021}	24.687 [24.687]	25.656 (25.656) [25.656] {25.656}	27.299 [27.299] {27.299}	30.970 {31.041}
ω_3	61.697	61.910	62.153	62.436 {62.436}	62.776 [62.776]	63.199 [63.199]	63.751 [63.751] {63.752}	64.527 [64.527]	65.747 [65.747] {65.747}	68.115 [68.115] {68.115}	75.653 {75.487}

are deemed correct. We see that only our initial value method gives the full range of accurate results, especially at low and high taper parameters.

II) When gravity is present

Having established the accuracy of our simple initial value method, the frequencies for a standing tapered column under gravity are computed. For the zero gravity case $(\beta = 0)$, our numerical values agree with the exact values from Eq. (5.7). For the uniform column the numerical values agree with the exact values of Eq. (5.2) and [12]. The frequencies become irrelevant when the column has buckled (Table 2). Table 4-7 shows the results, presented here for the first time.

Table 4: Frequencies for $m = 2$, $n = 0$. Asterisks denote the column has buckled.

β/c	0	0.1	0.3	0.5	0.7	0.9	0.999
0	3.5160 22.035 61.699	3.4466 21.128 58.775	3.2984 19.228 52.666	3.1336 17.169 46.072	2.9442 14.856 38.677	2.7100 12.010 29.497	2.5558 9.9714 22.392
2.5	2.9035 21.538 61.189	2.8176 20.613 52.249	2.6268 18.668 52.099	2.4070 16.549 45.448	2.1379 14.146 37.968	1.7633 11.1127 28.622	1.4609 8.8619 21.265
5	2.1199 21.031 60.669	1.9940 20.084 57.718	1.7043 18.090 51.524	1.3207 15.905 44.814	0.6587 13.399 37.245	*	*
7.5	0.7310 20.507 60.158	0.0749 19.540 57.182	*	*	*	*	*

Instead of graphs, tables are used to show the subtle differences in frequencies. Tables are also more suited for practical use and for comparison with future research.

6 Discussions and conclusions

Our novel initial value method, being accurate and more efficient than any of the existing methods, is most suitable in the study of beam vibrations.

Table 5: Frequencies for $m = 3$, $n = 1$. Asterisks denote the column has buckled.

β/c	0	0.1	0.3	0.5	0.7	0.9	0.999
0	3.5160	3.5587	3.6667	3.8238	4.0817	4.6307	5.3021
	22.035	21.338	19.881	18.317	16.625	14.931	15.176
	61.699	58.980	53.322	47.265	40.588	32.833	30.125
2.5	2.9035	2.9485	3.0621	3.2269	3.4971	4.0727	4.7805
	21.538	20.837	19.369	17.793	14.085	14.371	14.621
	61.189	58.470	52.806	46.741	40.051	32.273	29.500
5	2.1199	2.1706	2.3009	2.4873	2.7886	3.4201	4.1906
	212.031	20.323	18.842	17.252	15.526	13.789	14.043
	60.669	57.954	52.284	46.210	39.507	31.704	29.014
7.5	0.7310	0.8466	1.0938	1.3934	1.8155	2.6018	3.4967
	20.507	19.794	19.300	16.693	14.947	13.181	13.440
	60.158	57.433	51.756	45.672	38.955	31.123	28.290
10	*	*	*	*	*	1.3402	2.6165
						12.544	12.808
						30.532	45.685

Table 6: Frequencies for $m = 1$, $n = 1$. Asterisks denote the column has buckled.

β/c	0	0.1	0.3	0.5	0.7	0.9	0.999
0	3.5160	3.6310	3.9160	4.3152	4.9317	6.0704	7.1422
	22.035	22.254	22.786	23.519	24.687	27.299	30.970
	61.699	61.910	62.463	63.199	64.527	68.115	75.653
2.5	2.9035	3.0376	3.3638	3.8093	4.4793	5.6822	6.7905
	21.538	21.771	22.333	23.102	24.315	26.986	30.687
	61.189	61.416	61.976	62.777	64.153	67.804	75.213
5	2.1199	2.2935	2.6994	3.2238	3.9751	5.2652	6.4193
	21.031	21.276	21.870	22.677	23.937	26.669	30.405
	60.669	60.918	61.511	62.352	63.776	67.491	74.946
7.5	0.7310	1.1326	1.8024	2.5035	3.3960	4.8117	6.0250
	20.507	20.769	21.396	22.243	23.553	26.347	30.119
	60.158	60.415	61.042	61.924	63.397	67.178	74.614
10	*	*	*	1.4609	2.6941	4.3104	5.6029
				21.801	23.162	26.023	29.830
				61.492	63.015	66.862	74.501

Exact stability criteria are found for the pointy column and numerical stability boundaries are determined for the blunt tipped column. From Tables 4-7 we can see that when the gravity effect β increases, the frequencies decrease until the fundamental frequency becomes zero, at which state the column buckles.

