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Numerical Solution of Euler-Lagrange Equation with Caputo Derivatives

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Abstract. In this paper the fractional Euler-Lagrange equation is considered. The fractional equation with the left and right Caputo derivatives of order $\alpha \in (0,1]$ is transformed into its corresponding integral form. Next, we present a numerical solution of the integral form of the considered equation. On the basis of numerical results, the convergence of the proposed method is determined. Examples of numerical solutions of this equation are shown in the final part of this paper.

AMS subject classifications: 26A33, 34A08, 65R20, 70H03

Key words: Fractional Euler-Lagrange equation, fractional integral equation, numerical solution, Caputo derivatives.

1 Introduction

In the past few years, many applications in real phenomena have been found, where certain dynamics are described not only by integer but also by real order operators [4, 15, 16, 23, 29, 33, 34, 38]. An important issue is that the derivative of fractional order has a local property at any point of a domain only when order is an integer number. For non-integer cases, the fractional derivative is a nonlocal operator and depends on the past values of a function (left derivative) or future ones (right derivative). We refer the reader to a summary of fractional calculus theory in monographs [6, 18, 24, 27] and papers [1, 14, 19, 20, 26, 31, 36, 37] that cover various problems in this field.

One natural application of fractional operators is variational calculus. In this approach, one modifies the variational principle with replacing the integer order operators by a fractional one. Then, the minimisation of the action leads to the fractional differential

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equations which are known in the literature as the fractional Euler-Lagrange equations. Different approaches have been considered in recent years e.g., the Lagrangian or Hamiltonian approach with fractional integrals or fractional derivatives [1,3,5,7,19,21].

The main feature of the fractional Euler-Lagrange equations is that the fractional operator appearing in these equations contains simultaneously the left and right derivative. This is also a fundamental problem in finding solutions of equations of a variational type [6]. Consequently, numerical methods have been devoted to solving fractional variational problems [8–12,28,35].

In this paper we present a numerical solution of the Euler-Lagrange equation with Caputo derivatives in the finite time interval.

2 Fractional preliminaries

In this section, we introduce the fractional derivatives and integrals used in this work and some of their properties (see [18,25,27]). The left and right Caputo derivatives of order $\alpha\in(0,1]$ are defined as follows

$$ {}^{C}D^{\alpha}_{a^+}x(t):=I^{1-\alpha}_{a^+}Dx(t), \tag{2.1a}$$

$$ {}^{C}D^{\alpha}_{b^-}x(t):=-I^{1-\alpha}_{b^-}Dx(t), \tag{2.1b}$$

where D is an operator of the first order derivative and operators $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ are the left and right fractional integrals of order $\alpha>0$, respectively, defined by

$$ I^{\alpha}_{a^+}x(t):=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{x(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \qquad (t>a), \tag{2.2a}$$

$$ I^{\alpha}_{b^-}x(t):=\frac{1}{\Gamma(\alpha)}\int_t^b\frac{x(\tau)}{(\tau-t)^{1-\alpha}}d\tau, \qquad (t<b). \tag{2.2b}$$

If $\alpha=1$, then ${}^{C}D^{1}_{a^+}x=x'$ and ${}^{C}D^{1}_{b^-}x=-x'$.

The composition rules of the fractional operators (for $\alpha\in(0,1]$) are as follows [18,22]

$$ I^{\alpha}_{a^+}\,{}^{C}D^{\alpha}_{a^+}x(t)=x(t)-x(a), \tag{2.3a}$$

$$ I^{\alpha}_{b^-}\,{}^{C}D^{\alpha}_{b^-}x(t)=x(t)-x(b), \tag{2.3b}$$

and the fractional integral of a constant C

$$ I^{\alpha}_{a^+}C=C\frac{(t-a)^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.4}$$

3 The Euler-Lagrange equation and its equivalent integral form

We consider the problem of extremizing a functional with a Lagrangian depending on the independent variable t, function x, and its left Caputo fractional derivative of order $\alpha \in (0,1]$

$$S=\int_a^b L(t,x,{}^C D^{\alpha}_{a^+}x)dt \tag{3.1}$$

subject to the boundary conditions

$$x(a)=\xi_a, \quad x(b)=\xi_b. \tag{3.2}$$

By using results presented in [21] we obtain the following fractional Euler-Lagrange equation

$$\frac{\partial L(t,x,{}^C D^{\alpha}_{a^+}x)}{\partial x}+{}^C D^{\alpha}_{b^-}\frac{\partial L(t,x,{}^C D^{\alpha}_{a^+}x)}{\partial {}^C D^{\alpha}_{a^+}x}=0. \tag{3.3}$$

When the Lagrangian has the form

$$L(t,x,{}^C D^{\alpha}_{a^+}x)\equiv\frac{1}{2}({}^C D^{\alpha}_{a^+}x)^2-\Big(\frac{\omega^{2\alpha}}{2}x^2+f\cdot x\Big), \tag{3.4}$$

then we get the fractional Euler-Lagrange equation in the finite time interval $t\in[a,b]$

$${}^C D^{\alpha}_{b^-}\,{}^C D^{\alpha}_{a^+}x(t)-\omega^{2\alpha}x(t)=f(t), \tag{3.5}$$

where x and f are continues on $[0,b]$.

