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Superconvergence of Finite Element Methods for Optimal Control Problems Governed by Parabolic Equations with Time-Dependent Coefficients

Yuelong Tang[1] and Yanping Chen[2,*]

[1] *Department of Mathematics and Computational Science, Hunan University of Science and Engineering, Yongzhou 425100, Hunan, China.*
[2] *School of Mathematical Sciences, South China Normal University, Guangzhou 510631, Guangdong, China.*

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Abstract. In this article, a fully discrete finite element approximation is investigated for constrained parabolic optimal control problems with time-dependent coefficients. The spatial discretisation invokes finite elements, and the time discretisation a nonstandard backward Euler method. On introducing some appropriate intermediate variables and noting properties of the L^2 projection and the elliptic projection, we derive the superconvergence for the control, the state and the adjoint state. Finally, we discuss some numerical experiments that illustrate our theoretical results.

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1. Introduction

Optimal control problems arise extensively in many social, economic, scientific and engineering applications. Nowadays, finite element (FE) methods seem to be the most widely used numerical approach for solving optimal control problems — cf. Refs. [1, 6, 8, 11, 20, 25, 28, 29] for systematic discussions of such methods for PDE and optimal control problems.

In the FE approximation for elliptic optimal control problems, *a priori* error estimates have been established [15], *a posteriori* error estimates of both recovery and residual type derived [13, 18], and adaptive FE approximations for optimal control problems investigated [12]. Recently, *a posteriori* and *a priori* error estimates in mixed FE methods for elliptic optimal control problems have been obtained in Refs. [5] and [30], respectively;

*Corresponding author. *Email address:* `yanpingchen@scnu.edu.cn` (Y. Chen)

and a variational discretisation approximation for optimal control problems with control constraints has also been considered [9,10].

Parabolic optimal control problems are frequently met in mathematical models describing petroleum reservoir simulation, environmental modelling, groundwater contaminant transport, and many other applications that can be difficult to handle. *A priori* and *a posteriori* error estimates of FE methods for such optimal control problems have also been established in Refs. [16] and [19,31], respectively. Relevant *a priori* estimates for space-time FE discretisation have also recently been obtained [22,23], and a characteristic FE approximation for optimal control problems governed by transient advection-diffusion equations has been investigated [7].

There has been extensive research on the superconvergence of FE methods for elliptic optimal control problems. The superconvergence of linear, semilinear and bilinear elliptic optimal control problems was established in Refs. [24], [3] and [32], respectively; and for mixed FE approximation of Stokes optimal control problems in Ref. [17]. Some superconvergence results of mixed FE approximation for elliptic optimal control problems have also been obtained [2,4,21,33]. Recently, we have derived the superconvergence of fully discrete FE approximation for linear and semilinear parabolic control problems in Refs. [27] and [26], respectively. In this article, we investigate the superconvergence of fully discrete FE methods for an optimal control problem governed by parabolic equations with time-dependent coefficients and control constraints, and then undertake some numerical experiments to confirm our theoretical results.

We consider the following parabolic optimal control problem:

$$\begin{cases} \min\limits_{u\in K} \dfrac{1}{2}\displaystyle\int_0^T \left(\int_\Omega (y-y_d)^2 + \int_\Omega u^2 \right) dt\,, \\ y_t(x,t) - \mathrm{div}(A(x,t)\nabla y(x,t)) = f(x,t) + u(x,t)\,, \quad x\in\Omega,\ t\in J\,, \\ y(x,t) = 0\,, \quad x\in\partial\Omega,\ t\in J\,, \\ y(x,0) = y_0(x)\,, \qquad x\in\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, $J=[0,T]$ with $T>0$, the coefficient $A(x,t) = (a_{ij}(x,t))_{2\times 2} \in \left(W^{1,\infty}(J;W^{1,\infty}(\bar{\Omega}))\right)^{2\times 2}$ such that for any $\xi\in\mathbb{R}^2$ and $t\in J$, $(A(x,t)\xi)\cdot\xi \geq c\mid\xi\mid^2$ with $c>0$. Let $f(x,t)\in C(J;L^2(\Omega))$ and $y_0(x)\in H_0^1(\Omega)$. Further more, we assume that K is a nonempty closed convex set in $L^2(J;L^2(\Omega))$ defined by

$$K = \left\{ v(x,t)\in L^2(J;L^2(\Omega)) : a \leq v(x,t) \leq b, \quad a.e.\ \ (x,t)\in\Omega\times J \right\},$$

where a and b are constants.

Here we also adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$, set $H_0^1(\Omega) \equiv \left\{v\in H^1(\Omega) : v|_{\partial\Omega}=0\right\}$ and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$, and denote by $L^s(J;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))} = (\int_0^T \|v\|^s_{W^{m,q}(\Omega)} dt)^{\frac{1}{s}}$ for $s\in[1,\infty)$ and the standard modification for $s=\infty$. Similarly, one can define the space

$H^l(J;W^{m,q}(\Omega))$. The details can be found in [16]. In addition, c or C denotes a generic positive constant.

In Section 2, we define a fully discrete FE approximation for the model problem, and then give some useful intermediate error estimates in Section 3. In Section 4, we derive the superconvergence properties for the control, the state and the adjoint state. In Section 5, we give some applications of the results in Section 4. We then discuss numerical experiments that illustrate our theoretical results in Section 6, and draw our conclusion in Section 7.

2. Fully Discrete FE Approximation for the Optimal Control Problem

A fully discrete FE approximation for the above model problem is now considered. For ease of exposition, we denote $L^p(J;W^{m,q}(\Omega))$ by $L^p(W^{m,q})$, and $V = L^2(W)$ with $W = H_0^1(\Omega)$ and $X = L^2(U)$ where $U = L^2(\Omega)$. Moreover, we denote $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|_m$ and $\|\cdot\|$, respectively.

