Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **8**, No. 6, pp. 1084-1098

DOI: 10.4208/aamm.2015.m1248
December 2016

Bifurcations and Single Peak Solitary Wave Solutions of an Integrable Nonlinear Wave Equation

Wei Wang, Chunhai Li* and Wenjing Zhu

School of Mathematics and Computing Science, Guilin University of Electronic Technology, Jinji Road No. 1, Guilin 541004, China

Received 6 August 2015; Accepted (in revised version) 2 December 2015

Abstract. Dynamical system theory is applied to the integrable nonlinear wave equation $u_t \pm (u^3 - u^2)_x + (u^3)_{xxx} = 0$. We obtain the single peak solitary wave solutions and compacton solutions of the equation. Regular compacton solution of the equation correspond to the case of wave speed $c=0$. In the case of $c \neq 0$, we find smooth soliton solutions. The influence of parameters of the traveling wave solutions is explored by using the phase portrait analytical technique. Asymptotic analysis and numerical simulations are provided for these soliton solutions of the nonlinear wave equation.

AMS subject classifications: 35Q51, 35Q53

Key words: Bifurcation, solitary wave, compaction.

1 Introduction

It is well known that the study of nonlinear wave equations and their solutions are of great importance in many areas of physics. Travelling wave solution is an important type of solution for the nonlinear partial differential equation. Finding their traveling wave solutions of these equations has become a hot research topic for many scholars. Many methods have been used to investigate these types of equations, such as tanh-sech method [1], Lie group method [2], exp-function method, bifurcation method [3–9] and sine-cosine method.

Classically, the solitary wave solutions of nonlinear evolution equations are determined by analytic formulae and serve as prototypical solutions that model physical localized waves. For integrable systems, the solitary waves interact clearly, and are known as solitons. The appearance of non-analytic solitary wave solutions to new classes of nonlinear wave equations, including peakons [10–14], which have a corner at their crest,

*Corresponding author.
Email: chunhai2001@163.com (C. H. Li)

cuspons [11], having a cusped crest and compactons [15–26], which have compact support, has vastly increased the menagerie of solutions appearing in model equations, both integrable and non-integrable.

There are two important nonlinearly dispersive equations. One is the well-known Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + u_{xxx}, \tag{1.1}$$

which was proposed by Camassa and Holm [10] as a model equation for unidirectional nonlinear dispersive waves in shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties. The Camassa-Holm equation have been found to has peakons, cuspons and composite wave solutions [11]. The other is the $K(m,n)$ equation

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \tag{1.2}$$

which was discovered by Rosenau and Hyman [27], where a is a constant and both the convection term $(u^m)_x$ and the dispersion effect term $(u^n)_{xxx}$ are nonlinear. These equations arise in the process of understanding the role of nonlinear dispersion in the formation of structures like liquid drops. Rosenau and Hyman derived solutions called compactons for Eq. (1.2). Xu and Tian [28] introduced the osmosis $K(2,2)$ equation

$$u_t + (u^2)_x - (u^2)_{xxx} = 0, \tag{1.3}$$

where the negative coefficient of dispersion term denotes the contracting dispersion.

In the present work, we consider the following integrable nonlinear wave equation

$$u_t + a(u^3 - u^2)_x + (u^3)_{xxx} = 0, \tag{1.4}$$

where $a = \pm 1$. It's a simple model used for cubic dispersion of presentation. In [29], Rosenau had studied the impact of a non-convex convection on formation of compactons by using this model. Note that whereas the $K(3,3)$ has four local conserved quantities [27]: $\int u dx$, $\int u^4 dx$, $\int u \cos x dx$ and $\int u \sin x dx$, Eq. (1.4) inherits from $K(3,3)$ only two conserved quantities: $\int u dx$, $\int u^4 dx$.

Here, by using bifurcation theory of dynamical system, we consider bifurcation problem of the single peak solitary wave solutions and compacton solutions for the Eq. (1.4).

We look for travelling wave solutions of Eq. (1.4) in the form of $u(x,t) = u(\xi)$ with $\xi = x - ct$, where c is the wave speed. Substituting the traveling wave solution $u(x,t) = u(x-ct)$ into Eq. (1.4), we have the following equation:

$$-cu_\xi + a(u^3 - u^2)_\xi + (u^3)_{\xi\xi\xi} = 0. \tag{1.5}$$

Integrating (1.5) once and setting the integration constant as g, we have

$$-cu + a(u^3 - u^2) + (u^3)_{\xi\xi} = g. \tag{1.6}$$

Substituting

$$\frac{du}{d\xi}=y$$

into the Eq. (1.6), then we have the following equivalent planar system

$$\begin{cases} \dfrac{du}{d\xi}=y, \\ \dfrac{dy}{d\xi}=\dfrac{-a(u^3-u^2)+cu-6uy^2+g}{3u^2}. \end{cases} \tag{1.7}$$

The system (1.7) has the first integral:

$$H(u,y)=\frac{3}{2}u^4y^2-\frac{g}{3}u^3-\frac{c}{4}u^4+\frac{a}{6}u^6-\frac{a}{5}u^5=h. \tag{1.8}$$

On the singular straight line $u=0$, the second equation in (1.7) is discontinuous. Such system (1.7) is called a singular system. In other words, $u_{\xi\xi}$ is not been defined on the straight lines in the phase plane (u,y). It derives that the differential system (1.7) could has some non-smooth behavior or breaking properties of traveling wave solution.

This paper is organized as follows. In Section 2, we analyze the bifurcations of phase portraits of system (1.7) with $a=\pm 1$. In Section 3, we give the parametric representations of the smooth solitary wave solutions and compacton solution of Eq. (1.4). A short conclusion is given in Section 4.