In particular, when $\alpha=1$, then ${}^C D^{1}_{b^-}\,{}^C D^{1}_{a^+}=-D^2$ and Eq. (3.5) becomes

$$D^2x(t)+\omega^2x(t)=-f(t). \tag{3.6}$$

Now, we can transform Eq. (3.5) into an integral equation [9]. Such an approach for equation (3.6) has been considered in [32]. We integrate Eq. (3.5) two times by using the right fractional integral operator (2.2b) and the left fractional integral operator (2.2a), respectively

$$I^{\alpha}_{a^+}I^{\alpha}_{b^-}\,{}^C D^{\alpha}_{b^-}\,{}^C D^{\alpha}_{a^+}x(t)-\omega^{2\alpha}I^{\alpha}_{a^+}I^{\alpha}_{b^-}x(t)=I^{\alpha}_{a^+}I^{\alpha}_{b^-}f(t). \tag{3.7}$$

Next, using the property (2.3b) we get

$$I^{\alpha}_{a^+}\Big({}^C D^{\alpha}_{a^+}x(t)-{}^C D^{\alpha}_{a^+}x(t)\Big|_{t=b}\Big)-\omega^{2\alpha}I^{\alpha}_{a^+}I^{\alpha}_{b^-}x(t)=I^{\alpha}_{a^+}I^{\alpha}_{b^-}f(t). \tag{3.8}$$

In the above equation, the value ${}^C D^{\alpha}_{a^+}x(t)|_{t=b}$ occurs to be a constant. The application of the composition rule (2.3a) and the fractional integral of a constant (2.4) leads to the following equation

$$x(t)-x(a)-{}^C D^{\alpha}_{a^+}x(t)|_{t=b}\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-\omega^{2\alpha}I^{\alpha}_{a^+}I^{\alpha}_{b^-}x(t)=I^{\alpha}_{a^+}I^{\alpha}_{b^-}f(t). \tag{3.9}$$

The unknown value ${}^C D^{\alpha}_{a^+} x(t)|_{t=b}$ can be determined due to the boundary condition. We substitute $t=b$ into Eq. (3.9)

$$x(b)-x(a)-{}^C D^{\alpha}_{a^+} x(t)|_{t=b}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}-\omega^{2\alpha} I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t)|_{t=b}=I^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t)|_{t=b} \tag{3.10}$$

and we obtain

$${}^C D^{\alpha}_{a^+} x(t)|_{t=b}=\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}(x(b)-x(a)-\omega^{2\alpha} I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t)|_{t=b}-I^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t)|_{t=b}). \tag{3.11}$$

Substituting the right-hand side of the formula (3.11) into Eq. (3.9) one get the integral form of Eq. (3.5) in the following form

$$\begin{aligned} &x(t)-\omega^{2\alpha}\Big(I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t)-\Big(\frac{t-a}{b-a}\Big)^{\alpha} I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t)\Big|_{t=b}\Big) \\ =&\Big(\frac{t-a}{b-a}\Big)^{\alpha}(\xi_b-\xi_a-I^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t)|_{t=b})+I^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t)+\xi_a. \end{aligned} \tag{3.12}$$

4 Numerical solution

In order to solve Eq. (3.12) we present a numerical scheme. We start with the introduction of the homogeneous grid of nodes $t_i=a+i\Delta t$ ($i=0,1,\cdots,n$), with the constant time step $\Delta t=(b-a)/n$, where $n+1$ is a number of nodes. For every grid node t_i we obtain the following equation

$$\begin{aligned} &x(t_i)-\omega^{2\alpha}\Big(I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t)|_{t=t_i}-\Big(\frac{t_i-t_0}{t_n-t_0}\Big)^{\alpha} I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t)|_{t=t_n}\Big) \\ =&\Big(\frac{t_i-t_0}{t_n-t_0}\Big)^{\alpha}(\xi_b-\xi_a-I^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t)|_{t=t_n})+I^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t)|_{t=t_i}+\xi_a. \end{aligned} \tag{4.1}$$