Let

$$a(v,w) = \int_\Omega (A(x,t)\nabla v)\cdot\nabla w\,, \quad \forall\, v\,,\, w \in W\,,$$

$$(f_1,f_2) = \int_\Omega f_1\cdot f_2\,, \quad \forall\, f_1, f_2 \in U\,.$$

It follows from the assumptions on $A(x,t)$ that

$$a(v,v) \geq c\|v\|_1^2, \quad |a(v,w)| \leq C\|v\|_1\|w\|_1\,, \quad \forall\, v,w \in W\,.$$

Thus a possible weak formula for the model problem (1.1) reads:

$$\begin{cases} \min\limits_{u\in K} \dfrac{1}{2}\displaystyle\int_0^T \left(\int_\Omega (y-y_d)^2 + \int_\Omega u^2\right) dt\,, \\ (y_t,w)+a(y,w) = (f+u,w)\,, \quad \forall\, w\in W,\ t\in J\,, \\ y(x,0) = y_0(x)\,, \quad x\in\Omega\,. \end{cases} \tag{2.1}$$

It is well known that the problem (2.1) has the unique solution $(y,u) \in \left(H^1(L^2)\cap V\right)\times K$ if and only if there is an adjoint state $p \in H^1(L^2)\cap V$ such that the triplet (y,p,u) satisfies the following optimality conditions (e.g. see [15,19]):

$$\begin{aligned} (y_t,w)+a(y,w) &= (f+u,w)\,, \quad \forall\, w\in W,\ t\in J\,, \\ y(x,0) &= y_0(x)\,, \quad x\in\Omega\,, \end{aligned} \tag{2.2}$$

$$\begin{aligned} -(p_t,q)+a(q,p) &= (y-y_d,q)\,, \quad \forall\, q\in W,\ t\in J\,, \\ p(x,T) &= 0\,, \quad x\in\Omega\,, \end{aligned} \tag{2.3}$$

$$(u+p, v-u) \geq 0\,, \quad \forall\, v \in K, t \in J\,. \tag{2.4}$$

We introduce the point-wise projection operator

$$\Pi_{[a,b]}(g(x,t)) = \max(a, \min(b, -g(x,t)))\,, \quad \forall\,(x,t) \in \Omega \times J\,.$$

As in Ref. [10], it is easy to prove that (2.4) is equivalent to

$$u(x,t) = \Pi_{[a,b]}(p(x,t))\,. \tag{2.5}$$

Let $\mathscr{T}^h$ be regular triangulations of Ω such that $\bar{\Omega} = \cup_{\tau \in \mathscr{T}^h} \bar{\tau}$. Let $h = \max_{\tau \in \mathscr{T}^h}\{h_\tau\}$, where h_τ denotes the diameter of the element τ. Furthermore, set

$$U^h = \left\{\, v_h \in L^2(\Omega) : v_h|_\tau \text{ is constant on all } \tau \in \mathscr{T}^h \,\right\},$$
$$W^h = \left\{\, v_h \in C(\bar{\Omega}) : v_h|_\tau \in \mathbb{P}_1,\ \forall\, \tau \in \mathscr{T}^h, w_h|_{\partial\Omega} = 0 \,\right\},$$

where $\mathbb{P}_1$ is the space of polynomials up to order 1 and $K^h = \left\{\, v_h \in U^h : a \leq v_h \leq b \,\right\}$.

We now define the fully discrete FE approximation for the model problem (2.1). Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}^+$ and $t_n = n\Delta t$, $n = 0, 1, \cdots, N$. Set $\varphi^n = \varphi(x, t_n)$ and write

$$d_t \varphi^n = \frac{\varphi^n - \varphi^{n-1}}{\Delta t}\,, n = 1, 2, \cdots, N\,.$$

For $1 \leq p < \infty$ we define the discrete time-dependent norms

$$|||\varphi|||_{l^p(J;W^{m,q}(\Omega))} := \left(\Delta t \sum_{n=1-l}^{N-l} \left\|\varphi^n\right\|^p_{W^{m,q}(\Omega)}\right)^{\frac{1}{p}}\,,$$

where $l = 0$ for the control u and the state y and $l = 1$ for the adjoint state p, with the standard modification for $p = \infty$. For convenience, we denote $|||\cdot|||_{l^p(J;W^{m,q}(\Omega))}$ by $|||\cdot|||_{l^p(W^{m,q})}$ and let

$$l^p_D(W^{m,q}) := \left\{\, \varphi : |||\varphi|||_{l^p(W^{m,q})} < \infty \,\right\}, \qquad 1 \leq p \leq \infty\,.$$

Then a possible fully discrete FE approximation of (2.1) is:

$$\begin{cases} \min\limits_{u_h^n \in K^h} \dfrac{1}{2} \displaystyle\sum_{n=1}^{N} \Delta t \left(\int_\Omega \left(y_h^n - y_d^n\right)^2 + \int_\Omega \left(u_h^n\right)^2 \right)\,, \\ \left(d_t y_h^n, w_h\right) + a\left(y_h^n, w_h\right) = \left(f^n + u_h^n, w_h\right)\,, \quad \forall\, w_h \in W^h,\ n = 1, 2, \cdots, N\,, \\ y_h^0(x) = y_0^h(x)\,, \quad x \in \Omega\,, \end{cases} \tag{2.6}$$

where $y_0^h(x)$ is an appropriate approximation of $y_0(x)$.