2 Bifurcations of phase portraits of system (1.7)

In this section, we study all bifurcations of phase portrait in the parametric space. Denote

$$\frac{d\xi}{d\zeta}=3u^2,$$

then system (1.7) has the same topological phase portraits as the following polynomial system

$$\begin{cases} \dfrac{du}{d\zeta}=3u^2y, \\ \dfrac{dy}{d\zeta}=-au^3+au^2+cu-6uy^2+g, \end{cases} \tag{2.1}$$

except for the singular line $u=0$, which is a straight line solution for system (2.1). Easy to see that system (2.1) is a Hamiltonian system with Hamiltonian function $H(u,y)$ defined as the same as (1.8). For a given $h=H(u,y)$, (1.8) determine a set of invariant curves of system (2.1), which contains some different branches of curves. As h varies, (1.8) defines different families of orbits of system (2.1) with different dynamical behaviors. Denote that

$$f(u)=-au^3+au^2+cu+g. \tag{2.2}$$

It is easy to see that on the (u,y)-phase plane, the abscissas of equilibrium points of system (2.1) on the u-axis are the solutions of $f(u)=0$. Noticing that $f'(u)=-3au^2+2au+c$, $f'(u)$ has two solutions $u_1,u_2(u_1<u_2)$ if $4a(a+3c)>0$.

Let $M(u_\epsilon,y_\epsilon)$ be the coefficient matrix of the system (2.1) at an equilibrium point $E_\epsilon(u_\epsilon,y_\epsilon)$, and $J(u_\epsilon,y_\epsilon)$ be its Jacobian determinant. By the theory of planar dynamical system [6–9], if $J<0$, then the equilibrium point is a saddle point; if $J>0$ and $Tr(M(u_\epsilon,y_\epsilon))=0$, the equilibrium point is a center point; if $J>0$ and $(Tr(M(u_\epsilon,y_\epsilon)))^2-4J>0$, the equilibrium point is a node; if $J=0$ and the index of equilibrium point is zero, then the equilibrium is a cusp; if $J=0$ and the index of equilibrium point is not zero, then the equilibrium point is a high-order equilibrium point.

2.1 Type 1: The case of $a=1$

When $a=1$, system (2.1) becomes

$$\begin{cases} \dfrac{du}{d\zeta}=3u^2y, \\ \dfrac{dy}{d\zeta}=-u^3+u^2+cu-6uy^2+g, \end{cases} \tag{2.3}$$

with the first integral

$$H(u,y)=\frac{3}{2}u^4y^2-\frac{g}{3}u^3-\frac{c}{4}u^4+\frac{1}{6}u^6-\frac{1}{5}u^5=h. \tag{2.4}$$

Case I: $g=0$

In this case, $f(u)=u(-u^2+u+c)$.

(1) when $c<-1/4$, it is easy to know that the function $f(u)=0$ has no real roots, which implies that system (2.3) has no equilibrium points on the u-axis.

(2) When $c=-1/4$, the function $f(u)=0$ has one real root $u=1/2$. So there is a degenerate equilibrium point on the u-axis.

(3) When $c>-1/4$, there exist two equilibrium points $(\frac{1\pm\sqrt{1+4c}}{2},0)$ for system (2.3) on the u-axis.

Case II: $g\neq 0$

In this case, $f'(u)=-3u^2+2u+c$, $\Delta_1=4(1+3c)$.

(1) when $c\leq -1/3$, $\Delta_1\leq 0$, the function $f(u)=0$ has one real root, there exist one equilibrium point on the u-axis.

(2) When $c>-1/3$, the function $f'(u)=0$ has two real roots. Obviously, $u_1=\frac{1-\sqrt{1+3c}}{3}$, $u_2=\frac{1+\sqrt{1+3c}}{3}$, $(u_1<u_2)$,

$$f(u_1)=\frac{-2\sqrt{1+3c}+2+9c-6c\sqrt{1+3c}+27g}{27},$$

$$f(u_2)=\frac{2\sqrt{1+3c}+2+9c+6c\sqrt{1+3c}+27g}{27}.$$

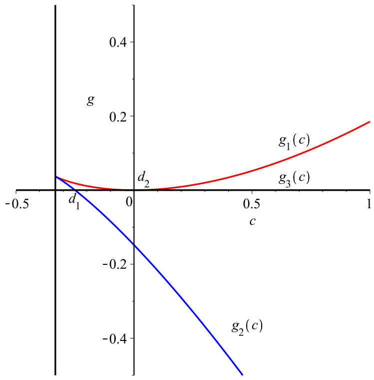

Figure 1: The bifurcation curves in the (c,g)-parameter plane when $a=1$.

Let $g_1(c)=f(u_1)=0$, $g_2(c)=f(u_2)=0$. Under this condition, if $g>g_1(c)$ or $g<g_2(c)$, there exists one equilibrium point on the u-axis; if $g_2(c)<g<g_1(c)$, there exist three equilibrium points on the u-axis; if $g=g_1(c)$ or $g=g_2(c)$, there exist two equilibrium points on the u-axis.

