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A Uniformly Stable Nonconforming FEM Based on Weighted Interior Penalties for Darcy-Stokes-Brinkman Equations

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Abstract. A nonconforming rectangular finite element method is proposed to solve a fluid structure interaction problem characterized by the Darcy-Stokes-Brinkman Equation with discontinuous coefficients across the interface of different structures. A uniformly stable mixed finite element together with Nitsche-type matching conditions that automatically adapt to the coupling of different sub-problem combinations are utilized in the discrete algorithm. Compared with other finite element methods in the literature, the new method has some distinguished advantages and features. The Boland-Nicolaides trick is used in proving the inf-sup condition for the multi-domain discrete problem. Optimal error estimates are derived for the coupled problem by analyzing the approximation errors and the consistency errors. Numerical examples are also provided to confirm the theoretical results.

AMS subject classifications: 65N30, 76S05

Key words: Fluid structure interactions, Darcy-Stokes-Brinkman equations, Stokes equations, Darcy flow, discontinuous coefficient, nonconforming rectangular element, interior penalty, inf-sup condition, error estimates.

1. Introduction

There are many applications of the fluid structure interaction between a fluid flow and a porous media, a fluid flow and another fluid flow, or a porous media and another porous media with different physical parameters. In this paper, we consider such a fluid structure interaction problem that is modeled by the Darcy-Stokes-Brinkman equations,

$$\eta\mathbf{u} + \nabla\cdot\big(p\,\mathbf{I} - \nu\nabla\mathbf{u}\big) = \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0, \qquad x\in\Omega\subset\mathbb{R}^2, \qquad a.e., \tag{1.1}$$

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where $\mathbf{u}(x)$ is the velocity, $p(x)$ is the pressure, and $\mathbf{f}(x)$ is an external force. We assume that $\nu(x)$ and $\eta(x)$ are nonnegative coefficients satisfying $\nu(x)+\eta(x)=\mu(x)$ with $0<m\le\mu(x)\le M$ a.e. on Ω. Note that if $\eta=0$, the equations become a Stokes flow while it is a Darcy flow if $\nu=0$, see Figure 1 for an illustration of different set-up of our interest in this paper. We use the non-slip boundary condition if the part of the boundary $\partial\Omega$ bordered with the flow with $\nu>0$, otherwise we use $\mathbf{u}\cdot\mathbf{n}=0$, where $\mathbf{n}$ is the normal direction.

Without loss of generality, we refer to normalized coefficients such that $m\le 1\le M$. In this case, the ratio $M/m\ge 1$ quantifies the spatial heterogeneity of the problem. We also denote $\mathbf{f}(x)$ as a vector-valued forcing term and $\mathbf{I}$ as the identity matrix. When both $\nu(x)$ and $\eta(x)$ are positive for any $x\in\Omega$, equation (1.1), complemented with boundary conditions, represents a standard problem called a generalized Stokes equation. In this paper, we are interested in the local singular limit case, i.e., when $\nu(x)\to 0$ or $\eta(x)\to 0$ in a sub-region of the domain. In the case of $\nu(x)=0$, a rigorous formulation of problem (1.1) requires us to differentiate between Stokes and Darcy subproblems and to introduce interface conditions. The aim of this work is to provide a finite element discretization scheme for the local singular limit cases. This will be achieved starting from the multi-domain formulation (2.1)-(2.7).

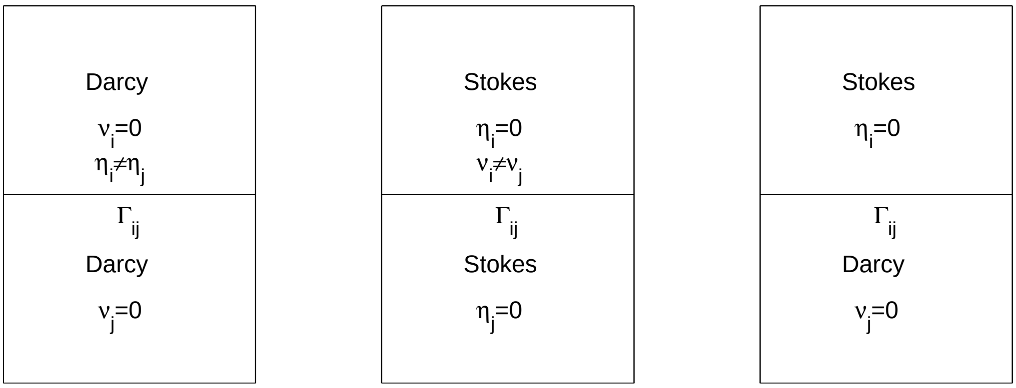

Figure 1: Diagrams of the fluid-structure interaction problem with three typical couplings.

There is rich literature on the coupling of viscous and inviscid sub-problems and applications [1, 2, 5, 9–13, 15–17, 19–21, 23–26, 30, 31]. It is desirable to develop a unified discretization framework for the problem. One difficulty is the treatment of interface conditions. Various approaches have been developed in the literature including Lagrange multipliers and mortar elements to satisfy the discrete interface conditions [5, 16, 20, 21, 25]. Generalizing the analysis in [9], D'angelo and Zunino [12] do with the coupling based on matching conditions due to the Nitsche method. This scheme is also particularly effective for the treatment of realistic applications, because interface conditions of practical interest, such as the ones proposed by Beavers and Joseph [3] and Saffman [27] for the coupling of free flows with porous media can naturally be embedded into the scheme. A finite difference approach that utilizes fast

Poisson solvers is proposed in [22].

It is challenging to design finite elements that are robust and optimally convergent for both viscous and inviscid problems. To guarantee the convergence, the discretization of viscous problems requires the inf-sup condition for the velocity and pressure spaces, while for inviscid problems some control of the divergence of velocities is also necessary. This difficulty was addressed in [23], where a new nonconforming element with nine degrees of freedom was proposed. The stabilized method is used in [9,12,15,26] to construct the unified stable mixed element. Recently, an article [32] by Zhang et al. propose a uniformly stable nonconforming rectangular element for viscous and inviscid problems that is simpler than that of [23]. In [29] and [28], Wang *et al.* and Shi *et al.* use this element for the planar elasticity problem and the conduction-convection problem respectively.

In this paper, we use a uniformly stable nonconforming rectangular mixed element with a weighted interior penalty formulation that automatically adapts to the coupling of different combinations of the heterogeneous problem. This non-conforming mixed element is the *lowest* order stable finite element for the coupled problems. Thus it is relatively easier to construct and implement compared with other approaches. Compared with the work in [12], our method has no stabilized term for the pressure. The analysis is following the similar procedure as that in [12]. But we use a different nonconforming finite element that alternate the proof to some extent. Another difference is that we construct a new auxiliary function in Section 4. In the numerical experiments, we considered three cases: Darcy-Darcy, Stokes-Stokes, Stokes-Darcy couplings, and their discrete meshes are non-matching on the common interface.

This paper is organized as follows. In §2, we introduce the multi-domain formulation and its weak formulation. In §3, we present a nonconforming rectangular finite element discretization. In §4, we derive the discrete inf-sup condition and the error estimates of the finite element approximation. Numerical examples are provided in §5 to verify the error analysis. Conclusions are given in §6.

2. A Multi-domain formulation

We use a multi-domain approach by considering a partition of Ω in N non-overlapping subregions such that $\bar{\Omega} = \cup_{i=1}^{N} \bar{\Omega}_i$, and we denote by $\mathbf{n}_i$ the outer unit normal of Ω_i. We assume that each subregion is characterized by a different value of the viscosity $\nu_i \geq 0$ and the hydraulic resistance $\eta_i \geq 0$, satisfying the assumption $\nu_i + \eta_i > 0$. For simplicity, we assume that ν_i and η_i are constants on each subregion. We can decompose the domain into sub-domains of three types, the Darcy flow with $\nu = 0$ and $\eta > 0$; the Stokes flow with $\nu > 0$ and $\eta = 0$; and Brinkman's flow with $\nu > 0$ and $\eta > 0$. Let $\mathcal{N}_i = \{j = 1, \cdots, N;\ j \neq i,\ \partial\Omega_i \cap \partial\Omega_j \neq \emptyset\}$ be the set of indices relative to the neighboring subregions of Ω_i. We define $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ for any $i = 1, \cdots, N$ and $j \in \mathcal{N}_i$. The normal directions $\mathbf{n}_{ij}$ along $\Gamma_{ij} = \Gamma_{ji}$ is assigned according to a predefined rule, for example, the normal direction is defined as pointing to the Ω_i side if $\nu_i \geq \nu_j$. Note that the arbitrariness of $\mathbf{n}_{ij}$ will not affect the setup of the method. On the boundary of Ω, we consider the outward unit normal vector $\mathbf{n}=\mathbf{n}_i$ on $\partial\Omega\cap\partial\Omega_i$.

Let $\mathbf{T}(\mathbf{u},p)=p\mathbf{I}-\nu\nabla\mathbf{u}$. Then, our multidomain problem requires us to find N couples $(\mathbf{u}_i,p_i)$ such that

$$\eta_i\mathbf{u}_i+\nabla\cdot\mathbf{T}(\mathbf{u}_i,p_i)=\mathbf{f}_i,\quad \nabla\cdot\mathbf{u}_i=0 \quad \text{in } \Omega_i, \tag{2.1}$$
$$\mathbf{u}_i=\mathbf{0} \quad \text{if } \nu_i>0 \quad \text{on } \partial\Omega\cap\partial\Omega_i, \tag{2.2}$$
$$\mathbf{u}_i\cdot\mathbf{n}_i=0 \quad \text{if } \nu_i=0 \quad \text{on } \partial\Omega\cap\partial\Omega_i, \tag{2.3}$$
$$\mathbf{u}_i=\mathbf{u}_j \quad \text{if } \nu_i\nu_j>0 \quad \text{on } \Gamma_{ij}, \tag{2.4}$$
$$\mathbf{u}_i\cdot\mathbf{n}_{ij}=\mathbf{u}_j\cdot\mathbf{n}_{ij} \quad \text{if } \nu_i\nu_j=0 \quad \text{on } \Gamma_{ij}, \tag{2.5}$$
$$\mathbf{T}(\mathbf{u}_i,p_i)\cdot\mathbf{n}_{ij}=\mathbf{T}(\mathbf{u}_j,p_j)\cdot\mathbf{n}_{ij} \quad \text{if } \nu_i\nu_j>0,\ \text{or } \nu_i=\nu_j=0 \quad \text{on } \Gamma_{ij}, \tag{2.6}$$
$$\mathbf{T}(\mathbf{u}_i,p_i)\cdot\mathbf{n}_{ij}=p_j\mathbf{n}_{ij}+\mathbf{n}_{ij}\times(\kappa_{ij}\mathbf{u}_i\times\mathbf{n}_{ij}) \quad \text{if } \nu_i>0,\nu_j=0 \quad \text{on } \Gamma_{ij}, \tag{2.7}$$

where, for any subregion Ω_i, we have listed the governing equations (see (2.1)), external boundary conditions (see (2.2)-(2.3)), and interface conditions for velocities, mass fluxes and stresses (see (2.4), (2.5) and (2.6)-(2.7), respectively). In particular, (2.7) is the so-called Beavers-Joseph-Saffman law, where $\kappa_{ij}\geq 0$ denotes a given friction coefficient associated to the interface Γ_{ij}. We note that when $\kappa_{ij}=0$, such an equation implies a free slip interface condition. This multi-domain problem includes three typical coupled models (see Figure 2).