The control problem (2.6) has the unique solution $\left(y_h^n, u_h^n\right) \in W^h \times K^h$, $n = 1, 2, \cdots, N$ if and only if there is an adjoint state $p_h^{n-1} \in W^h$, $n = 1, 2, \cdots, N$ such that the triplet

$(y_h^n, p_h^{n-1}, u_h^n) \in W^h \times W^h \times K^h$, $n = 1, 2, \cdots, N$, satisfies the following optimality conditions (e.g. see [19]):

$$\begin{aligned} \left(d_t y_h^n, w_h\right) + a\left(y_h^n, w_h\right) &= \left(f^n + u_h^n, w_h\right), \quad \forall w_h \in W^h,\ n = 1, 2, \cdots, N, \\ y_h^0(x) &= y_0^h(x), \quad x \in \Omega, \end{aligned} \tag{2.7}$$

$$\begin{aligned} -\left(d_t p_h^n, q_h\right) + a\left(q_h, p_h^{n-1}\right) &= \left(y_h^n - y_d^n, q_h\right), \quad \forall q_h \in W^h,\ n = N, \cdots, 2, 1, \\ p_h^N(x) &= 0, \quad x \in \Omega, \end{aligned} \tag{2.8}$$

$$\left(u_h^n + p_h^{n-1}, v - u_h^n\right) \geq 0, \quad \forall v \in K^h,\ n = 1, 2, \cdots, N. \tag{2.9}$$

As in Ref. [20], it is easy to prove that (2.9) is equivalent to

$$u_h^n = \Pi_{[a,b]}\left(\pi^c\left(p_h^{n-1}\right)\right), \quad n = 1, 2, \cdots, N, \tag{2.10}$$

where π^c denotes the element integral average operator (e.g. see [13]).

3. Error Estimates of Intermediate Variables

We now give some error estimates of intermediate variables. For any $v \in K$, let $(y(v), p(v)) \in (H^1(L^2) \cap L^2(H^1)) \times (H^1(L^2) \cap L^2(H^1))$ be the solution of

$$\begin{aligned} \left(y_t(v), w\right) + a\left(y(v), w\right) &= \left(f + v, w\right), \quad \forall w \in W,\ t \in J, \\ y(v)(x, 0) &= y_0(x), \quad x \in \Omega, \end{aligned} \tag{3.1}$$

$$\begin{aligned} -\left(p_t(v), q\right) + a\left(q, p(v)\right) &= \left(y(v) - y_d, q\right), \quad \forall q \in W,\ t \in J, \\ p(v)(x, T) &= 0, \quad x \in \Omega. \end{aligned} \tag{3.2}$$

For any $v \in K$, a pair $\left(y_h^n(v), p_h^{n-1}(v)\right) \in W^h \times W^h$, $n = 1, 2, \cdots, N$, satisfies the following system:

$$\begin{aligned} \left(d_t y_h^n(v), w_h\right) + a\left(y_h^n(v), w_h\right) &= \left(f^n + v^n, w_h\right), \quad \forall w_h \in W^h,\ n = 1, 2, \cdots, N, \\ y_h^0(v)(x) &= y_0^h(x), \quad x \in \Omega, \end{aligned} \tag{3.3}$$

$$\begin{aligned} -\left(d_t p_h^n(v), q_h\right) + a\left(q_h, p_h^{n-1}(v)\right) &= \left(y_h^n(v) - y_d^n, q_h\right), \quad \forall q_h \in W^h,\ n = N, \cdots, 2, 1, \\ p_h^N(v)(x) &= 0, \quad x \in \Omega. \end{aligned} \tag{3.4}$$

Consequently, we have $(y, p) = (y(u), p(u))$ and $(y_h, p_h) = (y_h(u_h), p_h(u_h))$.

We define the following standard $L^2(\Omega)$-orthogonal projection $Q_h: U \to U^h$, which for all $\psi \in X$ satisfies

$$(\psi^n - Q_h \psi^n, v_h) = 0, \qquad \forall v_h \in U^h; \tag{3.5}$$

and elliptic projection operators $R_h: W \to W^h$, which for any $\phi \in V$ satisfies

$$a\left(\phi^n - R_h\phi^n, w_h\right) = (A(x,t_n)\nabla(\phi^n - R_h\phi^n), \nabla w_h) = 0\,, \quad \forall\, w_h \in W^h\,. \tag{3.6}$$

We have the following approximation properties:

$$\|\psi^n - Q_h\psi^n\|_{-s} \le Ch^{1+s}|\psi^n|_1\,, \quad \forall\, \psi^n \in H^1(\Omega), s = 0,1\,, \tag{3.7}$$

$$\|\phi^n - R_h\phi^n\| \le Ch^2\|\phi^n\|_2\,, \quad \forall\, \phi^n \in H^2(\Omega)\,. \tag{3.8}$$

Lemma 3.1. *Let* $\left(y_h(u), p_h(u)\right)$ *and* $\left(y_h\left(Q_hu\right), p_h\left(Q_hu\right)\right)$ *be the discrete solutions of* (3.3) *and* (3.4) *with* $v = u$ *and* $v = Q_hu$*, respectively. Suppose that* $u \in l_D^2(H^1)$*. Then*

$$|||y_h(Q_hu) - y_h(u)|||_{l^2(H^1)} + |||p_h(Q_hu) - p_h(u)|||_{l^2(H^1)} \le Ch^2|||u|||_{l^2(H^1)}\,. \tag{3.9}$$

Proof. Setting $v = Q_hu$ and $v = u$ in (3.3), we obtain the error equation

$$\begin{aligned}\left(d_ty_h^n(Q_hu) - d_ty_h^n(u), w_h\right) + a\left(y_h^n(Q_hu) - y_h^n(u), w_h\right) &= \left(Q_hu^n - u^n, w_h\right)\,,\\ &\forall\, w_h \in W^h, n = 1,2,\cdots,N\,.\end{aligned} \tag{3.10}$$