Thus, the following three bifurcation curves of system (2.3) in the (c,g)-parameter plane are obtained

$$g_1(c):g=\frac{(2\sqrt{1+3c}+6c\sqrt{1+3c})-2-9c}{27}, \tag{2.5a}$$

$$g_2(c):g=\frac{-(2\sqrt{1+3c}+6c\sqrt{1+3c})-2-9c}{27}, \tag{2.5b}$$

$$g_3(c):g=0. \tag{2.5c}$$

Moreover, the curves $g_2(c)$ and $g_3(c)$ intersect at the point $d_1=(-1/4,0)$, the curves $g_1(c)$ and $g_3(c)$ are tangent at the point $d_2=(0,0)$. These bifurcation curves divide the (c,g)-parametric plane into fourteen regions (see Fig. 1):

$$A_1:\Big\{(c,g)|c>-\frac{1}{3},\ g_1(c)<g\Big\}\bigcup\Big\{(c,g)|c\le-\frac{1}{3},\ g>0\Big\}\bigcup\Big\{(c,g)|c>-\frac{1}{3},\ 0<g<g_2(c)\Big\},$$

$$A_2:\Big\{(c,g)|c\le-\frac{1}{4},\ g<0\Big\}\bigcup\Big\{(c,g)|-\frac{1}{4}<c,\ g<g_2(c)\Big\},$$

$$A_3:\Big\{(c,g)|-\frac{1}{3}<c<0,\ g=g_1(c)\Big\},\qquad A_4:\{(c,g)|0<c,\ g=g_1(c)\},$$

$$A_5:\Big\{(c,g)|-\frac{1}{3}<c<-\frac{1}{4},\ g=g_2(c)\Big\},\qquad A_6:\Big\{(c,g)|-\frac{1}{4}<c,\ g=g_2(c)\Big\},$$

$$A_7:\Big\{(c,g)|-\frac{1}{3}<c<-\frac{1}{4},\ g_2(c)<g<g_1(c)\Big\}\bigcup\Big\{(c,g)|-\frac{1}{4}\le c<0,\ 0<g<g_1(c)\Big\},$$

$$A_8:\Big\{(c,g)|0<c,\ 0<g<g_1(c)\},\qquad A_9:\Big\{(c,g)|-\frac{1}{4}<c,\ g_2(c)<g<0\Big\},$$

$$A_{10}:\Big\{d_1:(c,g)=\Big(-\frac{1}{4},0\Big)\Big\},\qquad A_{11}:\Big\{(c,g)|-\frac{1}{4}<c<0,\ g=0)\Big\},$$
$$A_{12}:\{d_2:(c,g)=(0,0)\},\qquad A_{13}:\{(c,g)|0<c,\ g=0)\},$$
$$A_{14}:\Big\{(c,g)|c<-\frac{1}{4},\ g=0\Big\}.$$

In this case, the phase portraits of system (2.3) can be shown in Fig. 2.

Remark 2.1. When $(c,g)\in A_{14}$, system (2.3) has no equilibrium point on the u-axis. So we don't give its phase portraits.

2.2 Type 2: The case of $a=-1$

When $a=-1$, system (2.1) becomes

$$\begin{cases} \dfrac{du}{d\zeta}=3u^2y, \\ \dfrac{dy}{d\zeta}=u^3-u^2+cu-6uy^2+g, \end{cases} \tag{2.6}$$

with the first integral

$$H(u,y)=\frac{3}{2}u^4y^2-\frac{g}{3}u^3-\frac{c}{4}u^4-\frac{1}{6}u^6+\frac{1}{5}u^5=h. \tag{2.7}$$

Case I: $g=0$

In this case, $f(u)=u(u^2-u+c)$.

(1) when $c>1/4$, it is easy to know that the function $f(u)=0$ has no real root, this implies that system (2.6) has no equilibrium points on the u-axis.

(2) When $c=1/4$, the function $f(u)=0$ has one real root $u=1/2$. So this is a degenerate equilibrium point on the u-axis.

(3) When $c<1/4$, there exist two equilibrium points $(\frac{1\pm\sqrt{1-4c}}{2},0)$ for system (2.6) on the u-axis.

Case II: $g\neq 0$

In this case, $f'(u)=3u^2-2u+c$, $\Delta_2=4(1-3c)$.

(1) when $c\geq 1/3$, $\Delta_2\leq 0$, the function $f(u)=0$ has one real root, there exist one equilibrium point on the u-axis.

(2) When $c<1/3$, the function $f'(u)=0$ has two real roots. Obviously, $u_3=\frac{1-\sqrt{1-3c}}{3}$, $u_4=\frac{1+\sqrt{1-3c}}{3}$, $(u_3<u_4)$,

$$f(u_3)=\frac{2\sqrt{1-3c}-2+9c-6c\sqrt{1-3c}+27g}{27},$$
$$f(u_4)=\frac{-2\sqrt{1-3c}-2+9c+6c\sqrt{1-3c}+27g}{27}.$$