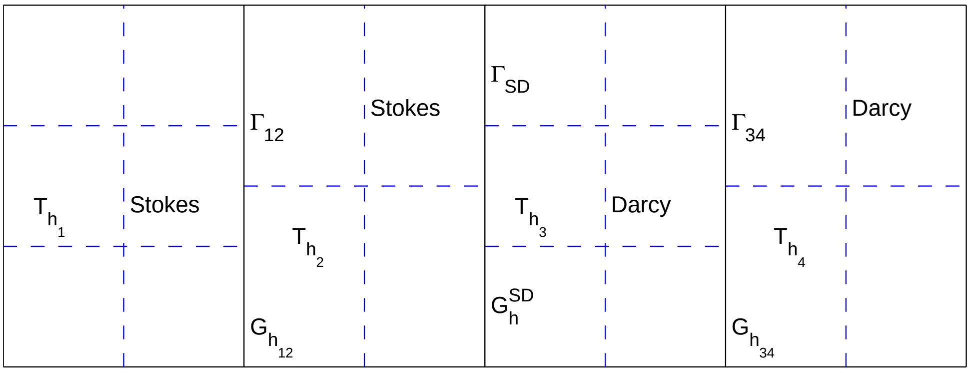

Figure 2: A diagram of the multi-domain approach.

To proceed, we address the variational formulation of problem (2.1)-(2.7), which will be the starting point to set up a discretization scheme based on finite elements. We address the interfaces of the domain with $\nu_i\nu_j>0$ first. Since the interface conditions (2.4)-(2.5) and (2.6)-(2.7) prescribe the continuity of velocities and stresses for any Γ_{ij} with $\nu_i\nu_j>0$, as well as the continuity of the normal velocities and pressures when $\nu_i=\nu_j=0$, it is possible to cluster all subregions characterized by $\nu_i>0$ into a single subregion, associated with a generalized Stokes problem with piecewise constant coefficients. We proceed similarly for the subdomains with $\nu_i=0$, resorting to a Darcy problem with variable hydraulic resistance. More precisely, we set $\Omega_s=$

$\bigcup_{\{i:\nu_i>0\}}\Omega_i,\ \Omega_d=\bigcup_{\{i:\nu_i=0\}}\Omega_i$ and $\Gamma_{sd}=\bigcup_{\{i,j:\nu_i>0,\nu_j=0\}}\Gamma_{ij}$ with $\mathbf{n}_{sd}:=\mathbf{n}_{ij}$ and $\kappa_{sd}:=\kappa_{ij}$ on $\Gamma_{sd}\cap\Gamma_{ij}$. We define

$$
\begin{aligned}
\mathbf{V}_s &:= \Big\{\mathbf{v}\in\mathbf{H}^1(\Omega_s)\big|\ \mathbf{v}|_{\partial\Omega\cap\partial\Omega_s}=\mathbf{0}\Big\}, && Q_s:=L^2(\Omega_s),\\
\mathbf{V}_d &:= \Big\{\mathbf{v}\in\mathbf{H}(div,\Omega_d)\big|\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega\cap\partial\Omega_d}=0\Big\}, && Q_d:=H^1(\Omega_d),
\end{aligned}
$$

where $\mathbf{H}(div,\Omega_d):=\{\mathbf{v}\in\mathbf{L}^2(\Omega_d)|\ \nabla\cdot\mathbf{v}\in L^2(\Omega_d)\}$, and we aim to find $(\mathbf{u}_s,p_s)\in\mathbf{V}_s\times Q_s$ and $(\mathbf{u}_d,p_d)\in\mathbf{V}_d\times Q_d$ with $\int_{\Omega_s}p_s+\int_{\Omega_d}p_d=0$ such that

$$
\begin{aligned}
&\int_{\Omega_s}(\nu\nabla\mathbf{u}_s:\nabla\mathbf{v}_s+\eta\mathbf{u}_s\mathbf{v}_s-\nabla\cdot\mathbf{v}_sp_s-\nabla\cdot\mathbf{u}_sq_s)\\
&\quad+\int_{\Gamma_{sd}}\kappa_{sd}(\mathbf{u}_s\times\mathbf{n}_{sd})\cdot(\mathbf{v}_s\times\mathbf{n}_{sd})+\int_{\Omega_d}(\eta\mathbf{u}_d\mathbf{v}_d-\nabla\cdot\mathbf{v}_dp_d-\nabla\cdot\mathbf{u}_dq_d)\\
&\quad+\int_{\Gamma_{sd}}(\mathbf{v}_s\cdot\mathbf{n}_{sd}-\mathbf{v}_d\cdot\mathbf{n}_{sd})p_d+\int_{\Gamma_{sd}}(\mathbf{u}_s\cdot\mathbf{n}_{sd}-\mathbf{u}_d\cdot\mathbf{n}_{sd})q_d\\
=&\int_{\Omega_s}\mathbf{f}_s\mathbf{v}_s+\int_{\Omega_d}\mathbf{f}_d\mathbf{v}_d,\qquad\forall\mathbf{v}_s\in\mathbf{V}_s,\quad\mathbf{v}_d\in\mathbf{V}_d,\quad q_s\in Q_s,\quad q_d\in Q_d,
\end{aligned}
\tag{2.8}
$$

where the additional regularity of the pressure on Ω_d, namely $p_d\in H^1(\Omega_d)$, is required to make sure that the interface terms $\int_{\Gamma_{sd}}(\mathbf{v}_s\cdot\mathbf{n}_{sd}-\mathbf{v}_d\cdot\mathbf{n}_{sd})p_d$ and $\int_{\Gamma_{sd}}(\mathbf{u}_s\cdot\mathbf{n}_{sd}-\mathbf{u}_d\cdot\mathbf{n}_{sd})q_d$ are well defined. Finally, we denote $(\mathbf{u},p)\in\mathbf{V}\times Q$ (with $\mathbf{V}:=\mathbf{V}_s\oplus\mathbf{V}_d$ and $Q:=(Q_s\oplus Q_d)\cap L_0^2(\Omega)$ being $L_0^2(\Omega):=\{q\in L^2(\Omega)|\ \int_\Omega q=0\}$) as the global weak solution of (2.8) such that $\mathbf{u}|_{\Omega_s}:=\mathbf{u}_s,\ p|_{\Omega_s}:=p_s$, and $\mathbf{u}|_{\Omega_d}:=\mathbf{u}_d,\ p|_{\Omega_d}:=p_d$. Existence and uniqueness of the solution $(\mathbf{u},p)$ is established in [13,21].

3. A nonconforming finite element method

Now we discuss how to discretize (2.8). For simplification of the presentation, we borrow most of the notations from [12]. Keep in mind that we use a different finite element and the discrete form, which are simpler than that in [12]. We assume that Ω and Ω_i are convex polygonal domains and we consider partitioning each sub-domain Ω_i into a family of conforming partition $\mathcal{T}_{h_i}$ of affine rectangles K. Let the family $\mathcal{T}_{h_i}$ be shape-regular and quasi-uniform, and let h_i be the local mesh characteristic parameter, while $h=\max_{i=1,\cdots,N}h_i$ with the assumption $h\ll 1$, and $\mathcal{T}_h=\cup_{i=1}^N\mathcal{T}_{h_i}$. Nevertheless, we do not require that the neighboring partitions $\mathcal{T}_{h_i}$ be conformal to their interface. More precisely, for any $K\in\mathcal{T}_{h_i}$, we difine $h_K=diam(K)$ and $h_E=diam(E)$ with $E\in\partial K$. We denote by $\mathcal{B}_{h_i}$ the trace meshes at the external boundaries, by $\mathcal{F}_{h_i}$ the set of all interior edges of $\mathcal{T}_{h_i}$, and by $\mathcal{G}_{h_{ij}}$ the intersection of the trace meshes. In short, we use the following notations:

$$\begin{aligned}
&\mathcal{B}_{h_i} := \{E|\ E = \partial K \cap \partial\Omega \quad \forall K \in \mathcal{T}_{h_i}\}, \qquad \mathcal{B}_h = \cup_{i=1}^N \mathcal{B}_{h_i},\\
&\mathcal{F}_{h_i} := \{E|\ E = \partial K_r \cap \partial K_s \quad \forall K_r \neq K_s \in \mathcal{T}_{h_i}\}, \qquad \mathcal{F}_h = \cup_{i=1}^N \mathcal{F}_{h_i},\\
&\mathcal{G}_{h_{ij}} := \{E|\ E = \partial K_i \cap \partial K_j \quad \forall K_i \in \mathcal{T}_{h_i} \quad \forall K_j \in \mathcal{T}_{h_j}\}.
\end{aligned}$$

We assume that $\mathcal{G}_{h_{ij}}$ is non-degenerate; namely, there exists $0 < \sigma < \infty$ such that for any $E \in \mathcal{G}_{h_{ij}}$ we have $diam(K_i) + diam(K_j) \leq \sigma diam(E)$ for $K_i \in \mathcal{T}_{h_i}$ and $K_j \in \mathcal{T}_{h_j}$ such that $\partial K_i \cap \partial K_j = E$. For any $E \in \mathcal{F}_{h_i}$ with $E = \partial K_r \cap \partial K_s$, $K_r \neq K_s \in \mathcal{T}_{h_i}$, we define $\mathbf{n}_E$ as the outer unit normal vector of K_r if $r > s$ and of K_s otherwise.