We note that

$$\begin{aligned}&\left(d_ty_h^n(Q_hu) - d_ty_h^n(u), y_h^n(Q_hu) - y_h^n(u)\right)\\ \ge&\frac{1}{2\Delta t}\left(\left\|y_h^n(Q_hu) - y_h^n(u)\right\|^2 - \left\|y_h^{n-1}(Q_hu) - y_h^{n-1}(u)\right\|^2\right)\,,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\left(Q_hu^n - u^n, y_h^n(Q_hu) - y_h^n(u)\right) &\le C\left\|Q_hu^n - u^n\right\|_{-1}\left\|y_h^n(Q_hu) - y_h^n(u)\right\|_1\\ &\le Ch^2\|u^n\|_1\left\|y_h^n(Q_hu) - y_h^n(u)\right\|_1\\ &\le C(\delta)h^4\|u^n\|_1^2 + \delta\left\|y_h^n(Q_hu) - y_h^n(u)\right\|_1^2\,.\end{aligned} \tag{3.12}$$

By choosing $w_h = y_h^n(Q_hu) - y_h^n(u)$ and multiplying both sides of Eq. (3.10) by $2\Delta t$, and then summing n from 1 to N, we get

$$\begin{aligned}&\left\|y_h^N(Q_hu) - y_h^N(u)\right\|^2 + c\sum_{n=1}^N\Delta t\left\|y_h^n(Q_hu) - y_h^n(u)\right\|_1^2\\ \le&C(\delta)h^4\sum_{n=1}^N\Delta t\|u^n\|_1^2 + \delta\sum_{n=1}^N\Delta t\left\|y_h^n(Q_hu) - y_h^n(u)\right\|_1^2\,.\end{aligned} \tag{3.13}$$

Hence we have

$$|||y_h(Q_hu) - y_h(u)|||_{l^2(H^1)} \le Ch^2|||u|||_{l^2(H^1)}\,. \tag{3.14}$$

Setting $v = Q_hu$ and $v = u$ in the equation in (3.4), we have the error equation

$$\begin{aligned}&-\left(d_tp_h^n(Q_hu) - d_tp_h^n(u), q_h\right) + a\left(q_h, p_h^{n-1}(Q_hu) - p_h^{n-1}(u)\right)\\ &\quad= \left(y_h^n(Q_hu) - y_h^n(u), q_h\right)\,, \quad \forall\, q_h \in W^h, n = N,\cdots,2,1\,.\end{aligned} \tag{3.15}$$

Similarly, we derive

$$|||p_h(Q_h u) - p_h(u)|||_{l^2(H^1)} \leq C|||y_h(Q_h u) - y_h(u)|||_{l^2(L^2)} \,. \tag{3.16}$$

Consequently, inequality (3.9) follows from the inequalities (3.14) and (3.16). □

Lemma 3.2. *For any $v \in K$, let $(y(v), p(v))$ and $(y_h(v), p_h(v))$ be the solutions of* (3.1) *and* (3.2)*, and* (3.3) *and* (3.4)*, respectively. Assume that $y(v), p(v) \in l_D^2(H^2) \cap H^1(H^2) \cap H^2(L^2)$ and $y_d \in H^1(L^2)$. Then*

$$|||R_h y(v) - y_h(v)|||_{l^2(H^1)} + |||R_h p(v) - p_h(v)|||_{l^2(H^1)} \leq C\left(\Delta t + h^2\right). \tag{3.17}$$

Proof. From (3.1) and (3.3), we obtain

$$\left(y_t^n(v) - d_t y_h^n(v), w_h\right) + a\left(y^n(v) - y_h^n(v), w_h\right) = 0\,,\ \forall\, w_h \in W^h,\ n = 1, 2, \cdots, N\,. \tag{3.18}$$

By using the definition of R_h, we get

$$\begin{aligned} &\left(d_t R_h y^n(v) - d_t y_h^n(v), w_h\right) + a\left(R_h y^n(v) - y_h^n(v), w_h\right) \\ &\quad = \left(d_t R_h y^n(v) - d_t y^n(v) + d_t y^n(v) - y_t^n(v), w_h\right). \end{aligned} \tag{3.19}$$

It is notable that

$$\begin{aligned} &\left(d_t R_h y^n(v) - d_t y^n(v), R_h y^n(v) - y_h^n(v)\right) \\ \leq &\left\|d_t R_h y^n(v) - d_t y^n(v)\right\| \left\|R_h y^n(v) - y_h^n(v)\right\| \\ \leq &Ch^2 \left\|d_t y^n(v)\right\|_2 \left\|R_h y^n(v) - y_h^n(v)\right\| \\ \leq &Ch^2 (\Delta t)^{-1} \int_{t_{n-1}}^{t_n} \left\|y_t(v)\right\|_2 dt \left\|R_h y^n(v) - y_h^n(v)\right\| \\ \leq &Ch^2 (\Delta t)^{-\frac{1}{2}} ||y_t(v)||_{L^2(t_{n-1}, t_n; H^2(\Omega))} \left\|R_h y^n(v) - y_h^n(v)\right\|, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\left(d_t y^n(v) - y_t^n(v), R_h y^n(v) - y_h^n(v)\right) \\ = &(\Delta t)^{-1}\left(y^n(v) - y^{n-1}(v) - \Delta t\, y_t^n(v), R_h y^n(v) - y_h^n(v)\right) \\ \leq &(\Delta t)^{-1}\left\|y^n(v) - y^{n-1}(v) - \Delta t\, y_t^n(v)\right\| \left\|R_h y^n(v) - y_h^n(v)\right\| \\ = &(\Delta t)^{-1}\left\|\int_{t_{n-1}}^{t_n} (t_{n-1} - s)(y_{tt}(v))(s) ds\right\| \left\|R_h y^n(v) - y_h^n(v)\right\| \\ \leq &C(\Delta t)^{\frac{1}{2}} ||y_{tt}(v)||_{L^2(t_{n-1}, t_n; L^2(\Omega))} \left\|R_h y^n(v) - y_h^n(v)\right\|. \end{aligned} \tag{3.21}$$