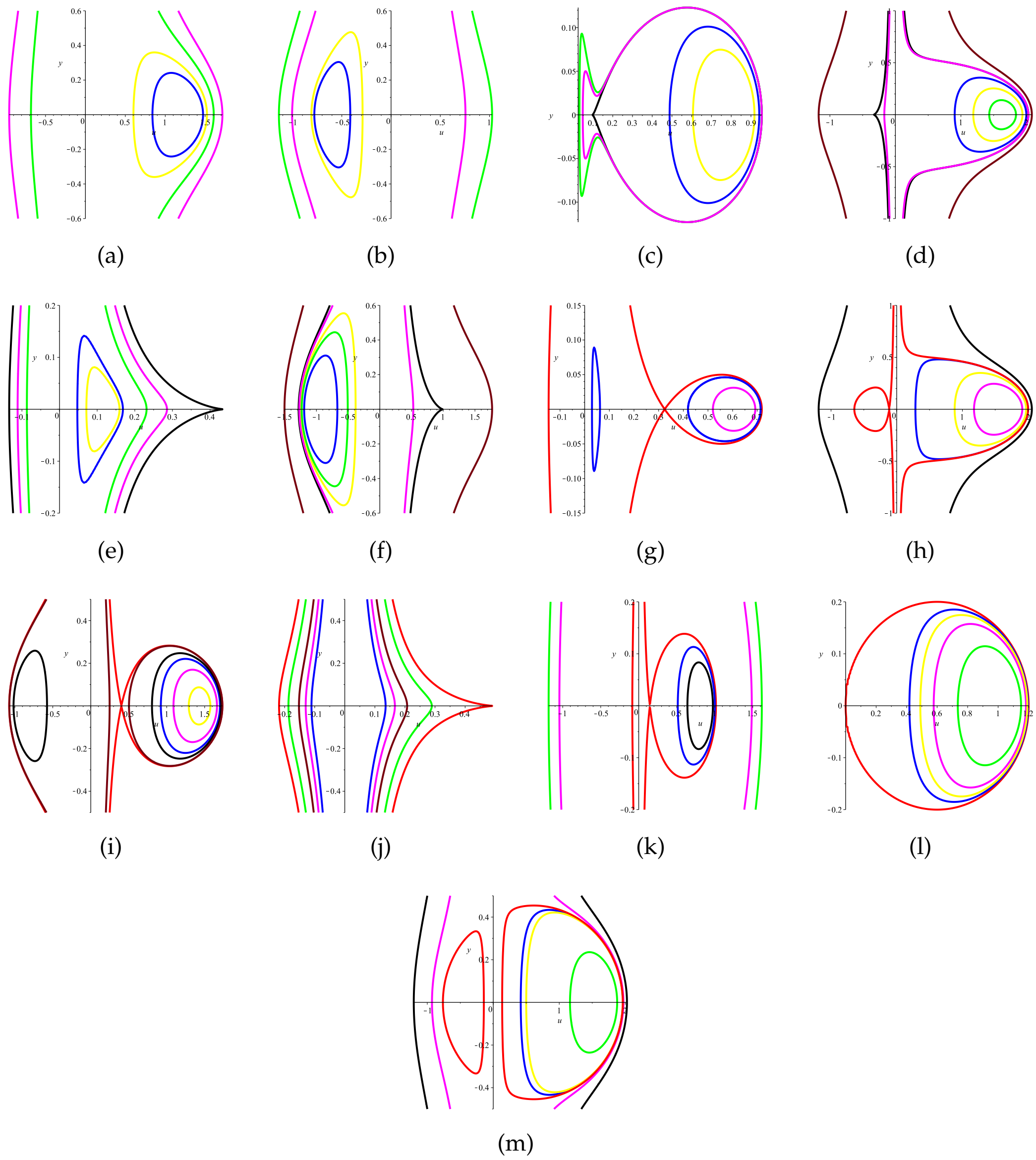

Figure 2: The phase portraits of system (2.3) when $a=1$. (a) $(c,g)\in A_1$. (b) $(c,g)\in A_2$. (c) $(c,g)\in A_3$. (d) $(c,g)\in A_4$. (e) $(c,g)\in A_5$. (f) $(c,g)\in A_6$. (g) $(c,g)\in A_7$. (h) $(c,g)\in A_8$. (i) $(c,g)\in A_9$. (j) $(c,g)\in A_{10}$. (k) $(c,g)\in A_{11}$. (l) $(c,g)\in A_{12}$. (m) $(c,g)\in A_{13}$.

Let $L_1(c)=f(u_3)=0$, $L_2(c)=f(u_4)=0$. Under this condition, if $g<L_1(c)$ or $g>L_2(c)$, there exists one equilibrium point on the u-axis; if $L_1(c)<g<L_2(c)$, there exist three equilibrium points on the u-axis; if $g=L_1(c)$ or $g=L_2(c)$, there exist two equilibrium points on the u-axis.

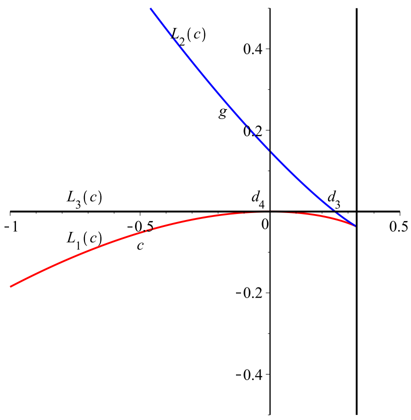

Figure 3: The bifurcation curves in the (c,g)-parameter plane when $a=-1$.

Then, we obtain three bifurcation curves of system (2.6) on the (c,g)-parameter plane

$$L_1(c): g=\frac{-2\sqrt{1-3c}+2-9c+6c\sqrt{1-3c}}{27}, \tag{2.8a}$$

$$L_2(c): g=\frac{2\sqrt{1-3c}+2-9c-6c\sqrt{1-3c}}{27}, \tag{2.8b}$$

$$L_3(c): g=0. \tag{2.8c}$$

In addition, the curves $L_2(c)$ and $L_3(c)$ intersect at the point $d_3=(1/4,0)$, the curves $L_1(c)$ and $L_3(c)$ are tangent at the point $d_4=(0,0)$. These bifurcation curves divide the (c,g)-parametric plane into fourteen regions (see Fig. 3):

$B_1:\left\{(c,g)|g<L_1(c),\, c<\frac{1}{3}\right\}\bigcup\left\{(c,g)|c\geq\frac{1}{3},\, g<0\right\}\bigcup\left\{(c,g)|0<c<\frac{1}{3},\, L_2(c)<g<0\right\}$,

$B_2:\left\{(c,g)|\frac{1}{4}\leq c,\, g<0\right\}\bigcup\left\{(c,g)|c<\frac{1}{4},\, L_2(c)<g\right\}$,

$B_3:\left\{(c,g)|0<c<\frac{1}{3},\, g=L_1(c)\right\}$, $\quad B_4:\{(c,g)|c<0,\, g=L_1(c)\}$,

$B_5:\left\{(c,g)|\frac{1}{4}<c<\frac{1}{3},\, g=L_2(c)\right\}$, $\quad B_6:\left\{(c,g)|c<\frac{1}{4},\, g=L_2(c)\right\}$,