We define the velocity finite element space on the element K as follows

$$\mathbf{V}_K = \Big\{\mathbf{v} = (v_1, v_2)^T |\ v_1 \in span\{1, x, y, y^2\},\ v_2 \in span\{1, x, y, x^2\}\Big\},$$

and the local discrete velocity spaces are defined as

$$\begin{aligned}
\mathbf{V}_{h_i} = \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega_i)\Big| \qquad &\mathbf{v}_h|_K \in \mathbf{V}_K,\ \forall K \in \mathcal{T}_{h_i};\ \text{for } K_r, K_s \in \mathcal{T}_{h_i},\\
&\text{if } \partial K_r \cap \partial K_s = E, \text{then } \textstyle\int_E \mathbf{v}_h|_{\partial K_r} = \int_E \mathbf{v}_h|_{\partial K_s}\Big\}.
\end{aligned}$$

The approximation space for the pressure on Ω_i is

$$Q_{h_i} = \Big\{q_h \in L^2(\Omega_i)\Big|\ q_h|_K \in P_0(K),\ \forall K \in \mathcal{T}_{h_i}\Big\},$$

with $P_0(K)$ being the space of constant functions on K. We also introduce $\mathbf{V}_h := \bigoplus_{i=1}^N \mathbf{V}_{h_i}$ and $Q_h = (\bigoplus_{i=1}^N Q_{h_i}) \cap L_0^2(\Omega)$. At the discrete level, we will address the boundary conditions by means of penalty techniques. For this reason, we do not require the discrete functions to vanish at the external boundary.

Since the functions $\mathbf{v}_h \in \mathbf{V}_h$ may be discontinuous across Γ_{ij}, we define the neighboring values of $\mathbf{v}_h$ as

$$\mathbf{v}_h^{\mp}(\mathbf{x}) = \lim_{\delta\to 0^+} \mathbf{v}(\mathbf{x} \mp \delta\mathbf{n}_{ij}) \qquad a.e. \text{ on } \Gamma_{ij}.$$

Set $[\mathbf{v}_h] := \mathbf{v}_h^- - \mathbf{v}_h^+$, $\{\mathbf{v}_h\} := \frac{1}{2}(\mathbf{v}_h^- + \mathbf{v}_h^+)$ and

$$\{\mathbf{v}_h\}_w := w_i\mathbf{v}_{h_i} + w_j\mathbf{v}_{h_j}, \qquad \{\mathbf{v}_h\}^w := w_j\mathbf{v}_{h_i} + w_i\mathbf{v}_{h_j},$$

with $w_i + w_j = 1$ a.e. on Γ_{ij}. For $\mathbf{v}_h$ is discontinuous across $\mathcal{F}_h$, we also define the jump $[\mathbf{v}_h]$ with appropriate sign through any $E \in \mathcal{F}_h$. We say that the averages $\{\cdot\}_w$ and $\{\cdot\}^w$ are conjugate because they satisfy the following identity: $[ab] = \{a\}_w[b] + [a]\{b\}^w$. We also apply similar definitions for any other quantity depending on $(\mathbf{v}_h, q_h)$. Let Γ be the collection of the local interfaces, i.e., $\Gamma := \bigcup_{\{i=1,\cdots,N, j\in\mathcal{N}_i, i<j\}} \Gamma_{ij}$ with unit normal vector $\mathbf{n}_\Gamma := \mathbf{n}_{ij}$ on $\Gamma \cap \Gamma_{ij}$. Similarly, we denote by $\mathcal{G}_h$ the collection of all the local trace meshes $\mathcal{G}_{h_{ij}}$. We also introduce $\mathcal{G}_h^{sd} = \bigcup_{\{i,j:\nu_i>0,\nu_j=0\}} \mathcal{G}_{h_{ij}}$, i.e., the collection of all the trace meshes lying on Γ_{sd}. Then, for simplicity, we apply the following abridged notation:

$$\int_{\mathcal{B}_{h_i}} \mathbf{v}_h \cdot \mathbf{n}_i := \sum_{E\in\mathcal{B}_{h_i}} \int_E \mathbf{v}_h \cdot \mathbf{n}_i, \qquad \int_{\mathcal{T}_{h_i}} \mathbf{v}_h := \sum_{K\in\mathcal{T}_{h_i}} \int_K \mathbf{v}_h.$$

Summing them up over all the sub-regions, these notations are easily extended to $\mathcal{B}_h, \mathcal{T}_h$. Furthermore, we set

$$\int_{\mathcal{G}_{h_{ij}}} \mathbf{v}_h \cdot \mathbf{n}_{ij} := \sum_{E\in\mathcal{G}_{h_{ij}}} \int_E \mathbf{v}_h \cdot \mathbf{n}_{ij}, \qquad \int_{\mathcal{G}_h} \mathbf{v}_h \cdot \mathbf{n}_\Gamma := \sum_{i=1}^N \sum_{j\in\mathcal{N}_i, i<j} \int_{\mathcal{G}_{h_{ij}}} \mathbf{v}_h \cdot \mathbf{n}_{ij}.$$

For any $\mathbf{u}_{h_i}, \mathbf{v}_{h_i} \in \mathbf{V}_{h_i}$ and $p_{h_i}, q_{h_i} \in Q_{h_i}$ and, given a constant parameter γ_E to be discussed later on, we define the local bilinear forms

$$\begin{aligned} a_{h_i}(\mathbf{u}_{h_i}, \mathbf{v}_{h_i}) :=& \int_{\mathcal{T}_{h_i}} \big(\nu_i \nabla_h \mathbf{u}_{h_i} : \nabla_h \mathbf{v}_{h_i} + \eta_i \mathbf{u}_{h_i} \cdot \mathbf{v}_{h_i}\big) \\ &- \int_{\mathcal{B}_{h_i}} \big(\nu_i \frac{\partial \mathbf{u}_{h_i}}{\partial \mathbf{n}_i} \cdot \mathbf{v}_{h_i} + \nu_i \frac{\partial \mathbf{v}_{h_i}}{\partial \mathbf{n}_i} \cdot \mathbf{u}_{h_i}\big) + \int_{\mathcal{B}_{h_i}} \nu_i \frac{\gamma_E}{h_E} \mathbf{u}_{h_i} \cdot \mathbf{v}_{h_i}, \end{aligned} \tag{3.1}$$

$$b_{h_i}(\mathbf{v}_{h_i}, p_{h_i}) := -\int_{\mathcal{T}_{h_i}} \nabla_h \cdot \mathbf{v}_{h_i} p_{h_i} + \int_{\mathcal{B}_{h_i}} \mathbf{v}_{h_i} \cdot \mathbf{n}_i p_{h_i}, \tag{3.2}$$

where ∇_h and $\nabla_h \cdot$ are the gradient operator and the divergence operator respectively taken piecewise over $\mathcal{T}_{h_i}, i = 1, \cdots, N$. Denote

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{i=1}^N a_{h_i}(\mathbf{u}_{h_i}, \mathbf{v}_{h_i}), \qquad b_h(\mathbf{v}_h, p_h) := \sum_{i=1}^N b_{h_i}(\mathbf{v}_{h_i}, p_{h_i}).$$

In this paper, we define the weights $w_i = \frac{\nu_j}{\nu_i + \nu_j}$. Then, we introduce the bilinear forms responsible for the coupling conditions:

$$\begin{aligned} c_h(\mathbf{u}_h, \mathbf{v}_h) :=& \int_{\mathcal{G}_h} \{\nu\}_w \frac{\gamma_E}{h_E} [\mathbf{u}_h][\mathbf{v}_h] + \int_{\mathcal{G}_h^{sd}} \kappa_{sd}(\{\mathbf{u}_h\}^w \times \mathbf{n}_{sd}) \cdot (\{\mathbf{v}_h\}^w \times \mathbf{n}_{sd}) \\ &- \int_{\mathcal{G}_h} \Big(\{\nu \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}_\Gamma}\}_w \cdot [\mathbf{v}_h] + \{\nu \frac{\partial \mathbf{v}_h}{\partial \mathbf{n}_\Gamma}\}_w \cdot [\mathbf{u}_h]\Big), \end{aligned} \tag{3.3}$$

$$d_h(\mathbf{v}_h, p_h) := \int_{\mathcal{G}_h} [\mathbf{v}_h] \cdot \mathbf{n}_\Gamma \{p_h\}_w, \tag{3.4}$$

$$J_h(\mathbf{u}_h, \mathbf{v}_h) = \int_{\mathcal{G}_h} \frac{\gamma_E}{h_E} ([\mathbf{u}_h] \cdot \mathbf{n}_\Gamma)([\mathbf{v}_h] \cdot \mathbf{n}_\Gamma) + \int_{\mathcal{B}_h} \frac{\gamma_E}{h_E} (\mathbf{u}_h \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n}), \tag{3.5}$$

where γ_E are constant parameters that should be suitably chosen to ensure the stability of the method. Owing to the definitions of $\{\cdot\}^w$ and the weights, we notice that $\{\mathbf{u}_h\}^w = \mathbf{u}_{h_i}$ if $\nu_i > 0, \nu_j = 0$, while $\{\mathbf{u}_h\}^w = \mathbf{u}_{h_j}$ in the opposite case. The term $\int_{\mathcal{G}_h^{sd}} \kappa_{sd}(\{\mathbf{u}_h\}^w \times \mathbf{n}_{sd}) \cdot (\{\mathbf{v}_h\}^w \times \mathbf{n}_{sd})$ in (3.3) is equivalent to the Beavers-Joseph-Saffman condition (2.7), also applied in (2.8).

Then, we define

$$
\begin{aligned}
A_h(\mathbf{u}_h, \mathbf{v}_h) &:= a_h(\mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{u}_h, \mathbf{v}_h), \\
B_h(\mathbf{v}_h, p_h) &:= b_h(\mathbf{v}_h, p_h) + d_h(\mathbf{v}_h, p_h),
\end{aligned}
$$

and we denote the right-hand side by $F_h(\mathbf{v}_h) := \sum_{i=1}^N \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_{h_i}$. The mixed formulation of the discrete problem reads as follows: given a sufficiently regular $F_h(\cdot)$, find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$
\begin{aligned}
&A_h(\mathbf{u}_h, \mathbf{v}_h) + J_h(\mathbf{u}_h, \mathbf{v}_h) + B_h(\mathbf{v}_h, p_h) = F_h(\mathbf{v}_h) \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\
&B_h(\mathbf{u}_h, q_h) = 0 && \forall q_h \in Q_h.
\end{aligned} \tag{3.6}
$$

Introducing the product space $\mathbf{W}_h = \mathbf{V}_h \times Q_h$, the right-hand side $G_h(\mathbf{v}_h, q_h) = (F_h(\mathbf{v}_h), 0)$, and the bilinear form

$$
C_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = A_h(\mathbf{u}_h, \mathbf{v}_h) + J_h(\mathbf{u}_h, \mathbf{v}_h) + B_h(\mathbf{v}_h, p_h) - B_h(\mathbf{u}_h, q_h),
$$

problem (3.6) is equivalent to the following: given a sufficiently regular $G_h(\cdot)$, find $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ such that

$$
C_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = G_h(\mathbf{v}_h, q_h) \qquad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h. \tag{3.7}
$$

4. Convergence analysis of the proposed FE method

In this section we aim to analyze the stability and the convergence of the problem (3.7). We show that our proposed FE method is stable and has optimal convergence order.