We introduce notations $x_i=x(t_i)$ and $f_i=f(t_i)$ (the values of functions $x(t)$ and $f(t)$ at the node t_i). In our previous works [9,10,13] we have determined the discrete form of the composition of operators. On the basis of our earlier results, we present the final discrete forms (being the approximation of $I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t)|_{t=t_i}$ and $I^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t)|_{t=t_i}$, for $i=0,1,\cdots,n$) as

$$I^{\alpha}_{a^+} I^{\alpha}_{b^-} x(t)|_{t=t_i}\approx\sum_{j=0}^{n} x_j z^{(\alpha)}_{i,j} \quad \text{and} \quad I^{\alpha}_{a^+} I^{\alpha}_{b^-} f(t)|_{t=t_i}\approx\sum_{j=0}^{n} f_j z^{(\alpha)}_{i,j}, \tag{4.2}$$

where $z_{i,j}^{(\alpha)}=\sum_{k=0}^{\min(i,j)} u_{i,k}^{(\alpha)} v_{k,j}^{(\alpha)}$ and coefficients $u_{i,k}^{(\alpha)}$ and $v_{k,j}^{(\alpha)}$ look as follows:

$$
\begin{aligned}
u_{i,k}^{(\alpha)}=&\frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+2)}\\
&\times\begin{cases}
0, & \text{for } i=0 \text{ and } k=0,\\
(i-1)^{\alpha+1}-i^{\alpha+1}+i^{\alpha}(\alpha+1), & \text{for } i>0 \text{ and } k=0,\\
(i-k+1)^{\alpha+1}-2(i-k)^{\alpha+1}+(i-k-1)^{\alpha+1}, & \text{for } i>0 \text{ and } 0<k<i,\\
1, & \text{for } i>0 \text{ and } k=i,
\end{cases}
\end{aligned}
\tag{4.3a}
$$

$$
\begin{aligned}
v_{k,j}^{(\alpha)}=&\frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+2)}\\
&\times\begin{cases}
0, & \text{for } k=n \text{ and } j=n,\\
(n-k-1)^{\alpha+1}-(n-k)^{\alpha+1}+(n-k)^{\alpha}(\alpha+1), & \text{for } k<n \text{ and } j=n,\\
(j-k+1)^{\alpha+1}-2(j-k)^{\alpha+1}+(j-k-1)^{\alpha+1}, & \text{for } k<n \text{ and } k<j<n,\\
1, & \text{for } k<n \text{ and } j=k.
\end{cases}
\end{aligned}
\tag{4.3b}
$$

In order to compute values of operators (4.2) at every node t_i, we need to use values of functions at all nodes of the domain. From the computational point of view, it leads to an increase calculation time of discrete values of function.

Now, we present the numerical scheme of the integral equation (3.12). If we substitute (4.2) into (4.1), then the solution can be written as the system of $n+1$ linear equations:

$$
x_i-\omega^{2\alpha}\Big(\sum_{j=0}^{n} x_j z_{i,j}^{(\alpha)}-\Big(\frac{i}{n}\Big)^{\alpha}\sum_{j=0}^{n} x_j z_{n,j}^{(\alpha)}\Big)=\Big(\frac{i}{n}\Big)^{\alpha}\Big[\xi_b-\xi_a-\sum_{j=0}^{n} f_j z_{n,j}^{(\alpha)}\Big]+\sum_{j=0}^{n} f_j z_{i,j}^{(\alpha)}+\xi_a, \tag{4.4}
$$

for $i=0,\cdots,n$.

The Eqs. (4.4) can also be written in the matrix form

$$
\mathbf{A}\cdot\mathbf{x}=\mathbf{b}, \tag{4.5}
$$

where $\mathbf{x}=[x_0,x_1,\cdots,x_n]^{\mathrm{T}}$ and the coefficients in matrices $\mathbf{A}$ and $\mathbf{b}$ look as follows:

$$
A_{i,j}=\delta_{i,j}-\omega^{2\alpha}\Big(z_{i,j}{}^{(\alpha)}-\Big(\frac{i}{n}\Big)^{\alpha} z_{n,j}^{(\alpha)}\Big), \tag{4.6a}
$$

$$
b_i=\Big(\frac{i}{n}\Big)^{\alpha}\Big[\xi_b-\xi_a-\sum_{j=0}^{n} f_j z_{n,j}^{(\alpha)}\Big]+\sum_{j=0}^{n} f_j z_{i,j}^{(\alpha)}+\xi_a, \tag{4.6b}
$$

for $i=0,\cdots,n$, $j=0,\cdots,n$ and

$$
\delta_{i,j}=\begin{cases} 1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}
$$

5 Example of computations

We present the results of computations obtained by our numerical scheme to the considered fractional Euler-Lagrange equation (3.5). We use the Gaussian elimination algorithm (the LUP decomposition method [30]) to solve the system of linear equation (4.5). We present several examples of calculations for different values of parameters α, ω and for different types of a function $f(t)$. In all examples we assumed $a=0$, $b=1$. The time domain $t\in[0,1]$ has been divided into $n=1000$ subintervals. The values of the parameters used in the solution of Eq. (3.5) are given in the plot legends.