$B_7:\left\{(c,g)|0<c\leq\frac{1}{4},\, L_1(c)<g<0\right\}\bigcup\left\{(c,g)|\frac{1}{4}<c<\frac{1}{3},\, L_1(c)<g<L_2(c)\right\}$,

$B_8:\left\{(c,g)|c<\frac{1}{4},\, 0<g<L_2(c)\right\}$, $\quad B_9:\{(c,g)|c<0,\, L_1(c)<g<0\}$,

$B_{10}:\left\{d_3:(c,g)=\left(\frac{1}{4},0\right)\right\}$, $\quad B_{11}:\left\{(c,g)|0<0<\frac{1}{4},\, g=0\right\}$,

$B_{12}:\{d_4:(c,g)=(0,0)\}$, $\quad B_{13}:\{(c,g)|c<0,\, g=0)\}$,

$B_{14}:\left\{(c,g)|c>\frac{1}{4},\, g=0\right\}$.

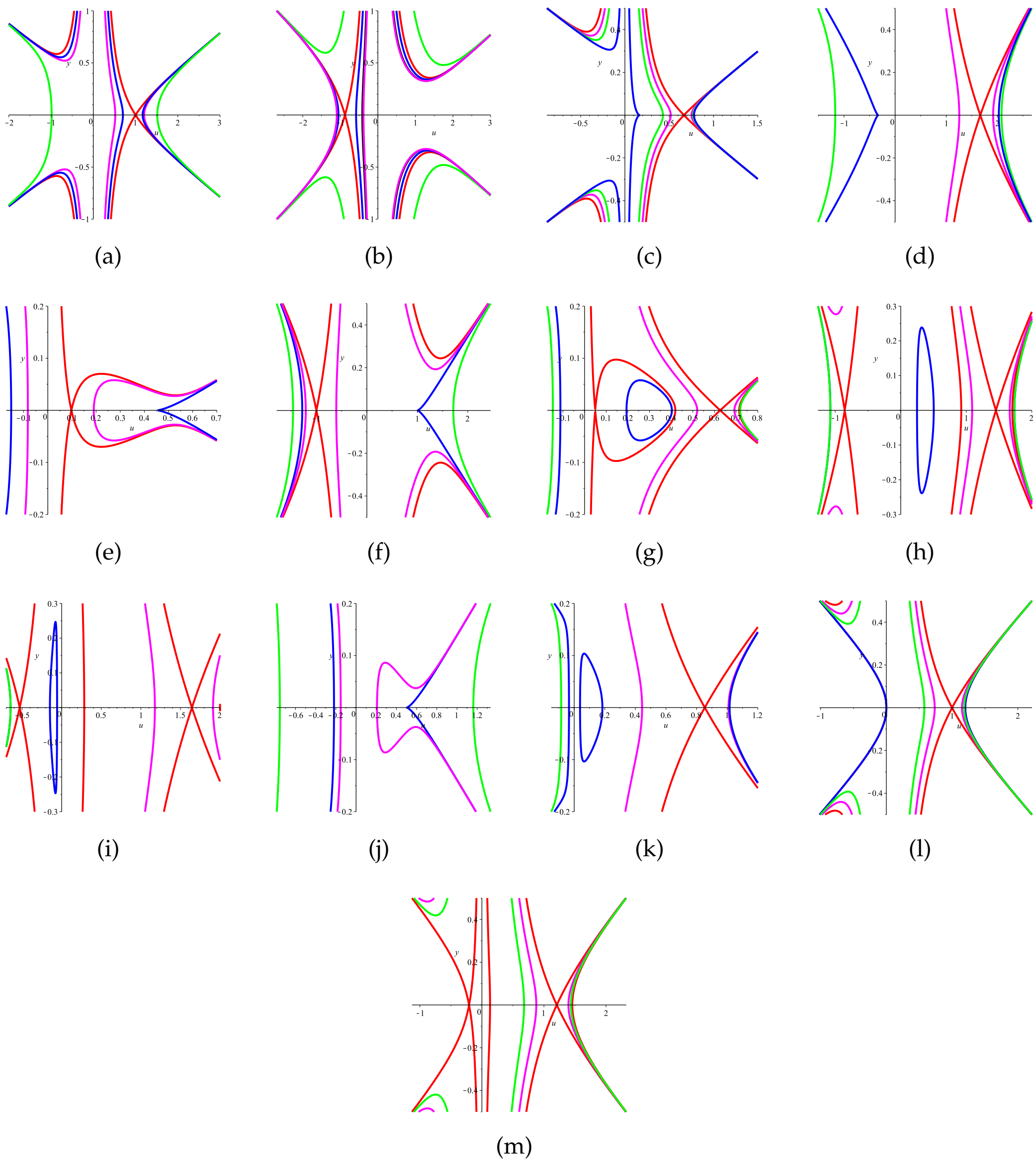

Figure 4: The phase portraits of system (2.6) when $a=-1$. (a) $(c,g)\in B_1$. (b) $(c,g)\in B_2$. (c) $(c,g)\in B_3$. (d) $(c,g)\in B_4$. (e) $(c,g)\in B_5$. (f) $(c,g)\in B_6$. (g) $(c,g)\in B_7$. (h) $(c,g)\in B_8$. (i) $(c,g)\in B_9$. (j) $(c,g)\in B_{10}$. (k) $(c,g)\in B_{11}$. (l) $(c,g)\in B_{12}$. (m) $(c,g)\in B_{13}$.

In this case, the phase portraits of system (2.6) can be shown in Fig. 4.

Remark 2.2. When $(c,g)\in B_{14}$, system (2.6) has no equilibrium points on the u-axis. So we don't give its phase portraits.