First, for any $\mathbf{v}_h \in \mathbf{V}_h$, we introduce the following norm:

$$
\|\mathbf{v}_h\|^2_{\mp\frac{1}{2},h,\Gamma} = \sum_{E \in \mathcal{G}_h} h_E^{\pm 1} \|\mathbf{v}_h\|^2_{0,E},
$$

where $\|\cdot\|_{0,\Sigma}$ denotes the standard norm in $L^2(\Sigma)$. The definition can be straightforwardly extended to $\mathcal{B}_h$. Then, we introduce the following norms in $\mathbf{V}_h$ and $\mathbf{W}_h$, respectively

$$
\begin{aligned}
|||\mathbf{v}_h|||^2 &:= \|\eta^{\frac{1}{2}} \mathbf{v}_h\|^2_{0,\Omega} + \|\nu^{\frac{1}{2}} \nabla_h \mathbf{v}_h\|^2_{0,\cup\Omega_i} + \|\nu^{\frac{1}{2}} \mathbf{v}_h\|^2_{\frac{1}{2},h,\partial\Omega} + \|\{\nu\}_w^{\frac{1}{2}} [\mathbf{v}_h]\|^2_{\frac{1}{2},h,\Gamma} \\
&\quad + \|\kappa_{sd}^{\frac{1}{2}} \{\mathbf{v}_h\}^w \times \mathbf{n}_{sd}\|^2_{0,\Gamma_{sd}}, \\
|||(\mathbf{v}_h, q_h)|||^2 &:= |||\mathbf{v}_h|||^2 + \|\nabla_h \cdot \mathbf{v}_h\|^2_{0,\cup\Omega_i} + \|[\mathbf{v}_h] \cdot \mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\mathbf{v}_h \cdot \mathbf{n}\|^2_{\frac{1}{2},h,\partial\Omega} + \|q_h\|^2_{0,\Omega},
\end{aligned}
$$

where $\|\mathbf{v}_h\|^2_{0,\cup\Omega_i}$ denotes $\sum_{i=1}^N \|\mathbf{v}_{h_i}\|^2_{0,\Omega_i}$, and

$$\|\nabla_h \mathbf{v}_h\|^2_{0,\Omega_i} = \int_{\mathcal{T}_{h_i}} |\nabla_h \mathbf{v}_{h_i}|^2, \quad \|\nabla_h \cdot \mathbf{v}_h\|^2_{0,\Omega_i} = \int_{\mathcal{T}_{h_i}} |\nabla_h \cdot \mathbf{v}_{h_i}|^2.$$

We shall also make use of the broken Sobolev space $\mathcal{H}^1(\Omega) = \bigoplus_{i=1}^N \mathcal{H}^1(\Omega_i)$ equipped with the broken seminorm $|v|^2_{h,\cup\Omega_i} := \sum_{i=1}^N |v|^2_{h,\Omega_i}$ and $|v|_{h,\Omega_i} = \|\nabla_h v\|_{0,\Omega_i}$. We also set

$$\mathcal{H}^1_0(\Omega) = \Big\{ v \in \mathcal{H}^1(\Omega) |\ v|_{\partial\Omega} = 0,\ \forall K \in \mathcal{T}_h(\Omega) \Big\}.$$

Owing to the assumption $\nu_i + \eta_i = \mu_i \geq m > 0$ and exploiting Poincar$\acute{e}$-Friedrichs inequalities (see [7]), we obtain

$$m\|\mathbf{v}_h\|^2_{0,\Omega_i} \leq \|\nu^{\frac{1}{2}} \nabla_h \mathbf{v}_h\|^2_{0,\Omega_i} + \|\eta^{\frac{1}{2}} \mathbf{v}_h\|^2_{0,\Omega_i} + \|\nu^{\frac{1}{2}} \mathbf{v}_h\|^2_{\frac{1}{2},h,\partial\Omega_i},$$

which easily implies that $|||\mathbf{v}_h|||^2 \geq m\|\mathbf{v}_h\|^2_{0,\Omega}$. We will also make use of the following inverse inequalities (see [8]) that hold true for any $E \in \partial K, K \in \mathcal{T}_{h_i}, i = 1,\cdots,N$, provided that the mesh family is shape-regular:

$$h_E^{\frac{1}{2}} \|v_h\|_{0,E} \lesssim \|v_h\|_{0,K}, \tag{4.1}$$
$$h_K \|\nabla v_h\|_{0,K} \lesssim \|v_h\|_{0,K}. \tag{4.2}$$

Here and in what follows, the symbol " $\lesssim$ " denotes an inequality involving a positive constant C independent of the size of the mesh, h, the viscosity, ν, and the hydraulic resistance η.

In order to ensure that the bilinear forms (3.1)-(3.5) make sense for the exact weak solution of the problem, we require the following additional regularity:

$$(\mathbf{u}_s, p_s) \in \mathbf{H}^{\frac{3}{2}+\epsilon}(\Omega_s) \times H^{\frac{1}{2}+\epsilon}(\Omega_s) \qquad \forall \epsilon > 0. \tag{4.3}$$

To study the convergence of the discrete solution, we need the approximating properties of the finite element spaces. As in [28] and [32], we define the interpolation operator $\pi_{h_i} : \mathbf{H}^1(\Omega_i) \to \mathbf{V}_{h_i}$, for any $\mathbf{v} \in \mathbf{H}^1(\Omega_i)$

$$\int_E \pi_{h_i} \mathbf{v} = \int_E \mathbf{v}, \qquad \forall E \in \partial K,\ \forall K \in \mathcal{T}_{h_i}. \tag{4.4}$$

Set $\pi_h = \Pi_{i=1}^N \pi_{h_i} : [\mathcal{H}^1(\Omega)]^2 \to \mathbf{V}_h$. Since the operator π_{h_i} preserves the linear polynomials locally, it follows from a standard scaling argument, using the Bramble-Hilbert lemma, that the following estimates hold

Lemma 4.1. (Interpolation errors [32]) *For* $s = 1,\ 2$, *we have*

$$\|\mathbf{v} - \pi_h \mathbf{v}\|_{\mathbf{L}^2(K)} \leq C h_K^s |\mathbf{v}|_{\mathbf{H}^s(K)} \qquad \forall K \in \mathcal{T}_h, \tag{4.5}$$
$$|\mathbf{v} - \pi_h \mathbf{v}|_{\mathbf{H}^1(K)} \leq C h_K |\mathbf{v}|_{\mathbf{H}^2(K)} \qquad \forall K \in \mathcal{T}_h, \tag{4.6}$$
$$\|\nabla \cdot (\mathbf{v} - \pi_h \mathbf{v})\|_{\mathbf{L}^2(K)} \leq C h_K |\nabla \cdot \mathbf{v}|_{\mathbf{H}^1(K)} \qquad \forall K \in \mathcal{T}_h. \tag{4.7}$$

4.1. Well-posedness of the discrete formulation

The first step in analyzing the method proposed here consists of observing that it is by construction consistent with problem (2.8), and that the bilinear form $C_h(\cdot,\cdot)$ is bounded and positive. These properties can be proved similarly to those in [12].

Lemma 4.2. (Consistency) *Let* $(\mathbf{u},p)$ *be the solution of the problem* (2.8) *with the regularity assumption* (4.3)*, and let* $(\mathbf{u}_h,p_h)\in\mathbf{W}_h$ *be the solution of the problem* (3.7)*. Then we have*

$$\begin{aligned}&C_h((\mathbf{u}-\mathbf{u}_h,p-p_h),(\mathbf{v}_h,q_h))\\ =&\int_{\mathcal{F}_h}\Big(\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot[\mathbf{v}_h]-[\mathbf{v}_h\cdot\mathbf{n}]p\Big)\qquad\forall(\mathbf{v}_h,q_h)\in\mathbf{W}_h.\end{aligned}\tag{4.8}$$

Proof. First, we observe that (4.8) is equivalent to

$$\begin{aligned}C_h((\mathbf{u},p),(\mathbf{v}_h,q_h))=&G_h(\mathbf{v}_h,q_h)+\int_{\mathcal{F}_h}\Big(\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot[\mathbf{v}_h]-[\mathbf{v}_h\cdot\mathbf{n}]p\Big)\\ &\forall(\mathbf{v}_h,q_h)\in\mathbf{W}_h.\end{aligned}\tag{4.9}$$

Since $(\mathbf{u},p)$ is the solution in the weak sense of (2.1)-(2.7), we have $J_h(\mathbf{u},\mathbf{v}_h)=0$ and $B_h(\mathbf{u},q_h)=0$. As a result we obtain

$$\begin{aligned}C_h((\mathbf{u},p),(\mathbf{v}_h,q_h))&=a_h(\mathbf{u},\mathbf{v}_h)+c_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)+d_h(\mathbf{v}_h,p)\qquad(4.10)\\ =a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)&+\int_{\mathcal{G}_h^{sd}}\kappa_{sd}(\{\mathbf{u}\}^w\times\mathbf{n}_{sd})\cdot(\{\mathbf{v}_h\}^w\times\mathbf{n}_{sd})+\int_{\mathcal{G}_h}\{\mathbf{T}(\mathbf{u},p)\mathbf{n}_\Gamma\}_w[\mathbf{v}_h].\end{aligned}$$

Furthermore, by Green's formula we obtain

$$\begin{aligned}a_h(\mathbf{u},\mathbf{v}_h)+b_h(\mathbf{v}_h,p)&=\int_{\mathcal{T}_h}(\nu\nabla\mathbf{u}:\nabla_h\mathbf{v}_h+\eta\mathbf{u}\cdot\mathbf{v}_h-\nabla_h\cdot\mathbf{v}_hp)+\int_{\mathcal{B}_h}\big[\big(\mathbf{T}(\mathbf{u},p)\mathbf{n}\big)\mathbf{v}_h\big]\\ &=\int_{\mathcal{T}_h}\mathbf{f}\cdot\mathbf{v}_h-\int_{\mathcal{F}_h}\big[\big(\mathbf{T}(\mathbf{u},p)\mathbf{n}\big)\mathbf{v}_h\big]-\int_{\mathcal{G}_h}\big[\big(\mathbf{T}(\mathbf{u},p)\mathbf{n}_\Gamma\big)\mathbf{v}_h\big].\end{aligned}\tag{4.11}$$

By virtue of the regularity of $(\mathbf{u},p)$ and $[ab]=\{a\}[b]+[a]\{b\}$, we have

$$\begin{aligned}\int_{\mathcal{F}_h}\big[\big(\mathbf{T}(\mathbf{u},p)\mathbf{n}\big)\mathbf{v}_h\big]&=\int_{\mathcal{F}_h}\{\mathbf{T}(\mathbf{u},p)\mathbf{n}\}[\mathbf{v}_h]=\int_{\mathcal{F}_h}\mathbf{T}(\mathbf{u},p)\mathbf{n}[\mathbf{v}_h]\\ &=-\int_{\mathcal{F}_h}\Big(\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot[\mathbf{v}_h]-[\mathbf{v}_h\cdot\mathbf{n}]p\Big).\end{aligned}\tag{4.12}$$