Similar to Lemma 3.1, from (3.19)–(3.21) and Young's inequality, we have

$$\begin{aligned} &\left\|R_h y^N(v) - y_h^N(v)\right\|^2 + c\sum_{n=1}^N \Delta t \left\|R_h y^n(v) - y_h^n(v)\right\|_1^2 \\ \leq &C(\delta)\left(h^4 ||y_t(v)||_{L^2(H^2)}^2 + (\Delta t)^2 ||y_{tt}(v)||_{L^2(L^2)}^2\right) + \delta \sum_{n=1}^N \Delta t \left\|R_h y^n(v) - y_h^n(v)\right\|^2. \end{aligned} \tag{3.22}$$

Consequently, we get

$$|||R_h y(v) - y_h(v)|||_{L^2(H^1)} \le C\left(h^2\|y_t(v)\|_{L^2(H^2)} + \Delta t\|y_{tt}(v)\|_{L^2(L^2)}\right). \tag{3.23}$$

From (3.2) and (3.4), we obtain

$$\begin{aligned}&-\left(p_t^{n-1}(v) - d_t p_h^n(v), q_h\right) + \left(A(x,t_{n-1})\nabla q_h, \nabla p^{n-1}(v)\right) - a\left(q_h, p_h^{n-1}(v)\right)\\ &\quad= \left(y^{n-1}(v) - y_h^n(v) - y_d^{n-1} + y_d^n, q_h\right), \quad \forall q_h \in W^h,\ n = N,\cdots,2,1.\end{aligned} \tag{3.24}$$

From the definition of R_h, we derive

$$\begin{aligned}&-\left(d_t R_h p^n(v) - d_t p_h^n(v), q_h\right) + a\left(q_h, R_h p^{n-1}(v) - p_h^{n-1}(v)\right)\\ &\quad= \left(-d_t R_h p^n(v) + p_t^{n-1}(v) + y^{n-1}(v) - y_h^n(v) + y_d^n - y_d^{n-1}, w_h\right)\\ &\quad+ \left(\left(A(x,t_n) - A(x,t_{n-1})\right)\nabla q_h, \nabla p^{n-1}(v)\right),\end{aligned} \tag{3.25}$$

where $A(x,t) = (a_{ij}(x,t))_{2\times 2} \in \left(W^{1,\infty}(W^{1,\infty})\right)^{2\times 2}$. Similarly, we can prove that

$$\begin{aligned}&|||R_h p(v) - p_h(v)|||^2_{L^2(H^1)}\\ &\quad\le C(\delta)\left(|||R_h y(v) - y_h(v)|||^2_{l^2(H^1)} + h^4|||y(v)\|^2_{l^2(H^2)} + h^4\|p_t(v)\|^2_{L^2(H^2)}\right)\\ &\quad+ C(\delta)(\Delta t)^2\left(|||p(v)|||^2_{l^2(H^1)} + \|p_{tt}(v)\|^2_{L^2(L^2)} + \|y_t(v)\|^2_{L^2(L^2)} + \|(y_d)_t\|^2_{L^2(L^2)}\right).\end{aligned} \tag{3.26}$$

From inequalities (3.23) and (3.26), we get inequality (3.17). □

4. Superconvergence Analysis

We now discuss superconvergence between the FE solution and projections of the exact solution, and begin by deriving the superconvergence of the control variable. Let u be the solution of (2.2)–(2.4). For a fixed t^* ($0 \le t^* \le T$), we divide Ω into the following subsets:

$$\begin{aligned}&\Omega^+ = \{\ \cup\tau\ :\ \tau \subset \Omega,\ a < u(\cdot,t^*) < b\ \},\\ &\Omega^0 = \{\ \cup\tau\ :\ \tau \subset \Omega,\ u(\cdot,t^*)|_\tau = a \ \text{ or }\ u(\cdot,t^*)|_\tau = b\ \},\\ &\Omega^- = \Omega \setminus \left(\Omega^+ \cup \Omega^0\right).\end{aligned}$$

It is easy to see that the above three subsets do not intersect and $\Omega = \bar{\Omega}^+ \cup \bar{\Omega}^0 \cup \bar{\Omega}^-$. We now assume that u and $\mathcal{T}_h$ are regular such that meas $\left(\Omega^-\right) \le Ch$ (e.g. see [23]).