3 Single peak solitary wave and compacton solutions

3.1 Single peak solitary wave solutions of Eq. (1.4)

In this section, we study single peak solitary wave solutions of Eq. (1.4) by using the phase portraits given in the Section 2. Learning from Eq. (1.8), we get

$$(u_\xi)^2 = \frac{2h}{3u^4} + \frac{2g}{9u} + \frac{c}{6} - \frac{au^2}{9} + \frac{2au}{15}. \tag{3.1}$$

To study single peak solitary wave solutions, we impose the boundary condition

$$\lim_{\xi \to \pm\infty} u(\xi) = A, \tag{3.2}$$

where A is a constant. In fact, the constant A is equal to the horizontal coordinate of saddle point. Substituting the boundary condition (3.2) into (3.1), Eq. (3.1) becomes

$$(u_\xi)^2 = \frac{(u-A)^2 V(u)}{90u^4}, \tag{3.3}$$

where

$$\begin{aligned} V(u) = & -10au^4 + (-20aA + 12a)u^3 + (15c - 30aA^2 + 24aA)u^2 \\ & + (-20aA^3 + 16aA^2 + 10Ac)u + (-10aA^4 + 8aA^3 + 5A^2c). \end{aligned}$$

The fact that both sides of Eq. (3.3) are nonnegative implies $V(u) \geq 0$. If $3aA^2 - 2aA - c \leq 0$, then Eq. (3.3) reduces to

$$(u_\xi)^2 = \frac{(u-A)^2 (B_1 - u)(u - B_2)((u-k_1)^2 + k_2^2)}{90u^4}, \tag{3.4}$$

where B_1, B_2, k_1, k_2 are real constants and $B_1 > B_2$.

Definition 3.1. A function $u(\xi)$ is said to be a single peak soliton solution of the Eq. (1.4) if $u(\xi)$ satisfies the following conditions:

(C1) $u(\xi)$ is continuous on R and has a unique peak point ξ_0, where $u(\xi)$ attains its global maximum or minimum value;

(C2) $u(\xi) \in C^3(R - \{\xi_0\})$ satisfies Eq. (1.4) on $R - \{\xi_0\}$;

(C3) $\lim_{\xi \to \pm\infty} u(\xi) = A$.

Definition 3.2. A wave function $u(\xi)$ is called peakon if $u(\xi)$ is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} u'(\xi) = -\lim_{\xi \downarrow \xi_0} u'(\xi) = a$, $a \neq 0$, $a \neq \pm\infty$.

Definition 3.3. A wave function $u(\xi)$ is called cuspon if $u(\xi)$ is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} u'(\xi) = -\lim_{\xi \downarrow \xi_0} = +\infty$ (or $-\infty$).

Without loss of generality, we assume $\xi_0=0$.

Theorem 3.1. *Assume that that $u(\xi)$ is a single peak solitary wave solution for the Eq.* (1.4) *at the peak point $\xi_0=0$, Then we have $u(0)=0$ or $u(0)=B_1$ or $u(0)=B_2$.*

Proof. If $u(0)\neq 0$, then $u(\xi)\neq 0$ for any $\xi\in R$ since $u(\xi)\in C^3(R-\{0\})$. Differentiating both sides of Eq. (3.3) yields $u\in C^\infty(R)$.

If $u(0)\neq 0$, by Eq. (3.3) we see $u'(0)$ exists. According to the definition of peak point, we have $u'(0)=0$. Thus we obtain $u(0)=B_1$ or $u(0)=B_2$ from Eq. (3.4), since $u(0)=A$ contradicts the fact that 0 is the unique peak point. $\square$

By virtue of the above theorem, all single peak soliton solutions for the Eq. (1.4) must satisfy the following initial and boundary values problem

$$\begin{cases} (u_\xi)^2=\dfrac{(u-A)^2V(u)}{90u^4}, \\ u(0)\in\{0,B_1,B_2\}, \\ \lim_{\xi\to\pm\infty}u(\xi)=A. \end{cases} \tag{3.5}$$

Below, we will present some implicit formulas for the single peak solitary wave solutions for some specific cases.

3.1.1 Type 1: Smooth solitary wave solutions of system (1.4)

From (3.4), we have

$$(u_\xi)^2=\frac{(u-A)^2(B_1-u)(u-B_2)(u-d)(u-\overline{d})}{90u^4}, \tag{3.6}$$

where A is real double root, B_1, B_2 are reel roots, d, $\overline{d}$ are complex roots.

Suppose $a=1$. There exist smooth solitary wave solutions of system (1.4), which corresponds to the homoclinic orbits defined by $H(u,y)=h_1=H(A,0)$ in the Fig. 2(h). By the standard phase portrait analysis, we have $B_2<A<0<B_1$. On the interval $[B_2,A]$, we have the following smooth solitary wave solutions. From (3.6), we have

$$u_\xi=\frac{(u-A)\sqrt{(B_1-u)(u-B_2)(u-d)(u-\overline{d})}}{\sqrt{90}u^2}\text{sign}(\xi). \tag{3.7}$$

Further,

$$\frac{|\xi|}{\sqrt{90}}=\int_{B_2}^{u}\frac{u^2}{(u-A)\sqrt{(B_1-u)(u-B_2)(u-d)(u-\overline{d})}}du. \tag{3.8}$$

Eq. (3.8) can be reduced to

$$\frac{|\xi|}{\sqrt{90}}=I_1+AI_2+A^2I_3, \tag{3.9}$$

where

$$I_1=\int_{B_2}^{u}\frac{u}{\sqrt{(B_1-u)(u-B_2)(u-d)(u-\overline{d})}}du,$$
$$I_2=\int_{B_2}^{u}\frac{1}{\sqrt{(B_1-u)(u-B_2)(u-d)(u-\overline{d})}}du,$$
$$I_3=\int_{B_2}^{u}\frac{1}{(u-A)\sqrt{(B_1-u)(u-B_2)(u-d)(u-\overline{d})}}du.$$