Thanks to the algebraic identity $[ab]=\{a\}_w[b]+[a]\{b\}^w$ and to the interface conditions (2.6)-(2.7) (i.e., $[\mathbf{T}(\mathbf{u},p)\mathbf{n}]=\mathbf{0}$ on $\Gamma\setminus\Gamma_{sd}$ and $\mathbf{T}(\mathbf{u}_s,p_s)\mathbf{n}_s=p_d\mathbf{n}_s+\mathbf{n}_s\times(\kappa_{sd}\mathbf{u}_s\times\mathbf{n}_s)$ on Γ_{sd}), the last term of (4.11) is equivalent to

$$\begin{aligned}&\int_{\mathcal{G}_h}\big[\big(\mathbf{T}(\mathbf{u},p)\mathbf{n}_\Gamma\big)\mathbf{v}_h\big]\\ =&\int_{\mathcal{G}_h}\big\{\mathbf{T}(\mathbf{u},p)\mathbf{n}_\Gamma\big\}_w[\mathbf{v}_h]+\int_{\mathcal{G}_h^{sd}}\kappa_{sd}(\{\mathbf{u}\}^w\times\mathbf{n}_{sd})\cdot(\{\mathbf{v}_h\}^w\times\mathbf{n}_{sd}).\end{aligned}\tag{4.13}$$

Finally, we substitute (4.11)-(4.13) into (4.10) and we obtain (4.9). □

Lemma 4.3. (Boundedness [12]) *The bilinear form $C_h(\cdot,\cdot)$ satisfies*

$$C_h((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)) \lesssim |||(\mathbf{u}_h,p_h)|||\ |||(\mathbf{v}_h,q_h)||| \qquad \forall(\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)\in \mathbf{W}_h.$$

Lemma 4.4. (Positivity [12]) *For $\gamma_E \gtrsim 1$, there exists a constant $C_{pos}>0$ such that*

$$C_h((\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)) \gtrsim C_{pos}|||\mathbf{v}_h|||^2 + J_h(\mathbf{v}_h,\mathbf{v}_h) \qquad \forall(\mathbf{v}_h,q_h)\in \mathbf{W}_h.$$

To analyze the stability of our numerical method we follow the approach of Boland and Nicolaides (see [6] and also Chapter II, Section 1.4 in [18]) in order to split the inf-sup condition into a local condition on each subdomain and a global condition on a suitable subspace of $\mathbf{W}_h$. To this aim, we introduce $\tilde{Q}_{h_i} = Q_{h_i}\cap L^2_0(\Omega_i) := \{q_{h_i}\in Q_{h_i}|\ \int_{\Omega_i} q_{h_i} = 0\}$ and by consequence $Q_{h_i} = \tilde{Q}_{h_i}\oplus\mathbb{R}$, where $p_{h_i} = \tilde{p}_{h_i} + \bar{p}_{h_i}$ is the corresponding splitting of the pressure. Finally, let $\bar{Q}_h$ be the space of constant functions on each subdomain Ω_i that satisfy $\sum_{i=1}^N \bar{q}_{h_i}|\Omega_i| = 0$. We aim to prove a local inf-sup condition on $\mathbf{V}_{h_i}\times\tilde{Q}_{h_i}$, and a global one, relative to the subspace $\mathbf{V}_h\times\bar{Q}_h$. To this aim, we introduce the following lemmas.

Lemma 4.5. (Local inf-sup condition) *For any $\tilde{p}_{h_i}\in\tilde{Q}_{h_i}$, there exists $\tilde{\mathbf{v}}_{h_i}\in\mathbf{V}_{h_i}\cap[\mathcal{H}^1_0(\Omega_i)]^2$, such that*

$$b_{h_i}(\tilde{\mathbf{v}}_{h_i},\tilde{p}_{h_i}) = \|\tilde{p}_{h_i}\|^2_{0,\Omega_i}, \tag{4.14}$$

$$\|\tilde{\mathbf{v}}_{h_i}\|_{h,\Omega_i} \lesssim \|\tilde{p}_{h_i}\|_{0,\Omega_i}, \tag{4.15}$$

where $\|\mathbf{v}_h\|^2_{h,\Omega_i} = \|\mathbf{v}_h\|^2_{0,\Omega_i} + |\mathbf{v}_h|^2_{h,\Omega_i}$.

Proof. We observe that, by means of the surjectivity of the divergence operator from $\mathbf{H}^1_0(\Omega_i)$ to $L^2_0(\Omega_i)$, for any $\tilde{p}_{h_i}\in\tilde{Q}_{h_i}$ there exists a stable function $\mathbf{v}_i\in\mathbf{H}^1_0(\Omega_i)$ such that

$$-\nabla\cdot\mathbf{v}_i = \tilde{p}_{h_i}, \qquad \|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)} \lesssim \|\tilde{p}_{h_i}\|_{0,\Omega_i}.$$

We define an interpolation operator π_{h_i}: $\mathbf{H}^1_0(\Omega_i)\to\mathbf{V}_{h_i}\cap[\mathcal{H}^1_0(\Omega_i)]^2$, for any $\mathbf{v}_i\in\mathbf{H}^1_0(\Omega_i)$

$$\int_E \pi_{h_i}\mathbf{v}_i = \int_E \mathbf{v}_i \qquad \forall E\in\partial K,\ \forall K\in\mathcal{T}_{h_i}.$$

By Lemma 3.1 in [28] and the Poincar$\acute{e}$ inequality, we have

$$b_{h_i}(\mathbf{v}_i - \pi_{h_i}\mathbf{v}_i,\tilde{p}_{h_i}) = 0,$$

$$\|\pi_{h_i}\mathbf{v}_i\|_{h,\Omega_i} \lesssim \|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)}. \tag{4.16}$$

So we have the desired result from the Fortin rule. □

In order to prove the inf-sup condition on $\mathbf{V}_h\times\bar{Q}_h$, we define the conforming linear element space $\hat{V}_{h_i}$ with a refined partition of the partition $\mathcal{T}_{h_i}$ by connecting the diagonal line of each rectangular element in $\mathcal{T}_{h_i}$, note that $\hat{V}_{h_i}\times\hat{V}_{h_i}$ is a subspace of $\mathbf{V}_{h_i}$, $i=1,\cdots,N$, and construct the following functions.

Lemma 4.6. (Auxiliary functions [12]) *For any* $i=1,\cdots,N$, $j\in\mathcal{N}_i$ *there exist functions* $w^{(i)}_{\Gamma_{ij}}\in\hat{V}_{h_i}, w^{(j)}_{\Gamma_{ij}}\in\hat{V}_{h_j}$ *with* $w^{(i)}_{\Gamma_{ij}}=0$ *on* $\partial\Omega_i\setminus\Gamma_{ij}$, $w^{(j)}_{\Gamma_{ij}}=0$ *on* $\partial\Omega_j\setminus\Gamma_{ij}$, *and*

$$w_{\Gamma_{ij}}=w^{(i)}_{\Gamma_{ij}}+w^{(j)}_{\Gamma_{ij}}\in\big(\hat{V}_{h_i}\oplus\hat{V}_{h_j}\big)\cap\mathcal{H}^1_0(\Omega_i\cup\Omega_j)$$

such that

$$\int_{\mathcal{G}_{h_{ij}}} w^{(k)}_{\Gamma_{ij}}=1,\qquad k=i,j, \tag{4.17}$$

$$\|w^{(k)}_{\Gamma_{ij}}\|_{1,\Omega_k}\lesssim 1,\qquad k=i,j, \tag{4.18}$$

$$\|[w_{\Gamma_{ij}}]\|_{\frac{1}{2},h,\Gamma_{ij}}\lesssim 1. \tag{4.19}$$

Lemma 4.7. (Global inf-sup condition on subspace) *For any* $\bar{p}_h\in\bar{Q}_h$ *there exists* $\bar{\mathbf{v}}_h\in\mathbf{V}_h\cap[\mathcal{H}^1_0(\Omega)]^2$, *such that*

$$B_h(\bar{\mathbf{v}}_h,\bar{p}_h)\gtrsim\|\bar{p}_h\|^2_{0,\Omega}, \tag{4.20}$$

$$\|\bar{\mathbf{v}}_h\|_{h,\cup\Omega_i}\lesssim\|\bar{p}_h\|_{0,\Omega}, \tag{4.21}$$

$$\int_{\mathcal{G}_h}[\bar{\mathbf{v}}_h]\cdot\mathbf{n}_\Gamma=0, \tag{4.22}$$

$$\|[\bar{\mathbf{v}}_h]\cdot\mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}\lesssim\|\bar{p}_h\|_{0,\Omega}, \tag{4.23}$$

where $\|\bar{\mathbf{v}}_h\|^2_{h,\cup\Omega_i}=\sum_{i=1}^N\|\bar{\mathbf{v}}_h\|^2_{h,\Omega_i}$.

Proof. The proof is similar to that for Lemma 4.6 in [12], with the new nonconforming element used. Set

$$\bar{\mathbf{v}}_{h_{ij}}:=-\bar{p}_{h_i}|\Omega_i|\big(\pi_{h_i}(w^{(i)}_{\Gamma_{ij}}\mathbf{n}_i)+\pi_{h_j}(w^{(j)}_{\Gamma_{ij}}\mathbf{n}_i)\big),$$

where π_{h_i} is the interpolated operator defined in (4.4), and $\bar{\mathbf{v}}_h:=\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\bar{\mathbf{v}}_{h_{ij}}$. From the definition of π_{h_i} in (4.4) and (4.17), we have $\bar{\mathbf{v}}_{h_{ij}}\in[\mathcal{H}^1_0(\Omega_i\cup\Omega_j)]^2$, $\bar{\mathbf{v}}_h\in[\mathcal{H}^1_0(\Omega)]^2$, and

$$\begin{aligned}\int_{\mathcal{G}_{h_{ij}}}[\bar{\mathbf{v}}_{h_{ij}}\cdot\mathbf{n}_i]&=-\bar{p}_{h_i}|\Omega_i|\left(\int_{\mathcal{G}_{h_{ij}}}\pi_{h_i}(w^{(i)}_{\Gamma_{ij}}\mathbf{n}_i)\cdot\mathbf{n}_i-\int_{\mathcal{G}_{h_{ij}}}\pi_{h_j}(w^{(j)}_{\Gamma_{ij}}\mathbf{n}_i)\cdot\mathbf{n}_i\right)\\&=-\bar{p}_{h_i}|\Omega_i|\left(\int_{\mathcal{G}_{h_{ij}}}w^{(i)}_{\Gamma_{ij}}-\int_{\mathcal{G}_{h_{ij}}}w^{(j)}_{\Gamma_{ij}}\right)=0.\end{aligned}$$