The simulation results as the plots of $x(t)$ are given below. In Figs. 1, 2 and 3, we

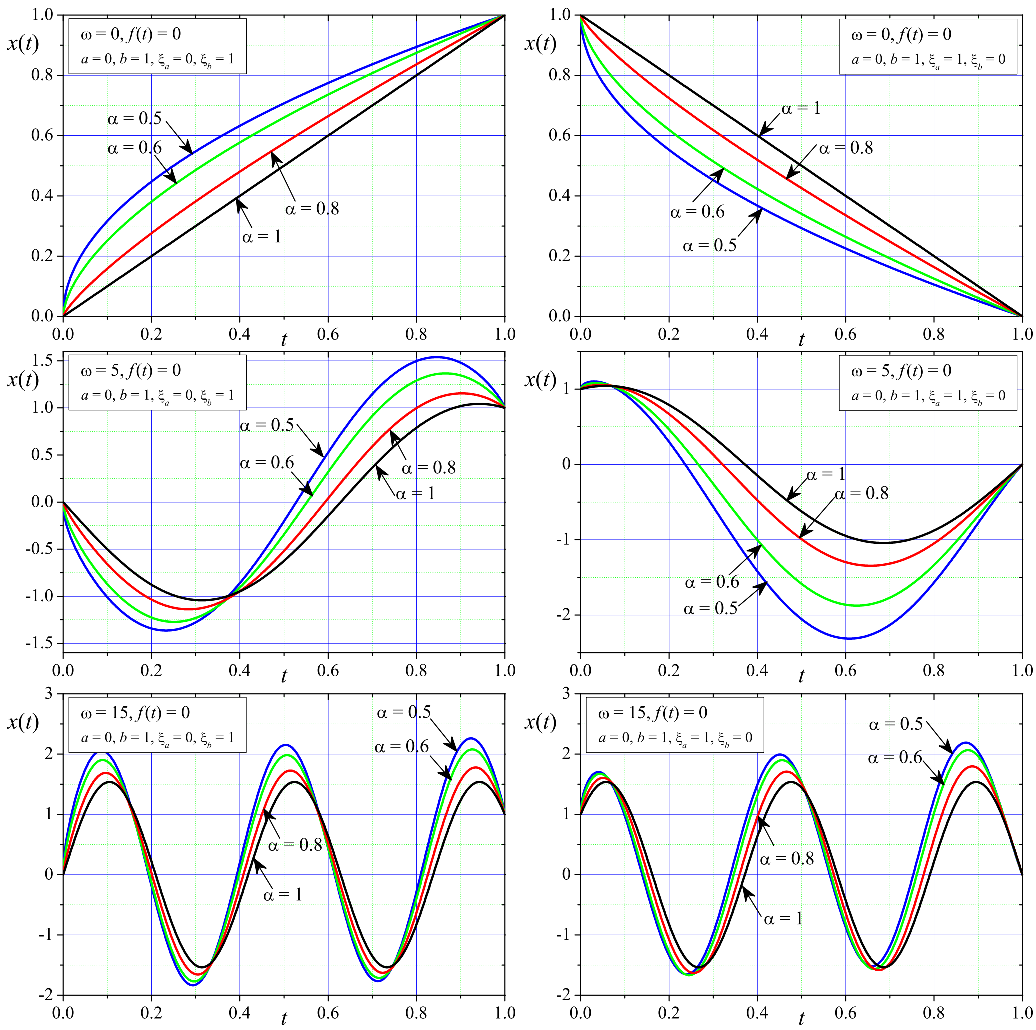

Figure 1: Numerical solution of Eq. (3.5) for $\alpha\in\{0.5,0.6,0.8,1\}$, $\omega\in\{0,5,15\}$, $f(t)=0$, $a=0$, $b=1$, and: left-side: $\xi_a=0$, $\xi_b=1$, right-side: $\xi_a=1$, $\xi_b=0$.