We obtain

$$I_1=\frac{g(B_1N+B_2M)}{M-N}F_1,\quad I_2=gcn^{-1}(\cos\varphi,\kappa),\quad I_3=\frac{M+N}{M(B_2-A)-N(B_1-A)}F_2,$$

where

$$b_1=\frac{d+\overline{d}}{2},\qquad a_1^2=-\frac{(d-\overline{d})^2}{4},\qquad M^2=(B_1-b_1)^2+a_1^2,$$
$$N^2=(B_2-b_1)^2+a_1^2,\qquad g=\frac{1}{\sqrt{MN}},\qquad \kappa^2=\frac{(B_1-B_2)^2-(M-N)^2}{4MN},$$
$$\kappa'=\sqrt{1-\kappa^2},\qquad \cos\varphi=cnu_1=\frac{(B_1-u)N-(u-B_2)M}{(B_1-u)N+(u_1-B_2)M},\qquad sdu=\frac{snu}{dnu},$$
$$\alpha_1=(B_2M-B_1N)/(B_1N+B_2M),\qquad \alpha_2=\beta_1=(M-N)/(M+N),$$
$$\beta_2=(B_2M-B_1N+AN-AM)/(B_1N+B_2M-AM-AN),$$
$$F_i=\alpha_iu_1+\frac{\beta_i-\alpha_i}{1-\beta_i^2}\Big[\Pi\Big(u_1,\frac{\beta_i^2}{\beta_i^2-1}\Big)-\beta_if_i\Big],$$
$$f_i=\sqrt{\frac{1-\beta_i^2}{k^2+k'^2\beta_i^2}}\arctan\Big(\sqrt{\frac{k^2+k'^2\beta_i^2}{1-\beta_i^2}}sdu_1\Big),$$

and $i=1,2$. $cn(u,k)$, $sn(u,k)$, $dn(u,k)$ are the Jacobian elliptic function, while $\Pi(\cdots)$ is the elliptic integral of the third kind.

Therefore, we have the parametric representation of solitary wave solution of (1.4) as Eq. (3.9). And the profile of smooth solitary wave is shown in Fig. 5.

3.2 Compacton solution of system (1.4)

When $a=1$, $A=c=0$, by the standard phase portrait analysis (see Fig. 2(l)), we have $B_2=0$, $B_1=6/5$. Eq. (3.3) becomes

$$(u_\xi)^2=\frac{u(6-5u)}{45}. \tag{3.10}$$

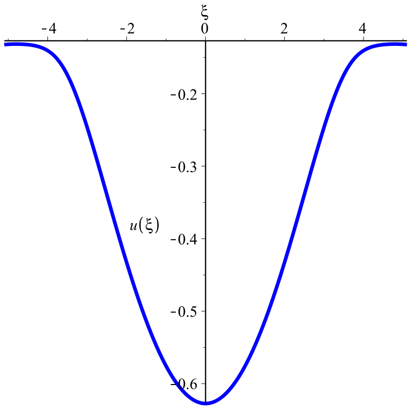

Figure 5: The profile of smooth solitary wave of $u(\xi)$ of Eq. (1.4).

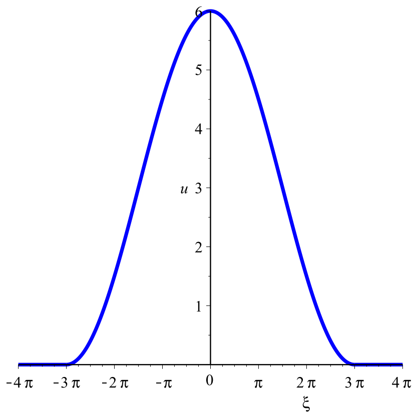

Figure 6: The profile of compacton of $u(\xi)$ of Eq. (1.4) when $a=1$, $A=c=0$.

From Eq. (3.10), we have

$$u_\xi=\frac{\sqrt{u(6-5u)}}{\sqrt{45}}\mathrm{sign}(\xi). \tag{3.11}$$

Integrating both sides of Eq. (3.11) on the interval $[0,6/5]$ leads to a compacton solution with compact support

$$u_2(\xi)=\begin{cases}\frac{1}{5}\Big(3+3\cos\Big(\frac{|\xi|}{3}\Big)\Big), & |\xi|\le 3\pi,\\ 0, & \text{otherwise},\end{cases} \tag{3.12}$$

with the properties

$$u_2(0)=0,\quad \lim_{\xi\to\pm\infty}u_2(\xi)=A=0,\quad u_2'(0)=0.$$

The profile of the compacton solution is shown in Fig. 6.

4 Conclusions

In this paper, we study the integrable nonlinear wave Eq. (1.4). By using the method of dynamical system, we have analyzed the numbers and relative position of the equilibrium points. Furthermore, we obtain the parametric representations of single peak solitary wave and compacton solution for the Eq. (1.4). Asymptotic analysis and numerical simulations are provided for smooth solitary wave and compacton solution of the equation.

Acknowledgments

This work are supported by the National Natural Science Foundation of China (No. 11361017), Science Foundation of the Education Office of Guangxi Province (No. KY2015ZD043, Guangxi Natural Science Foundation (No. 2014GXNSFBA118007, No. 2014GXNSFBA118017 and No. 2015GXNSFGA139004), Program for Innovative Research Team of Guilin University of Electronic Technology, Project of Outstanding Young Teachers' Training in Higher Education Institutions of Guangxi and Innovation Project of GUET Graduate Education (Nos. YJCXS201557).