So

$$\int_{\mathcal{G}_h}[\bar{\mathbf{v}}_h]\cdot\mathbf{n}_\Gamma=\sum_{i=1}^N\sum_{j\in\mathcal{N}_i,i<j}\int_{\mathcal{G}_{h_{ij}}}[\bar{\mathbf{v}}_{h_{ij}}]\cdot\mathbf{n}_{ij}=0,$$

which leads to (4.22), and that $\int_{\mathcal{G}_h}[\bar{\mathbf{v}}_{h_{ij}}]\cdot\mathbf{n}_\Gamma\{\bar{p}_h\}_w=0$. Now, we prove (4.20) below:

$$\begin{aligned}B_h(\bar{\mathbf{v}}_{h_{ij}},\bar{p}_h)&=-\int_{\mathcal{T}_h}\nabla_h\cdot\bar{\mathbf{v}}_{h_{ij}}\bar{p}_h+\int_{\mathcal{G}_h}[\bar{\mathbf{v}}_{h_{ij}}]\cdot\mathbf{n}_\Gamma\{\bar{p}_h\}_w\\&=-\bar{p}_{h_i}\int_{\mathcal{G}_{h_{ij}}}\bar{\mathbf{v}}_{h_{ij}}\cdot\mathbf{n}_i-\bar{p}_{h_j}\int_{\mathcal{G}_{h_{ij}}}\bar{\mathbf{v}}_{h_{ij}}\cdot\mathbf{n}_j\\&=\bar{p}_{h_i}^2|\Omega_i|-\bar{p}_{h_i}\bar{p}_{h_j}|\Omega_i|=\bar{p}_{h_i}|\Omega_i|(\bar{p}_{h_i}-\bar{p}_{h_j}).\end{aligned}$$

Thus

$$B_h(\bar{\mathbf{v}}_h,\bar{p}_h)=\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}B_h(\bar{\mathbf{v}}_{h_{ij}},\bar{p}_h)=\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\bar{p}_{h_i}|\Omega_i|(\bar{p}_{h_i}-\bar{p}_{h_j}),$$

which can be reinterpreted algebraically as $B_h(\bar{\mathbf{v}}_h,\bar{p}_h)=\bar{\mathbf{p}}_h^TB\bar{\mathbf{p}}_h$, where the vector $\bar{\mathbf{p}}_h=[\bar{p}_{h_i}]_{1\le i\le N}$ and the matrix $B=(b_{ij})_{1\le i,j\le N}$ is defined as follows:

$$b_{ij}=\begin{cases}|\Omega_i|card(\mathcal{N}_i) & \text{if}\quad j=i,\\-|\Omega_i| & \text{if}\quad j\in\mathcal{N}_i,\\0 & \text{otherwise.}\end{cases}$$

We also introduce the matrix $D=diag(|\Omega_1|,\cdots,|\Omega_N|)$ and observe that $\bar{\mathbf{p}}_h^TD\bar{\mathbf{p}}_h=\|\bar{p}_h\|_{0,\Omega}^2$. Then, the quantity $\gamma:=\min_{\bar{p}_h\in\bar{Q}_h}(\bar{\mathbf{p}}_h^TB\bar{\mathbf{p}}_h)(\bar{\mathbf{p}}_h^TD\bar{\mathbf{p}}_h)^{-1}$ is positive from the argument proposed in [4] (see Theorem 4.3) and it is independent of h. This deduces (4.20). Using (4.16), inequality (4.21) can be derived as follows:

$$\begin{aligned}\|\bar{\mathbf{v}}_h\|_{h,\cup\Omega_i}^2&\lesssim\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\|\bar{\mathbf{v}}_{h_{ij}}\|_{h,\cup\Omega_i}^2\\&\lesssim\sum_{i=1}^N\bar{p}_{h_i}^2|\Omega_i|^2\Big(\sum_{j\in\mathcal{N}_i}\|\pi_{h_i}(w_{\Gamma_{ij}}^{(i)}\mathbf{n}_i)\|_{h,\cup\Omega_i}^2\Big)\\&\lesssim\sum_{i=1}^N\bar{p}_{h_i}^2|\Omega_i|^2\|w_{\Gamma_{ij}}^{(i)}\|_{1,\Omega_i}^2\\&\lesssim\sum_{i=1}^N\bar{p}_{h_i}^2|\Omega_i|^2\lesssim\|\bar{p}_h\|_{0,\Omega}^2.\end{aligned}$$

We finally prove (4.23). For $i=1,\cdots,N$ and $j\in\mathcal{N}_i$, we can get

$$\|[\bar{\mathbf{v}}_h]\cdot\mathbf{n}_\Gamma\|_{\frac12,h,\Gamma}^2\lesssim\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\|[\bar{\mathbf{v}}_{h_{ij}}]\cdot\mathbf{n}_\Gamma\|_{\frac12,h,\Gamma}^2\lesssim\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\bar{p}_{h_i}^2|\Omega_i|^2\lesssim\|\bar{p}_h\|_{0,\Omega}^2,$$

where we have used the definition of $\bar{\mathbf{v}}_{h_{ij}}$ and (4.19) for the second inequality. □

Next, we introduce the global inf-sup condition for the bilinear form $B_h(\mathbf{v}_h,p_h)$.

Lemma 4.8. (Global inf-sup condition) *For any* $p_h \in Q_h$ *there exists* $\mathbf{v}_h \in \mathbf{V}_h \cap [\mathcal{H}_0^1(\Omega)]^2$*, such that*

$$B_h(\mathbf{v}_h, p_h) \gtrsim \|p_h\|_{0,\Omega}^2, \tag{4.24}$$
$$\|\mathbf{v}_h\|_{h,\cup\Omega_i} \lesssim \|p_h\|_{0,\Omega}, \tag{4.25}$$
$$|||\mathbf{v}_h||| \lesssim M^{\frac{1}{2}}\|p_h\|_{0,\Omega}. \tag{4.26}$$

Proof. Let $\tilde{\mathbf{v}}_h = \sum_{i=1}^N \tilde{\mathbf{v}}_{h_i}$ and $\tilde{p}_h = \sum_{i=1}^N \tilde{p}_{h_i}$ be as in Lemma 4.5 and let $\bar{\mathbf{v}}_h$ be as in Lemma 4.7. We set $\mathbf{v}_h := \tilde{\mathbf{v}}_h + \delta\bar{\mathbf{v}}_h$, where δ is a constant parameter to be chosen small enough. We note that $\mathbf{v}_h \in [\mathcal{H}_0^1(\Omega)]^2$ since $\tilde{\mathbf{v}}_{h_i} \in [\mathcal{H}_0^1(\Omega_i)]^2$ and $\bar{\mathbf{v}}_h \in [\mathcal{H}_0^1(\Omega)]^2$. Then, we follow the argument by Boland and Nicolaides. We observe that

$$B_h(\mathbf{v}_h, p_h) = B_h(\tilde{\mathbf{v}}_h, \tilde{p}_h) + \delta B_h(\bar{\mathbf{v}}_h, \bar{p}_h) + B_h(\tilde{\mathbf{v}}_h, \bar{p}_h) + \delta B_h(\bar{\mathbf{v}}_h, \tilde{p}_h). \tag{4.27}$$

Exploiting $\tilde{\mathbf{v}}_{h_i} \in [\mathcal{H}_0^1(\Omega_i)]^2$ and Lemma 4.7, we obtain that $B_h(\tilde{\mathbf{v}}_h, \bar{p}_h)$ vanishes:

$$\int_{\mathcal{T}_{h_i}} \nabla_h \cdot \tilde{\mathbf{v}}_{h_i}\bar{p}_{h_i} = \bar{p}_{h_i}\int_{\mathcal{B}_{h_i}} \tilde{\mathbf{v}}_{h_i} \cdot \mathbf{n}_i = 0,$$
$$\int_{\mathcal{G}_{h_{ij}}} [\tilde{\mathbf{v}}_h] \cdot \mathbf{n}_i \{\bar{p}_h\}_w = \{\bar{p}_h\}_w \int_{\mathcal{G}_{h_{ij}}} [\tilde{\mathbf{v}}_h] \cdot \mathbf{n}_i = 0,$$

while $B_h(\bar{\mathbf{v}}_h, \tilde{p}_h)$ can be estimated as follows:

$$\begin{aligned}\big|B_h(\bar{\mathbf{v}}_h, \tilde{p}_h)\big| &= \Big| -\int_{\mathcal{T}_h} \nabla_h \cdot \bar{\mathbf{v}}_h\tilde{p}_h + \int_{\mathcal{G}_h} [\bar{\mathbf{v}}_h] \cdot \mathbf{n}_\Gamma \{\tilde{p}_h\}_w\Big| \\ &\lesssim \|\tilde{p}_h\|_{0,\Omega}^2 + \|\bar{\mathbf{v}}_h\|_{h,\cup\Omega_i}^2 + \Big(\epsilon\|[\bar{\mathbf{v}}_h] \cdot \mathbf{n}_\Gamma\|_{\frac{1}{2},h,\Gamma}^2 + \frac{1}{\epsilon}\|\{\tilde{p}_h\}_w\|_{-\frac{1}{2},h,\Gamma}^2\Big) \\ &\lesssim \Big(1 + \frac{C}{\epsilon}\Big)\|\tilde{p}_h\|_{0,\Omega}^2 + (1 + C\epsilon)\|\bar{p}_h\|_{0,\Omega}^2.\end{aligned} \tag{4.28}$$

By substituting (4.28), (4.14), and (4.20) into (4.27), and suitably choosing δ and ϵ, we obtain (4.24), while (4.25) follows from the combination of (4.15) and (4.21).