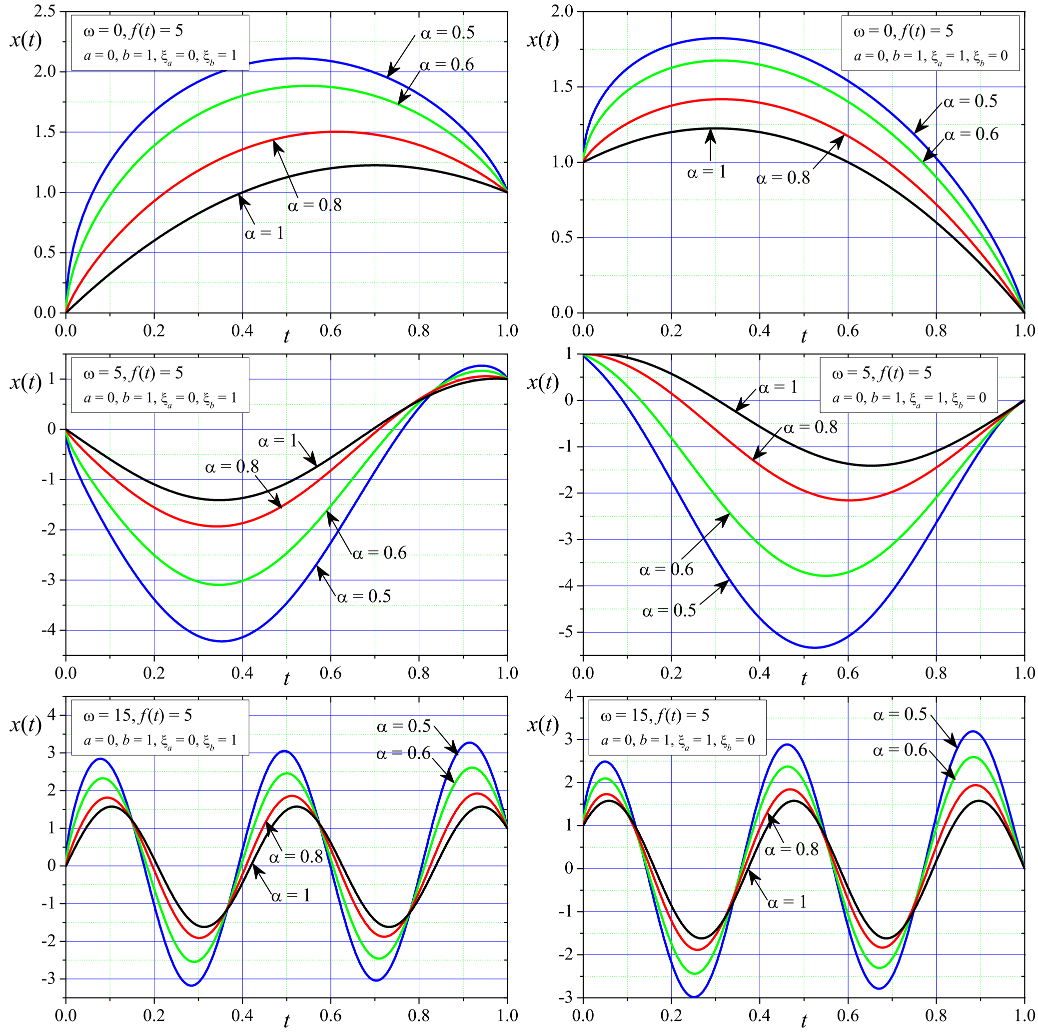

Figure 2: Numerical solution of Eq. (3.5) for $\alpha \in \{0.5, 0.6, 0.8, 1\}$, $\omega \in \{0, 5, 15\}$, $f(t) = 5$, $a = 0$, $b = 1$, and: left-side: $\xi_a = 0$, $\xi_b = 1$, right-side: $\xi_a = 1$, $\xi_b = 0$.

present results of the numerical solution of Eq. (3.5) for the fixed parameter ω and various values of order α. Whereas, in Fig. 4, we show results for the fixed order α and different values of the parameter ω. The behaviors of the solution of the fractional Euler-Lagrange equation are different from various values of order α, various values of parameter ω, and different forms of function f.

One can note that by fixing the parameter ω, and the function f and by changing the order of the Caputo derivative α we observe that the amplitude of oscillations increases when α decreases. On the other hand, for the fixed order α and for varying the parameter ω we can observe that the number of oscillations increases when the value of the