References

[1] A. WAZWAZ, *New soliton and kink solutions for the Gardner equation*, Commum. Nonlinear Sci. Numer. Simulat., 12 (2007), pp. 1395–1404.

[2] C. M. KHALIQUE AND A. BISWAS, *A Lie symmetry approach to nonlinear Schrödinger's equation with non-Kerr law nonlinearity*, Commum. Nonlinear Sci. Numer. Simulat., 14 (2009), pp. 4033–4040.

[3] A. Y. CHEN, J. B. LI AND W. T. HUANG, *The monotonicity and critical periods of periodic waves of the ϕ^6 field model*, Nonlinear Dyn., 63 (2011), pp. 205–215.

[4] J. B. ZHOU, L. X. TIAN AND X. H. FAN, *New exact travelling wave solutions for the equation with Osmosis dispersion*, Appl. Math. Comput., 217 (2010), pp. 1355–1366.

[5] W. G. RUI, *Different kinds of exact solutions with two-loop character of the two-component short pulse equations of the first kind*, Commun. Nonlinear Sci. Numer. Simulat., 18 (2013), pp. 2667–2678.

[6] A. Y. CHEN AND J. B. LI, *Single peak solitary wave solutions for the osmosis $K(2,2)$ equation under inhomogeneous houndary condition*, J. Math. Anal. Appl., 369 (2010), pp.758–766.

[7] J. B. LI AND H. H. DAI, Singular Nonlinear Traveling Wave Equations: Bifurcations and Exact Solutions, Science Press, Beijing, 2007.

[8] J. B. LI AND G. R. CHEN, *On a class of singular nonlinear traveling wave equations*, Int. J. Bifurcation Chaos., 17 (2007), pp. 4049–4065.

[9] J. B. LI, Y. ZHANG AND X. H. ZHAO, *On a class of singular nonlinear traveling wave equations, (II): an example of GCKdv equations*, Int. J. Bifurcation Chaos., 19 (2009), pp. 1995–2007.

[10] R. CAMASSA AND D. HOLM, *An integrable shallow water equation with peaked solution*, Phys. Rev. Lett., 71 (1993), pp. 1161–1164.

[11] J. LENELLS, *Traveling wave solutions of the Camassa-Holm equation*, J. Diff. Equ., 217 (2005), pp. 393–430.

[12] A. Y. CHEN, S. Q. WEN, S. Q. TANG, W. T. HUANG AND Z. J. QIAO, *Effects of quadratic singular curves in integrable equations*, Stud. Appl. Math., 134 (2015), pp. 24–61.

[13] Z. J. QIAO AND G. P. ZHANG, *On peaked and smooth solitons for the Camassa-Holm equation*, Europhy. Lett., 73 (2006), pp. 657–663.

[14] W. J. ZHU, A. Y. CHEN AND Q. H. LIU, *Periodic wave, solitary wave and compacton solutions of a nonlinear wave equation with degenerate dispersion*, Commun. Theor. Phys., 63 (2015), pp. 57–62.

[15] C. H. LI, S. Q. WEN AND A. Y. CHEN, *Single peak solitary wave and compacton solutions of the generalized two-component Hunter-Saxton system*, Nolinear Dyn., 79 (2015), pp. 1575–1585.

[16] L. ZHANG, L. CHEN AND X. HOU, *The effects of horizontal singular straight line in a generalized nonlinear Klein-Gordon model equation*, Nonlinear Dyn., 72 (2013), pp. 789–801.

[17] Y. WANG AND Q. BI, *Different wave solutions associated with singular lines on phase plane*, Nonlinear Dyn., 69 (2012), pp. 1705–1731.

[18] P. ROSENAU, *On nonanalytic solitary waves formed by a nonlinear dispersion*, Phys. Lett. A, 230 (1997), pp. 305–318.

[19] P. ROSENAU, *Nonlinear dispersion and compact structures*, Phys. Rev. Lett., 73 (1993), pp. 1737–1741.

[20] P. ROSENAU AND J. M. HYMAN, *Staley, multidimensional compacton*, Phys. Rev. Lett., 98 (2007), pp. 024101.

[21] P. ROSENAU, *Compactification of patterns by a singular convection or stress*, Phys. Rev. Lett., 99 (2007), pp. 234102.

[22] P. ROSENAU AND E. KASHDAN, *Emergence of compact structures in a Klein-Gordon model*, Phys. Rev. Lett., 104 (2010), pp. 034101.

[23] F. COOPER, H. SHEPARD AND P. SODANO, *Solitary wave in a class of generalized Korteweg-de Vries equations*, Phys. Rev. E, 48 (1993), pp. 4027–4032.

[24] J. B. ZHOU AND L. X. TIAN, *Soliton solution of the osmosis $K(2,2)$ equation*, Phys. Lett. A, 372 (2008), pp. 6232–6234.

[25] A. Y. CHEN, W. J. ZHU, Z. J. QIAO AND W. T. HUANG, *Algebraic traveling wave solutions of a non-local hydrodynamic-type model*, Math. Phys. Anal. Geom., 17 (2014), pp. 465–482.

[26] A. Y. CHEN, W. T. HUANG AND Y. A. XIE, *Nilpotent singular points and compactons*, Appl. Math. Comput., 236 (2014), pp. 300–310.

[27] P. ROSENAU AND J. M. HYMAN, *Compactons: solitons with finite wavelengths*, Phys. Rev. Lett., 70 (1993), pp. 564–567.

[28] C. H. XU AND L. X. TIAN, *The bifurcation and peakon for $K(2,2)$ equation with osmosis dispersion*, Chaos Solitons Fract., 40 (2009), pp. 893–901.

[29] P. ROSENAU AND A. ORON, *On compactons induced by a non-convex convection*, Nonlinear Sci. Numer. Simulat., 19 (2014), pp. 1329–1337.