Inequality (4.26) arises by observing that

$$\begin{aligned}|||\mathbf{v}_h|||^2 &= \|\eta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}^2 + \|\nu^{\frac{1}{2}}\nabla_h \mathbf{v}_h\|_{0,\cup\Omega_i}^2 \\ &\quad + \|\{\nu\}_w^{\frac{1}{2}}[\mathbf{v}_h]\|_{\frac{1}{2},h,\Gamma}^2 + \|\kappa_{sd}^{\frac{1}{2}}\{\mathbf{v}_h\}^w \times \mathbf{n}_{sd}\|_{0,\Gamma_{sd}}^2,\end{aligned} \tag{4.29}$$

where the first two terms on the right-hand side can be estimated as follows:

$$\sum_{i=1}^N \int_{\mathcal{T}_{h_i}} \Big(\eta_i|\mathbf{v}_h|^2 + \nu_i|\nabla_h \mathbf{v}_h|^2\Big) \lesssim \|\mu^{\frac{1}{2}}\mathbf{v}_h\|_{h,\cup\Omega_i}^2 \lesssim M\|\mathbf{v}_h\|_{h,\cup\Omega_i}^2 \lesssim M\|p_h\|_{0,\Omega}^2.$$

In regard to the last two terms of (4.29), we observe that $\mathbf{v}_h \times \mathbf{n}_{ij} = 0$ on Γ from the definition of $\mathbf{v}_h$. By consequence, $\kappa_{sd}^{\frac{1}{2}}\{\mathbf{v}_h\}^w \times \mathbf{n}_{sd} = \mathbf{0}$ on Γ and

$$\|\{\nu\}_w^{\frac{1}{2}}[\mathbf{v}_h]\|^2_{\frac{1}{2},h,\Gamma} = \|\{\nu\}_w^{\frac{1}{2}}[\mathbf{v}_h]\cdot\mathbf{n}_{sd}\|^2_{\frac{1}{2},h,\Gamma} \lesssim M\|p_h\|^2_{0,\Omega}.$$

The combination of the previous inequalities into (4.29) proves (4.26). $\square$

A classical result by Nečas on the solution of saddle-point boundary value problems, which we restrict here to the discrete case (see, for instance, Proposition 2.21 and Theorem 2.22 in [14]), ensures that the existence of a unique solution of problem (3.7) is a consequence of Theorem 4.1.

Theorem 4.1. (Stability) *Provided that $\gamma_E \gtrsim 1$, for any $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$, there exists $(\mathbf{w}_h, q_h) \in \mathbf{W}_h$ such that*

$$C_h((\mathbf{u}_h,p_h),(\mathbf{w}_h,q_h)) \gtrsim \frac{1}{M}|||(\mathbf{u}_h,p_h)|||\;|||(\mathbf{w}_h,q_h)|||. \tag{4.30}$$

Proof. Using the property $\nabla_h\cdot\mathbf{V}_{h_i}\subset Q_{h_i}$, we choose $(\mathbf{w}_h,q_h) = (\mathbf{u}_h+\delta_1\mathbf{v}_h, p_h+\delta_2\nabla_h\cdot\mathbf{u}_h)$ with $\delta_1,\delta_2>0$ being $\mathbf{v}_h$ as in Lemma 4.8. We split the proof of (4.30) into two parts. First, we prove $C_h((\mathbf{u}_h,p_h),(\mathbf{w}_h,q_h)) \gtrsim \frac{1}{M}|||(\mathbf{u}_h,p_h)|||^2$; then we show that $|||(\mathbf{w}_h,q_h)||| \lesssim |||(\mathbf{u}_h,p_h)|||$.

For the first part, we use the bilinearity of $C_h(\cdot,\cdot)$ to obtain

$$\begin{aligned} C_h((\mathbf{u}_h,p_h),(\mathbf{w}_h,q_h)) &= C_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)) + \delta_1 C_h((\mathbf{u}_h,p_h),(\mathbf{v}_h,0)) \\ &\quad + \delta_2 C_h((\mathbf{u}_h,p_h),(0,\nabla_h\cdot\mathbf{u}_h)). \end{aligned} \tag{4.31}$$

The first term on the right-hand side of (4.31) can be estimated as

$$C_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)) \geq C_{pos}|||\mathbf{u}_h|||^2 + J_h(\mathbf{u}_h,\mathbf{u}_h). \tag{4.32}$$

Applying Lemma 4.3 and Lemma 4.8 and the arithmetic/geometric inequality, the second term of (4.31) can be estimated as

$$\begin{aligned} C_h((\mathbf{u}_h,p_h),(\mathbf{v}_h,0)) &= A_h(\mathbf{u}_h,\mathbf{v}_h) + J_h(\mathbf{u}_h,\mathbf{v}_h) + B_h(\mathbf{v}_h,p_h) \\ &\geq (1-C_1\epsilon_1)\|p_h\|^2_{0,\Omega} - M\frac{C_1}{\epsilon_1}|||\mathbf{u}_h|||^2. \end{aligned} \tag{4.33}$$

The third term on the right-hand side of (4.31) is equivalent to

$$\begin{aligned} & C_h((\mathbf{u}_h,p_h),(0,\nabla_h\cdot\mathbf{u}_h)) \\ =& \int_{\mathcal{T}_h}(\nabla_h\cdot\mathbf{u}_h)^2 - \int_{\mathcal{G}_h}[\mathbf{u}_h]\cdot\mathbf{n}_\Gamma\{\nabla_h\cdot\mathbf{u}_h\}_w - \int_{\mathcal{B}_h}\mathbf{u}_h\cdot\mathbf{n}(\nabla_h\cdot\mathbf{u}_h) \\ \geq& (1-C_2\epsilon_2)\|\nabla_h\cdot\mathbf{u}_h\|^2_{0,\cup\Omega_i} - \frac{C_2}{\epsilon_2}\Big(\|[\mathbf{u}_h]\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\mathbf{u}_h\cdot\mathbf{n}\|^2_{\frac{1}{2},h,\partial\Omega}\Big). \end{aligned} \tag{4.34}$$

Then, combining (4.31)-(4.34), we obtain (4.30) as follows:

$$
\begin{aligned}
& C_h((\mathbf{u}_h,p_h),(\mathbf{w}_h,q_h)) \\
\gtrsim & \Big(C_{pos} - MC_1\frac{\delta_1}{\epsilon_1}\Big)|||\mathbf{u}_h|||^2 + \delta_2(1-C_2\epsilon_2)\|\nabla_h\cdot\mathbf{u}_h\|^2_{0,\cup\Omega_i} \\
& + \Big(\gamma_E - C_2\frac{\delta_2}{\epsilon_2}\Big)\big(\|[\mathbf{u}_h]\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma} + \|\mathbf{u}_h\cdot\mathbf{n}\|^2_{\frac{1}{2},h,\partial\Omega}\big) + \delta_1(1-C_1\epsilon_1)\|p_h\|^2_{0,\Omega} \\
\gtrsim & \frac{1}{M}|||(\mathbf{u}_h,p_h)|||^2,
\end{aligned}
\tag{4.35}
$$

for any $\gamma_E \gtrsim 1$, choosing $\epsilon_i < \frac{1}{C_i}$ for $i=1,2$, and δ_i such that $\delta_1 < \frac{C_{pos}\epsilon_1}{MC_1}$, $\delta_2 < \frac{\gamma_E\epsilon_2}{C_2}$. From the previous estimates for δ_1,δ_2, we conclude that $\frac{1}{M} \lesssim \delta_1 \lesssim \frac{1}{M}$ and $1 \lesssim \delta_2 \lesssim 1$, which imply (4.35).

For the second part of the proof, we observe that

$$
|||(\mathbf{w}_h,q_h)||| \lesssim |||(\mathbf{u}_h,p_h)||| + \delta_1|||(\mathbf{v}_h,0)||| + \delta_2|||(0,\nabla_h\cdot\mathbf{u}_h)|||. \tag{4.36}
$$

The second term on the right-hand side can be estimated owing to $\delta_1 \lesssim \frac{1}{M}$, (4.23) and (4.26), namely,

$$
\begin{aligned}
\delta_1|||(\mathbf{v}_h,0)|||^2 &= \delta_1\big[|||\mathbf{v}_h|||^2 + \|\nabla_h\cdot\mathbf{v}_h\|^2_{0,\cup\Omega_i} + \|[\mathbf{v}_h]\cdot\mathbf{n}_\Gamma\|^2_{\frac{1}{2},h,\Gamma}\big] \\
&\lesssim \delta_1 M\|p_h\|^2_{0,\Omega} \lesssim |||(\mathbf{u}_h,p_h)|||^2
\end{aligned}
$$

while for the third term we exploit $\delta_2 \lesssim 1$ to obtain

$$
\delta_2|||(0,\nabla_h\cdot\mathbf{u}_h)|||^2 = \delta_2\|\nabla_h\cdot\mathbf{u}_h\|^2_{0,\cup\Omega_i} \lesssim |||(\mathbf{u}_h,p_h)|||^2.
$$

The desired result can be obtained by substituting the two previous inequalities into (4.36). □

Inequality (4.30) shows that the stability of the scheme (3.7) depends on the coefficients of the problem only through the upper bound M; in particular, the scheme is robust with respect to the critical Stokes-Darcy transition, i.e., for vanishing viscosity $\nu \to 0$.

4.2. The Error estimates

Now we aim to study the convergence of $(\mathbf{u}_h,p_h)$ to $(\mathbf{u},p)$ when $h \to 0$. From Lemma 4.2, we have the standard second Strang's lemma.

Lemma 4.9. *Let* $(\mathbf{u},p)$ *be the solution of the problem* (2.8) *and* $(\mathbf{u}_h,p_h)$ *be the solution of the problem* (3.6). *Then, under the assumptions of Theorem* 4.1, the following error estimate holds

$$
\begin{aligned}
& |||(\mathbf{u}-\mathbf{u}_h,p-p_h)||| \\
\lesssim & \inf_{(\mathbf{v}_h,q_h)\in\mathbf{W}_h}|||(\mathbf{u}-\mathbf{v}_h,p-q_h)||| + \sup_{\mathbf{w}_h\in\mathbf{V}_h}\frac{\big|\int_{\mathcal{F}_h}\big(\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot[\mathbf{w}_h] - [\mathbf{w}_h\cdot\mathbf{n}]p\big)\big|}{|||\mathbf{w}_h|||}.
\end{aligned}
$$

To estimate the approximation errors, we first note that the distance of the pressure p from the space Q_h can be bounded in a standard way: Define L^2 projection operator $\theta_{h_i}: H^1(\Omega_i) \to Q_{h_i}$, if $p_i \in H^1(\Omega_i)$, then

$$\inf_{q_{h_i}\in Q_{h_i}} \|p_i - q_{h_i}\|_{0,\Omega_i} \leq \|p_i - \theta_{h_i} p_i\|_{0,\Omega_i} \lesssim h_i |p_i|_{H^1(\Omega_i)}.$$

Set $\theta_h = \Pi_{i=1}^N \theta_{h_i} : \mathcal{H}^1(\Omega) \to Q_h$. Recalling that $\mathcal{T}_{h_i}, i = 1, \cdots, N$ are a family of shape-regular and quasi-uniform triangulations and assuming $\mathbf{v} \in [\mathcal{H}^2(\Omega_i)]^2$, we have

$$\|\mathbf{v} - \pi_h \mathbf{v}\|_{\frac{1}{2},h,\Gamma} \lesssim h|\mathbf{v}|_{2,\cup\Omega_i},$$

where we have used the trace inequality (4.1) and Lemma 4.1. So it holds

Lemma 4.10. (Approximability). *For any* $(\mathbf{v}, q) \in [\mathcal{H}^2(\Omega)]^2 \times \mathcal{H}^1(\Omega)$*, we have*

$$|||(\mathbf{v} - \pi_h \mathbf{v}, q - \theta_h q)||| \lesssim Mh\Big(|\mathbf{v}|_{2,\cup\Omega_i} + |q|_{1,\cup\Omega_i}\Big).$$

The consistency error caused by the nonconforming rectangular element can be estimated by using the results in Lemma 4.2 of [28].