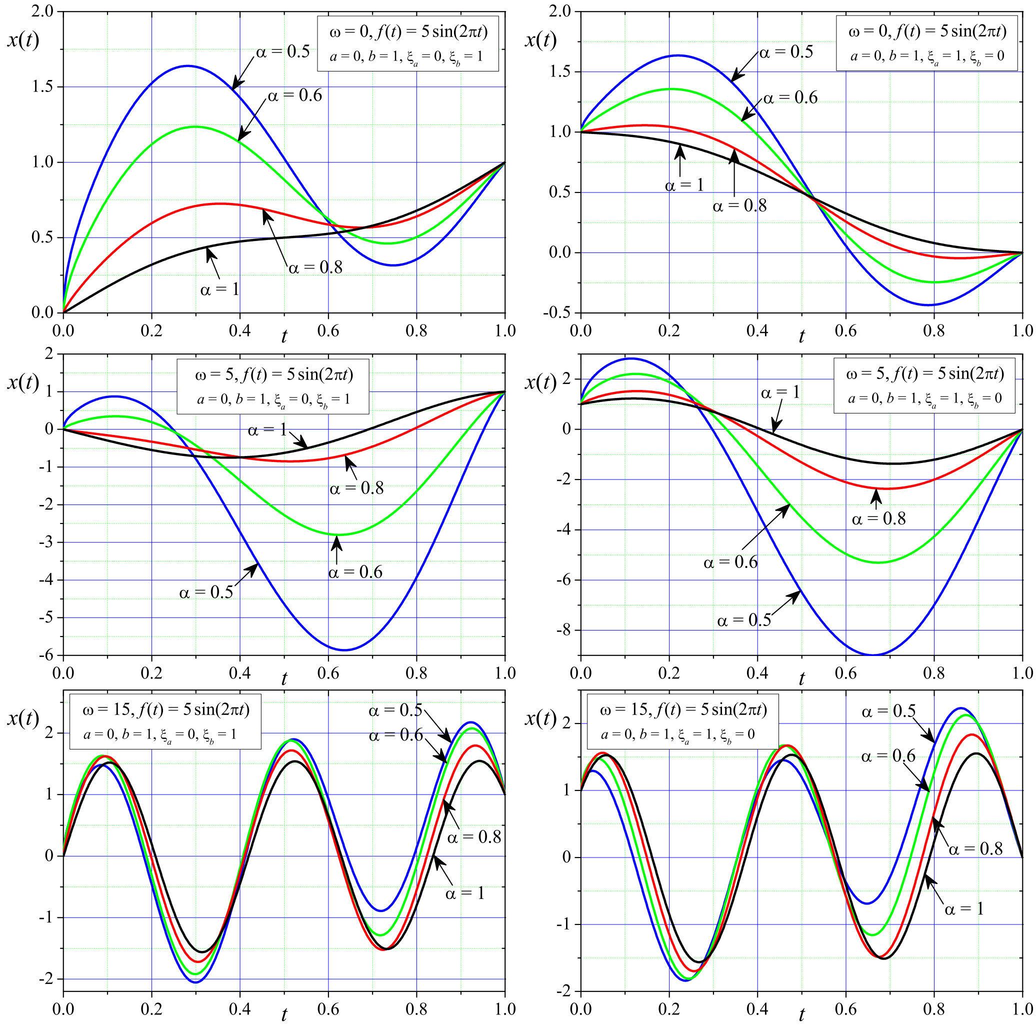

Figure 3: Numerical solution of Eq. (3.5) for $\alpha\in\{0.5,0.6,0.8,1\}$, $\omega\in\{0,5,15\}$, $f(t)=5\sin(2\pi t)$, $a=0$, $b=1$, and: left-side: $\xi_a=0$, $\xi_b=1$, right-side: $\xi_a=1$, $\xi_b=0$.

parameter ω increases. By changing only values of boundary conditions and simultaneously assuming that other parameters of the equation remaining unchanged, we obtain not symmetrical solutions for $\alpha<1$. Symmetry appears only when $\alpha=1$.

The analysis of the numerical solutions of the fractional Euler-Lagrange equation shows us that, taking into account the fractional order of the differential equation, it is possible to have more flexible models which can describe the dynamical properties of the real system in a better manner. Another important feature is that when we consider the fractional order derivative in the model, we deal with the memory effect of the model due to the kernel type in the fractional order operator.