We note that both the taper c and the cross section shape (m, n) of the column have large influences on the vibration frequencies.

Finally, we comment on the frequency spectrum peculiar to geometrically anisotropic tapered beams. In practice, a cantilever beam can oscillate in both A-A or B-B directions, which have different EI_0, but actual frequencies can only be compared with the same normalization. Consider the solid constant thickness tapered column

Table 7: Frequencies for $m = 4$, $n = 2$. Asterisks denote the column has buckled.

β/c	0	0.1	0.3	0.5	0.7	0.9	0.999
0	3.5160	3.6737	4.0669	4.6250	5.5093	7.2049	8.6810
	22.035	21.550	20.556	19.548	18.641	18.680	21.165
	61.699	58.189	54.015	48.579	42.810	37.124	40.031
2.5	2.9035	3.0821	3.5181	4.1209	5.0536	6.8082	8.3225
	21.538	21.062	20.085	19.097	18.212	18.277	20.792
	61.189	58.693	53.543	48.131	42.385	36.721	39.090
5	2.1199	2.2425	2.8639	3.5439	4.5510	6.3857	7.9461
	21.031	20.562	19.603	18.634	17.772	17.865	20.410
	60.669	58.192	53.067	47.678	41.955	36.312	38.674
7.5	0.7310	1.2137	2.0036	2.8499	3.9836	5.9319	7.5516
	20.507	20.048	19.108	18.160	17.322	17.444	20.022
	60.158	57.686	57.584	47.220	41.521	35.898	38.645
10	$*$	$*$	$*$	19.160	3.3180	5.4387	7.1334
				17.673	16.859	17.011	19.623
				46.758	41.082	35.481	37.966
15	$*$	$*$	$*$	$*$	1.1099	4.2784	6.2089
					15.892	16.112	18.798
					40.189	34.629	36.220
20	$*$	$*$	$*$	$*$	$*$	2.6339	5.1118
						15.160	5.1118
						33.758	35.300

shown in Fig. 1(c). The base cross section is a rectangle shown in Fig. 1(e), where the width and thickness are a and b respectively. For vibration about A-A $(m = 1, n = 1)$ and B-B $(m = 3, n = 1)$, the rigidities are respectively proportional to

$$EI_A \sim ab^3, \qquad EI_B \sim a^3 b. \tag{6.1}$$

Let $EI_0 = EI_A$, then the frequencies in Table 6 are unchanged. Using the same EI_A to normalize the frequencies in Table 5, we find from Eq. (6.1) the frequencies in the Table should be multiplied by the aspect ratio a/b.

As an example, let us take the column of Fig. 1(c) with $c = 0.5$, $\beta = 5$. The lowest three frequencies are listed in Table 8.

We take another example using the beam of Fig. 1(d), whose base cross section is shown in Fig. 1(f). For vibration about A-A $(m = 0, n = 0)$ and B-B $(m = 2, n = 0)$, the rigidities are proportional to

$$EI_A \sim b^3, \qquad EI_B \sim a^2 b. \tag{6.2}$$

Using EI_A for normalization, the frequency about A-A is unchanged (can be obtained from the $c = 0$ case) while those about B-B (Table 4) should be multiplied by the aspect ratio. The three lowest frequencies for the case $c = 0.5$, $\beta = 5$ are shown in Table 9.

We see vibrations in either direction can be excited. This property is peculiar to geometrically anisotropic beams but is seldom considered in the literature.

Table 8: Lowest frequencies for column of Fig. 1(c) for $c = 0.5$, $\beta = 5$. Asterisks show vibration is about the B-B axis, otherwise it is about the A-A axis.