Theorem 4.1. *Let u and u_h be the solutions of* (2.2)–(2.4) *and* (2.7)–(2.9)*, respectively. Assume that all the conditions in* Lemmas 3.1 *and* 3.2 *are valid, and the exact control and adjoint state solution satisfy*

$$u,\ p \in l^2_D(W^{1,\infty}).$$

Then

$$|||Q_h u - u_h|||_{l^2(L^2)} \le C\left(h^{\frac{3}{2}} + \Delta t\right). \tag{4.1}$$

Proof. Setting $v = u_h$ in (2.4) and $v = Q_h u^n$ in (2.9), we have

$$
\begin{aligned}
0 &\leq \left(u^n + p^n, u_h^n - u^n\right) + \left(u_h^n + p_h^{n-1}, Q_h u^n - u_h^n\right) \\
&= \left(u^n + p^n, Q_h u^n - u^n\right) + \left(u_h^n - u^n + p_h^{n-1} - p^n, Q_h u^n - u_h^n\right).
\end{aligned}
\tag{4.2}
$$

From the definition of Q_h and (4.2), we obtain

$$
\begin{aligned}
&|||Q_h u - u_h|||^2_{L^2(L^2)} \\
&= \sum_{n=1}^{N} \Delta t \left(Q_h u^n - u_h^n, Q_h u^n - u_h^n\right) = \sum_{n=1}^{N} \Delta t \left(u^n - u_h^n, Q_h u^n - u_h^n\right) \\
&\leq \sum_{n=1}^{N} \Delta t \left(u^n + p^n, Q_h u^n - u^n\right) + \sum_{n=1}^{N} \Delta t \left(p_h^{n-1}(Q_h u) - p_h^{n-1}(u), Q_h u^n - u_h^n\right) \\
&+ \sum_{n=1}^{N} \Delta t \left(p_h^{n-1}(u_h) - p_h^{n-1}(Q_h u), Q_h u^n - u_h^n\right) + \sum_{n=1}^{N} \Delta t \left(p_h^{n-1}(u) - p^{n-1}(u), Q_h u^n - u_h^n\right) \\
&+ \sum_{n=1}^{N} \Delta t \left(p^{n-1}(u) - p^n(u), Q_h u^n - u_h^n\right) \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 .
\end{aligned}
\tag{4.3}
$$

Note that

$$
\left(u^n + p^n, Q_h u^n - u^n\right) = \int_{\Omega^+} + \int_{\Omega^0} + \int_{\Omega^-} \left(u^n + p^n\right)\left(Q_h u^n - u^n\right) dx ,
\tag{4.4}
$$

and $\left(Q_h u^n - u^n\right)|_{\Omega^0} = 0$. From (2.5), we have $u^n + p^n = 0$ on Ω^+. Then

$$
\begin{aligned}
I_1 =& \Delta t \sum_{n=1}^{N} \int_{\Omega^-} \left(u^n + p^n\right)\left(Q_h u^n - u^n\right) dx \\
=& \Delta t \sum_{n=1}^{N} \int_{\Omega^-} \left(u^n + p^n - \pi^c(u^n + p^n)\right)\left(Q_h u^n - u^n\right) dx \\
\leq& Ch^2 \Delta t \sum_{n=1}^{N} \left\|u^n + p^n\right\|_{1,\Omega^-} \|u^n\|_{1,\Omega^-} \\
\leq& Ch^2 \Delta t \sum_{n=1}^{N} \left\|u^n + p^n\right\|_{W^{1,\infty}(\Omega^-)} \|u^n\|_{W^{1,\infty}(\Omega^-)} \operatorname{meas}\left(\Omega^-\right) \\
\leq& Ch^3 \left(|||u + p|||^2_{l^2(w^{1,\infty})} + |||u|||^2_{l^2(w^{1,\infty})}\right).
\end{aligned}
\tag{4.5}
$$

According to (3.3) and (3.4), we have

$$\begin{aligned} I_2 =& \sum_{n=1}^{N} \Delta t \left(p_h^{n-1}(Q_h u) - p_h^{n-1}(u), Q_h u^n - u_h^n \right) \\ \leq& C(\delta) \sum_{n=1}^{N} \Delta t \left\| p_h^{n-1}(Q_h u) - p_h^{n-1}(u) \right\|^2 + \delta \sum_{n=1}^{N} \Delta t \left\| Q_h u^n - u_h^n \right\|^2 \\ =& C(\delta) |||p_h(Q_h u) - p_h(u)|||_{l^2(L^2)}^2 + \delta |||Q_h u - u_h|||_{l^2(L^2)}^2 \\ \leq& C(\delta) h^4 + \delta |||Q_h u - u_h|||_{l^2(L^2)}^2. \end{aligned} \tag{4.6}$$

From Young's inequality and Lemma 3.1, we derive

$$\begin{aligned} I_3 =& - \sum_{n=1}^{N} \Delta t \left(y_h^n(u_h) - y_h^n(Q_h u), y_h^n(u_h) - y_h^n(Q_h u) \right) \\ =& - |||y_h(u_h) - y_h(Q_h u)|||_{l^2(L^2)}^2 \leq 0. \end{aligned} \tag{4.7}$$

By using (3.7), Young's inequality and Lemma 3.2, we get

$$\begin{aligned} I_4 =& \sum_{n=1}^{N} \Delta t \left(p_h^{n-1}(u) - R_h p^{n-1}(u) + R_h p^{n-1}(u) - p^{n-1}(u), Q_h u^n - u_h^n \right) \\ \leq& \sum_{n=1}^{N} \Delta t \left(\left\| p_h^{n-1}(u) - R_h p^{n-1}(u) \right\| + \left\| R_h p^{n-1}(u) - p^{n-1}(u) \right\| \right) \left\| Q_h u^n - u_h^n \right\| \\ \leq& C(\delta) \sum_{n=1}^{N} \Delta t \left\| p_h^{n-1}(u) - R_h p^{n-1}(u) \right\|^2 + C(\delta) h^4 \sum_{n=1}^{N} \Delta t \| p^{n-1}(u) \|_2^2 \\ &+ \delta \sum_{n=1}^{N} \Delta t \left\| Q_h u^n - u_h^n \right\|^2 \\ \leq& C(\delta) |||p_h(u) - R_h p(u)|||_{l^2(L^2)}^2 + C(\delta) h^4 |||p(u)|||_{l^2(H^2)}^2 + \delta |||Q_h u - u_h|||_{l^2(L^2)}^2 \\ \leq& C(\delta) \left(h^4 + (\Delta t)^2 \right) + \delta |||Q_h u - u_h|||_{l^2(L^2)}^2 . \end{aligned} \tag{4.8}$$