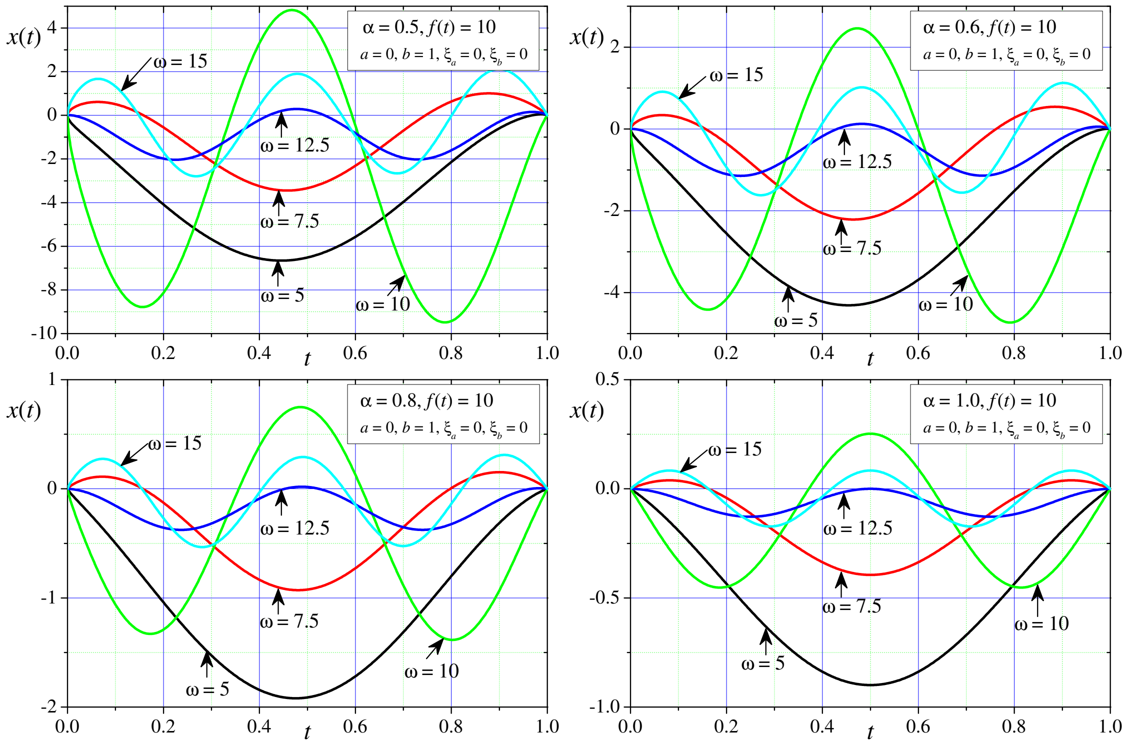

Figure 4: Numerical solution of Eq. (3.5) for $\alpha \in \{0.5, 0.6, 0.8, 1\}$, $\omega \in \{5, 7.5, 10, 12.5, 15\}$, $f(t) = 10$, $a = 0$, $b = 1$, $\xi_a = 0$, $\xi_b = 0$.

5.1 Estimating the rate of convergence

Convergence analysis of the numerical scheme (4.4) is important from the computational point of view, especially when an analytical solution of an equation with given parameters is not available. Here, we use the following formula (see Proposition [2]) for the rate of convergence $p = p_i(\Delta t, \alpha, \omega)$ at nodes t_i (i should be an even number), for fixed parameters α, ω and variable values of Δt

$$R_i^{(\Delta t,\alpha,\omega)} = \frac{x_i^{(\Delta t,\alpha,\omega)} - x_{i/2}^{(2\Delta t,\alpha,\omega)}}{x_{2i}^{(\Delta t/2,\alpha,\omega)} - x_i^{(\Delta t,\alpha,\omega)}} = 2^{p_i(\Delta t,\alpha,\omega)}, \tag{5.1}$$

from which we determine

$$p_i(\Delta t,\alpha,\omega) = \log_2 \frac{x_i^{(\Delta t,\alpha,\omega)} - x_{i/2}^{(2\Delta t,\alpha,\omega)}}{x_{2i}^{(\Delta t/2,\alpha,\omega)} - x_i^{(\Delta t,\alpha,\omega)}}. \tag{5.2}$$

In this order, we considered several cases for various parameters of the equation. We assumed in all cases: $\omega = \pi/2$, $f(t) = 10$, $a = 0$, $b = 1$. In Tables 1 and 2, we presented

Table 1: Numerical values of x at nodes t_i, $i \in \{n/4, n/2, 3n/4\}$ and rates of convergence p for parameters: $\omega = \pi/2$, $f(t) = 10$, $a = 0$, $b = 1$, $\xi_a = 1$ and $\xi_b = 0$.

		$t=0.25$		$t=0.5$		$t=0.75$	
α	$\Delta t = 1/n$	$x(t) \equiv x_{n/4}$	p	$x(t) \equiv x_{n/2}$	p	$x(t) \equiv x_{3n/4}$	p
0.5	1/100	5.751401814	–	6.000537754	–	4.053823043	–
	1/200	5.742775279	0.93	5.987934744	0.96	4.040117802	0.98
	1/400	5.738257173	0.97	5.981464417	0.98	4.033149420	0.99
	1/800	5.735945160	0.98	5.978187337	0.99	4.029637505	0.99
	1/1600	5.734775549	0.99	5.976538491	1.00	4.027875021	1.00
	1/3200	5.734187270	1.00	5.975711560	1.00	4.026992274	1.00
	1/6400	5.733892249	–	5.975297490	–	4.026550563	–
0.6	1/100	4.590617288	–	4.854742475	–	3.260517955	–
	1/200	4.588340770	1.07	4.851133657	1.12	3.256339986	1.14
	1/400	4.587255599	1.13	4.849469124	1.15	3.254449954	1.17
	1/800	4.586758949	1.16	4.848720577	1.17	3.253608947	1.18
	1/1600	4.586536532	1.18	4.848388617	1.18	3.253238194	1.19
	1/3200	4.586438124	1.19	4.848242559	1.19	3.253075620	1.19
	1/6400	4.586394883	–	4.848178587	–	3.253004551	–
0.8	1/100	3.065551324	–	3.330101157	–	2.243252313	–
	1/200	3.065526370	0.13	3.329983689	1.06	2.243025485	1.34
	1/400	3.065503628	1.03	3.329927196	1.29	2.242936187	1.43
	1/800	3.065492480	1.28	3.329904101	1.40	2.242903087	1.48
	1/1600	3.065487888	1.40	3.329895363	1.47	2.242891252	1.52
	1/3200	3.065486143	1.46	3.329892200	1.51	2.242887119	1.54
	1/6400	3.065485511	–	3.329891086	–	2.242885698	–
1	1/100	2.166255362	–	2.385747145	–	1.625061072	–
	1/200	2.166313163	2.00	2.385825124	2.00	1.625117515	2.00
	1/400	2.166327614	2.00	2.385844619	2.00	1.625131627	2.00
	1/800	2.166331227	2.00	2.385849493	2.00	1.625135155	2.00
	1/1600	2.166332130	2.00	2.385850712	2.00	1.625136037	2.00
	1/3200	2.166332356	1.99	2.385851016	2.00	1.625136257	2.00
	1/6400	2.166332412	–	2.385851093	–	1.625136313	–