Lemma 4.11. (Consistency error) *For any* $\mathbf{w}_h \in \mathbf{V}_h$*, we have*

$$\begin{aligned}&\Big|\int_{\mathcal{F}_h} \Big(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot [\mathbf{w}_h] - [\mathbf{w}_h \cdot \mathbf{n}]p\Big)\Big| \\ \lesssim & h(\nu^{\frac{1}{2}} |\mathbf{u}|_{2,\cup\Omega_i} + |p|_{1,\cup\Omega_i})|||\mathbf{w}_h|||, \quad \forall(\mathbf{u}, p) \in [\mathcal{H}^2(\Omega)]^2 \times \mathcal{H}^1(\Omega).\end{aligned}$$

Combining the previous lemmas, we conclude that the following optimal priori error estimate holds.

Theorem 4.2. (Convergence) *Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *be the solutions of the problems* (2.8) *and* (3.6) *respectively. Assume that* $(\mathbf{u}, p) \in [\mathcal{H}^2(\Omega)]^2 \times \mathcal{H}^1(\Omega)$*, then we have*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \lesssim Mh\Big(|\mathbf{u}|_{2,\cup\Omega_i} + |p|_{1,\cup\Omega_i}\Big).$$

5. Numerical experiments

In this section, we show some numerical experiments to validate our method and the analysis. We consider three coupling cases as in [12]: a Darcy-Darcy problem (P_{dd}), a Stokes-Stokes problem (P_{ss}), and a Stokes-Darcy problem (P_{sd}). The computational domain is $\Omega_1 = (0, 1) \times (0, 1)$, $\Omega_2 = (0, 1) \times (1, 2)$ and the interface is the line segment $y = 1$, $0 < x < 1$. The penalty parameter is chosen as $\gamma_E = 10$. The results are almost the same with modest choice of γ_E.

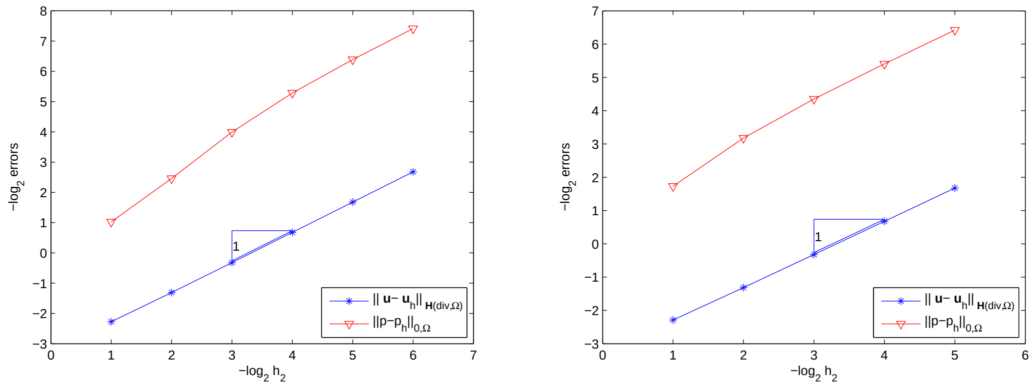

Figure 3: The $H(div)$ norm of $\mathbf{u}$ and L^2 norm of p errors for the example of P_{dd} versus mesh levels: $h_2/h_1 = \frac{1}{2}$(left), $h_2/h_1 = \frac{2}{1}$(right).

5.1. Darcy-Darcy coupling (P_{dd})

In this example, we consider the case of $\nu_1 = \nu_2 = 0, \eta_1 = \eta_2 = 1$, i.e. Darcy-Darcy coupled problem. The boundary data and the forcing terms are chosen such that the exact solution is given by

$$\mathbf{u}_d = (\pi\sin(\pi x), \pi\sin(\pi y))^T,$$
$$p_d = x^2 - \tfrac{1}{3}.$$

In Figure 3, we plot the errors of the velocity in $H(div)$ norm and the errors of the pressure in L^2 norm for the coupled Darcy-Darcy problem. The discrete meshes are non-matching on the common interface. In the left figure and the right figure, we consider the diameter of the discrete grid in Ω_1 and that in Ω_2 satisfying $h_1/h_2 = 2$ and $h_1/h_2 = 1/2$, respectively. Both results show that the errors decrease by half when the discrete grids refined once.

5.2. The Stokes-Stokes coupling (P_{ss})

In this example, we consider the case of $\nu_1 = \nu_2 = 1, \eta_1 = \eta_2 = 0$, i.e. Stokes-Stokes coupled problem. The boundary data and the forcing terms are chosen such that the exact solution is given by

$$\mathbf{u}_s = \left(2\pi\sin^2(\pi x)\sin(\pi y)\cos(\pi y), -2\pi\sin^2(\pi y)\sin(\pi x)\cos(\pi x)\right)^T,$$
$$p_s = \sin(\pi x) - \tfrac{2}{\pi}.$$

In Figure 4, we plot the errors of the velocity in H^1 semi-norm and the errors of the pressure in L^2 norm for the coupled Stokes-Stokes problem, the discrete meshes are nonmatching on the common interface. In the left figure and the right figure, we consider the diameter of the discrete grid in Ω_1 and that in Ω_2 satisfying $h_1/h_2 = 2$ and

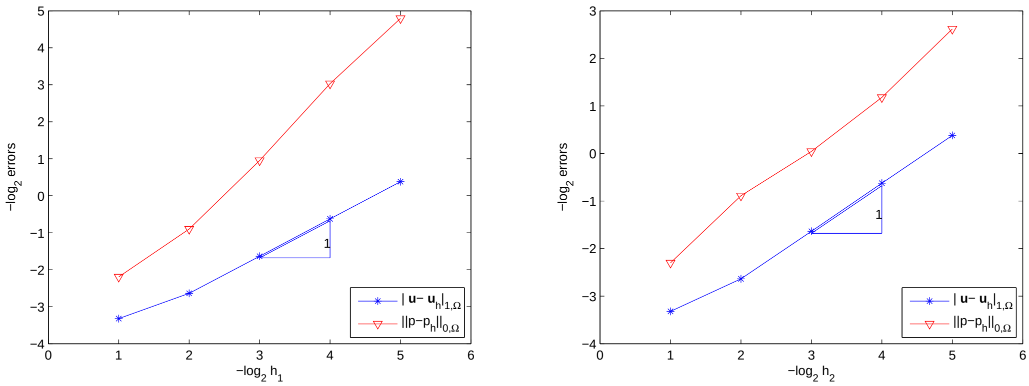

Figure 4: The H^1 semi-norm of $\mathbf{u}$ and L^2 norm of p errors for the example of P_{ss} versus mesh levels: $h_2/h_1 = \frac{1}{2}$(left), $h_2/h_1 = \frac{2}{1}$(right).

$h_1/h_2 = 1/2$, respectively. Both results show that the errors decrease by half when the discrete grids are refined once.

5.3. Stokes-Darcy coupling (P_{sd})

At last, we choose $\Omega_d = \Omega_1, \Omega_s = \Omega_2$ and set $\nu_1 = \eta_2 = 0$, i.e. Stokes-Darcy coupled problem, and set the physical parameters $\eta_1 = \nu_2 = \kappa_{12} = 1$ for simplicity. The boundary data and the forcing terms are chosen such that the exact solution is given by

$$
\begin{aligned}
\mathbf{u}_d &= \Big(-\tfrac{\pi^2}{8}\sin(\tfrac{\pi}{2}x)y,\ \tfrac{\pi}{4}\cos(\tfrac{\pi}{2}x)\Big)^T,\\
p_d &= -\tfrac{\pi}{4}\cos(\tfrac{\pi}{2}x)y,\\
\mathbf{u}_s &= \big(-\cos^2(\tfrac{\pi}{2}y)\sin(\tfrac{\pi}{2}x),\ \tfrac{1}{4}\cos(\tfrac{\pi}{2}x)(\sin(\pi y)+\pi y)\big)^T,\\
p_s &= -\tfrac{\pi}{4}\cos(\tfrac{\pi}{2}x)\,\big(y - 2\cos^2(\tfrac{\pi}{2}y)\big)\,.
\end{aligned}
$$

In Figure 5, we plot the errors between the exact solution and the finite element solution by using non-matching meshes with the mesh ratio being equal to $h_2/h_1 = \frac{1}{2}$ and $\frac{2}{1}$ separately. From the figures, we observe that the contraction factors are all around $\frac{1}{2}$ as the mesh is refined once. This clearly illustrates that the approximation order is linear. Therefore, we conclude that our numerical experiments confirm the error estimates in Section 4.

6. Conclusions

By using weighted interior penalties, we have proposed a unified nonconforming rectangular element method for incompressible flow problems modeled by the Darcy-

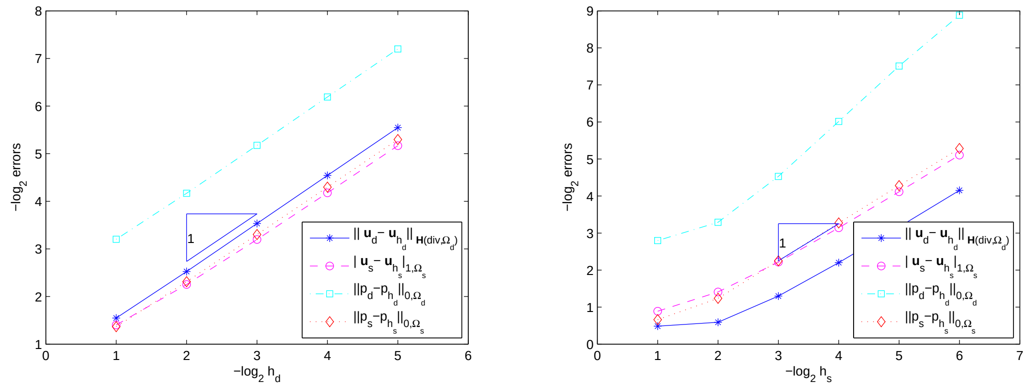

Figure 5: The error plots of finite element solutions in different norms for the example of P_{sd} versus mesh levels: $h_s/h_d = \frac{1}{2}$(left), $h_s/h_d = \frac{2}{1}$(right).

Stokes-Brinkman equations with discontinuous coefficients. The choice of the coefficients between different phases leads to different fluid-structure interactions such as Darcy-Stokes, Darcy-Darcy, Stokes-Stokes couplings. One of the advantage of the proposed new method is its flexibility; multi-domain heterogeneous problems are handled by the same method. We have shown the stability and optimal order error estimates for piecewise smooth solutions.

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