Noting that

$$\begin{aligned} I_5 =& \sum_{n=1}^{N} \Delta t \left(p^{n-1}(u) - p^n(u), Q_h u^n - u_h^n \right) \\ \leq& \sum_{n=1}^{N} \Delta t \int_{t_{n-1}}^{t_n} \left\| p_t(u) \right\| dt \left\| Q_h u^n - u_h^n \right\| \\ \leq& C(\delta) (\Delta t)^2 ||p_t(u)||_{L^2(L^2)}^2 + \delta |||Q_h u - u_h|||_{l^2(L^2)}^2 , \end{aligned} \tag{4.9}$$

inequality (4.1) follows from (4.3)–(4.9). $\square$

Remark 4.1. Because of the lower regularity, the 3/2 order superconvergence for h between the L^2-orthogonal projection and the FE solution for the control variable is optimal.

Theorem 4.2. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions* (2.2)–(2.4) *and* (2.7)–(2.9), *respectively. Assume that all the conditions in* Theorem 4.1 *are valid. Then*

$$|||R_h y - y_h|||_{l^2(H^1)} + |||R_h p - p_h|||_{l^2(H^1)} \le C\left(\Delta t + h^2\right) . \tag{4.10}$$

Proof. From (2.2) and (2.7), we have the error equation

$$\begin{aligned} &\left(y_t^n - d_t y_h^n, w_h\right) + a\left(y^n - y_h^n, w_h\right) = \left(u^n - u_h^n, w_h\right), \\ &\qquad\qquad \forall\, w_h \in W^h,\ n = 1, 2, \cdots, N . \end{aligned} \tag{4.11}$$

From the definition of R_h, we get

$$\begin{aligned} &\left(d_t R_h y^n - d_t y_h^n, w_h\right) + a\left(R_h y^n - y_h^n, w_h\right) \\ =&\left(d_t R_h y^n - d_t y^n + d_t y^n - y_t^n + u^n - u_h^n, w_h\right) , \quad n = 1, 2, \cdots, N . \end{aligned} \tag{4.12}$$

It is notable that

$$\begin{aligned} \left(u^n - u_h^n, R_h y^n - y_h^n\right) \le& C\left\|u^n - u_h^n\right\|_{-1}\left\|R_h y^n - y_h^n\right\|_1 \\ \le& C(\delta)\left\|u^n - u_h^n\right\|_{-1}^2 + \delta\left\|R_h y^n - y_h^n\right\|_1^2 , \end{aligned} \tag{4.13}$$

and the projection operator $\Pi_{[a,b]}$ is Lipschitz continuous with constant 1 and in addition $\pi^c\left(p_h^{n-1}\right) = Q_h\left(p_h^{n-1}\right)$. From (2.5) and (2.10), we have

$$\begin{aligned} \left\|u^n - u_h^n\right\|_{-1} =& \left\|\Pi_{[a,b]} p^n - \Pi_{[a,b]} \pi^c\left(p_h^{n-1}\right)\right\|_{-1} \le \left\|p^n - \pi^c\left(p_h^{n-1}\right)\right\|_{-1} \\ \le& \left\|p^n - p^{n-1}\right\|_{-1} + \left\|p^{n-1} - Q_h\left(p_h^{n-1}\right)\right\|_{-1} \\ \le& \left\|p^n - p^{n-1}\right\| + \left\|p^{n-1} - p_h^{n-1}\right\| + \left\|p_h^{n-1} - Q_h\left(p_h^{n-1}\right)\right\|_{-1} \\ \le& (\Delta t)^{\frac{1}{2}}\left\|p_t\right\|_{L^2(t_{n-1},t_n;L^2(\Omega))} + C\left(h^2 + \Delta t\right) + Ch^2\left|p_h^{n-1}\right|_1 , \end{aligned} \tag{4.14}$$

where we used $\left\|p^{n-1} - p_h^{n-1}\right\| \le C\left(h^2 + \Delta t\right)$ — cf. Theorem 1.5 in Ref. [28]. Similar to Lemma 3.2, from (4.12)–(4.14) we obtain

$$\begin{aligned} &|||R_h y - y_h|||_{l^2(H^1)} \\ \le& C(h^2\left(\left\|y_t\right\|_{L^2(H^2)} + |||p_h|||_{l^2(H^1)}\right) + C\Delta t\left(\left\|y_{tt}\right\|_{L^2(L^2)} + \left\|p_t\right\|_{L^2(L^2)}\right) . \end{aligned} \tag{4.15}$$

From (2.3) and (2.8) we obtain

$$\begin{aligned} &-\left(p_t^{n-1} - d_t p_h^n, q_h\right) + \left(A(x, t_{n-1})\nabla q_h, \nabla p^{n-1}\right) - a\left(q_h, p_h^{n-1}\right) \\ &\quad = \left(y^{n-1} - y_h^n - y_d^{n-1} + y_d^n, q_h\right) , \quad \forall\, q_h \in W^h,\ n = N, \cdots, 2, 1 ; \end{aligned} \tag{4.16}$$

and from the definition of R_h

$$
\begin{aligned}
&-\left(d_tR_hp^n-d_tp_h^n,q_h\right)+a\left(q_h,R_hp^{n-1}-p_h^{n-1}\right)\\
&\quad=\left(\left(A(x,t_n)-A(x,t_{n-1})\right)\nabla q_h,\nabla p^{n-1}\right)\\
&\quad+\left(-d_tR_hp^n+p_t^n+y^{n-1}-y_h^n-y_d^{n-1}+y_d^n,q_h\right)\ ,
\end{aligned}
\tag{4.17}
$$