numerical values at three selected nodes t_i, $i \in \{n/4, n/2, 3n/4\}$ for $\alpha \in \{0.5, 0.6, 0.8, 1\}$ and different combinations of boundary conditions: $\xi_a = 1$, $\xi_b = 0$ (in Table 1) and $\xi_a = 0$, $\xi_b = 1$ (in Table 2). Also, in both tables, the rates of convergence are shown. On the basis of analysis of values of p for increasing values of n, we can estimate the rate of convergence as $p = 2\alpha$.

6 Conclusions

In this paper the non-homogenous fractional Euler-Lagrange equation with Caputo derivatives of order $\alpha \in (0,1]$ was transformed into the integral form. Next, the numerical scheme for the integral form of equation was presented. We presented several examples

Table 2: Numerical values of x at nodes t_i, $i\in\{n/4,n/2,3n/4\}$ and rates of convergence p for parameters: $\omega=\pi/2$, $f(t)=10$, $a=0$, $b=1$, $\xi_a=0$ and $\xi_b=1$.

		$t=0.25$		$t=0.5$		$t=0.75$	
α	$\Delta t=1/n$	$x(t)\equiv x_{n/4}$	p	$x(t)\equiv x_{n/2}$	p	$x(t)\equiv x_{3n/4}$	p
0.5	1/100	5.680739373	–	6.527671682	–	4.990789515	–
	1/200	5.670654549	0.94	6.512988479	0.97	4.974863504	0.98
	1/400	5.665399277	0.97	6.505474754	0.98	4.966781748	0.99
	1/800	5.662716828	0.98	6.501675683	0.99	4.962713041	1.00
	1/1600	5.661361580	0.99	6.499765914	1.00	4.960672310	1.00
	1/3200	5.660680392	1.00	6.498808580	1.00	4.959650525	1.00
	1/6400	5.660338897	–	6.498329334	–	4.959139328	–
0.6	1/100	4.386745815	–	5.237761476	–	4.087854654	–
	1/200	4.384011561	1.09	5.233464617	1.13	4.082913068	1.15
	1/400	4.382722759	1.14	5.231496411	1.16	4.080686164	1.17
	1/800	4.382136355	1.16	5.230614665	1.18	4.079697454	1.18
	1/1600	4.381874585	1.18	5.230224478	1.19	4.079262151	1.19
	1/3200	4.381758976	1.19	5.230053016	1.19	4.079071419	1.19
	1/6400	4.381708229	–	5.229977970	–	4.078988078	–
0.8	1/100	2.665443020	–	3.495581958	–	2.907988670	–
	1/200	2.665382445	0.79	3.495406301	1.20	2.907692864	1.38
	1/400	2.665347375	1.19	3.495329637	1.35	2.907579569	1.45
	1/800	2.665332015	1.35	3.495299645	1.44	2.907538243	1.50
	1/1600	2.665325997	1.44	3.495288572	1.49	2.907523619	1.53
	1/3200	2.665323774	1.49	3.495284623	1.52	2.907518546	1.55
	1/6400	2.665322981	–	3.495283247	–	2.907516810	–
1	1/100	1.625061072	–	2.385747145	–	2.166255362	–
	1/200	1.625117515	2.00	2.385825124	2.00	2.166313163	2.00
	1/400	1.625131627	2.00	2.385844619	2.00	2.166327614	2.00
	1/800	1.625135155	2.00	2.385849493	2.00	2.166331227	2.00
	1/1600	1.625136037	2.00	2.385850712	2.00	2.166332130	2.00
	1/3200	1.625136257	1.99	2.385851016	2.00	2.166332356	2.00
	1/6400	1.625136313	–	2.385851093	–	2.166332412	–

of numerical solutions of considered equation for different values of parameters α, ω, different values of boundary conditions, and different functions $f(t)$. One can note that if the value of α decreases, then values of the amplitude of oscillations increase, and if the value of ω increases, then the oscillation frequency also increases. The analytical solution of this type of fractional differential equation (except for $\alpha=1$) is not yet known. Our proposed numerical method of solutions for $\alpha\to 1$ is consistent with the analytical solution for $\alpha=1$.

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