$a/b = 0.1$	$a/b = 1$	$a/b = 10$
0.2487*	2.4873*	3.2238
1.7282*	3.2238	22.677
3.2238	17.252*	24.873*

Table 9: Lowest frequencies for column of Fig. 1(d) for $c = 0.5$, $\beta = 5$. Asterisks show vibration is about the B-B axis, otherwise it is about the A-A axis.

$a/b = 0.1$	$a/b = 1$	$a/b = 10$
0.1321*	1.3207*	2.1199
1.5905*	2.1199	13.207*
2.1199	15.905*	21.031

It is possible to extend our analysis to other boundary conditions or tapers, but the effects are similar. If shear is included as in a Timoshenko column, the buckling loads will be lower and the vibration frequencies higher. However, exact stability criteria (as in Section 4) do not exist.

References

[1] A. G. GREENHILL, *Determination of the greatest height consistent with stability that a vertical pole or mast must be made, and of the greatest height to which a tree of given proportions can grow*, Proc. Camb. Phil. Soc., 4(2) (1881), pp. 65–73.

[2] C. M. WANG, C. Y. WANG AND J. N. REDDY, Exact Solutions for Buckling of Structural Members, CRC Press, Boca Raton, 2005.

[3] L. N. VIRGIN, S. T. SANTILLAN AND D. B. HOLLAND, *Effect of gravity on the vibration of vertical cantilevers*, Mech. Res. Commun., 34 (2007), pp. 312–317.

[4] A. N. DINNIK, Buckling and Torsion, Acad. Nauk. CCCP, Moscow, 1955.

[5] M. P. PAIDOUSSIS AND P. E. DOS TROIS MAISONS, *Free vibration of a heavy damped vertical cantilever*, J. Appl. Mech., 38 (1971), pp. 524–526.

[6] B. SCHAFER, *Free vibration of a gravity loaded clamped-free beam*, Ing. Arch., 55 (1985), pp. 66–80.

[7] T. YOKOYAMA, *Vibrations of a hanging Timoshenko beam under gravity*, J. Sound Vibr., 141 (1990), pp. 245–258.

[8] S. NAGULESWARAN, *Transverse vibration of a uniform Euler-Bernoulli beam under linearly varying axial force*, J. Sound Vibr., 146 (1991), pp. 191–198.

[9] D. J. WEI, S. X. YAN, Z. P. ZHANG AND X. F. LI, *Critical load for buckling of non-prismatic columns under self-weight and tip force*, Mech. Res. Commun., 37 (2010), pp. 554–558.

[10] I. A. KARNOVSKY AND O. I. LEBED, Non-Classical Vibrations of Arches and Beams, McGraw-Hill, New York, 2004.

[11] S. BARASCH AND Y. CHEN, *On the vibration of a rotating disk*, J. Appl. Mech., 39 (1972), pp. 1143–1144.

[12] C. Y. WANG, *Vibration of a standing heavy column with intermediate support*, J. Vibr. Acoust., (132) 2010, #044502.

[13] E. B. MAGRAB, Vibrations of Elastic Structural Members, Sijthoff and Noordhoff, Netherlands, 1979.

[14] J. M. GERE AND W. O. CARTER, *Critical buckling loads for tapered columns*, J. Struct. Div. ASCE., 88 (1962), pp.1–11.
[15] G. M. MURPHY, Ordinary Differential Equations and Their Solutions, Van Nostrand, Princeton, New Jersey, 1960.
[16] G. KIRCHKOFF, Gesammelte Abhandlungen, Sec. 18, Barth, Leipzig, 1882.
[17] E.T. CRANCH AND A. A. ADLER, *Bending vibrations of variable section beams*, J. Appl. Mech., 23 (1956), pp. 103–108.
[18] D. J. SANGER, *Transverse vibration of a class of non-uniform beams*, J. Mech. Eng. Sci., 10 (1968), pp. 111–120.
[19] H. C. WANG, *Generalized hypergeometric function solutions on the transverse vibration of a class of nonuniform beams*, J. Appl. Mech., 34 (1967), pp. 702–707.
[20] S. NAGULESWARAN, *Vibration of an Euler-Bernoulli beam of constant depth and with linearly varying breadth*, J. Sound Vibr., 153 (1992), pp. 509–522.
[21] B. DOWNS, *Transverse vibrations of cantilever beams having unequal breadth and depth tapers*, J. Appl. Mech., 44 (1977), pp. 737–742.