for any $q_h\in W^h$, $n=N,\cdots,2,1$. Similarly, we can prove that

$$
\begin{aligned}
&|||R_hp-p_h|||^2_{l^2(H^1)}\\
&\le C(\delta)\left(|||R_hy-y_h|||^2_{l^2(H^1)}+h^4|||y|||^2_{l^2(H^2)}+h^4||p_t||^2_{L^2(H^2)}\right)\\
&+C(\delta)(\Delta t)^2\left(|||p|||^2_{l^2(H^1)}+||p_{tt}||^2_{L^2(L^2)}+||y_t||^2_{L^2(L^2)}+||(y_d)_t||^2_{L^2(L^2)}\right).
\end{aligned}
\tag{4.18}
$$

From inequalities (4.15) and (4.18), we then derive inequality (4.10). □

5. Some Applications

We now consider some applications of the results derived in Section 4. Let $\mathscr{T}^H$ be regular triangulations of Ω with $H=2h$, such that $\bar\Omega=\cup_{T\in\mathscr{T}^H}\bar T$. Let $H=\max_{T\in\mathscr{T}^H}\{H_T\}$, where H_T denotes the diameter of the element T. Furthermore, we set

$$
W^H\ =\left\{\ w_h\in C(\bar\Omega)\ :\ w_h|_T\in\mathbb{P}_2,\ \forall\,T\in\mathscr{T}^H,w_h|_{\partial\Omega}=0\ \right\},
$$

where $\mathbb{P}_2$ is the space of polynomials up to order 2.

We first introduce a higher order interpolate operator $I_{2h}^2\ :\ C(\bar\Omega)\to W^H$ and $I_h\ :\ C(\bar\Omega)\to W^h$, the Lagrange interpolate operator (e.g. see [14]). They satisfy the following properties:

$$
\left\|v-I_{2h}^2v\right\|_1\le Ch^2\|v\|_3\,,\quad\forall\,v\in H^3(\Omega)\,,\tag{5.1}
$$

$$
\left\|I_{2h}^2v\right\|_1\le C\|v\|_1\,,\qquad\forall\,v\in W^h\,,\tag{5.2}
$$

$$
I_{2h}^2I_h=I_{2h}^2\,.\tag{5.3}
$$

Theorem 5.1. *Assume that all the conditions in* Theorem 4.2 *are valid, and suppose that* $y,p\in l_D^2(H^3)$. *Then*

$$
|||y-I_{2h}^2y_h|||_{l^2(H^1)}+|||p-I_{2h}^2p_h|||_{l^2(H^1)}\le C\left(h^2+\Delta t\right)\ .\tag{5.4}
$$

Proof. According to Theorem 2.1.1 in Ref. [14],

$$
\|I_hy-R_hy\|_1\le Ch^2\|y\|_3\,.\tag{5.5}
$$

From (5.1)–(5.3) and (5.5), for $n=1,2,\cdots,N$, we obtain

$$
\begin{aligned}
\left\|y^n-I_{2h}^2y_h^n\right\|_1=&\left\|y^n-I_{2h}^2y^n+I_{2h}^2\left(I_hy^n-R_hy^n\right)-I_{2h}^2\left(R_hy^n-y_h^n\right)\right\|_1\\
\le&\left\|y^n-I_{2h}^2y^n\right\|_1+C\left\|I_hy^n-R_hy^n\right\|_1+C\left\|R_hy^n-y_h^n\right\|_1\\
\le&Ch^2\left\|y^n\right\|_3+C\left\|R_hy^n-y_h^n\right\|_1\ ,
\end{aligned}
\tag{5.6}
$$

therefore

$$\sum_{n=1}^{N}\Delta t \left\|y^n - I_{2h}^2 y_h^n\right\|_1^2 \le Ch^4 \sum_{n=1}^{N}\Delta t \left\|y^n\right\|_3^2 + C\sum_{n=1}^{N}\Delta t \left\|R_h y^n - y_h^n\right\|_1^2 . \tag{5.7}$$

From inequality (4.10),

$$|||y - I_{2h}^2 y_h|||_{l^2(H^1)} \le C\left(h^2 + \Delta t\right) , \tag{5.8}$$

and similarly

$$|||p - I_{2h}^2 p_h|||_{l^2(H^1)} \le C\left(h^2 + \Delta t\right) . \tag{5.9}$$

Consequently, inequality (5.4) follows from inequalities (5.8) and (5.9). □

We now introduce a recovery operator G_h for the control. Let $G_h v$ be a continuous piecewise linear function, without zero boundary constraint. Similar to the Z-Z patch recovery in Refs. [34, 35], the values of $G_h v$ on the nodes are defined by least-squares argument on element patches surrounding the nodes. Let z be a node, $\omega_z = \cup_{\tau\in\mathcal{T}^h, z\in\bar{\tau}}\tau$, and v_z the linear function space on ω_z. Set $R_h v(z) = \sigma_z(z)$, where

$$E(\sigma_z) = \min_{\omega\in V_z} E(\omega)$$

and

$$E(\omega) = \sum_{\tau\in\omega_z}\left(\int_\tau \omega - \int_\tau v\right)^2 .$$

When $z \in \partial\Omega$, we should add a few extra neighbor elements to ω_z such that ω_z contains more than three elements. The details can be found in Ref. [13].

Theorem 5.2. *Let u and u_h be the solutions of* (2.2)–(2.4) *and* (2.7)–(2.9)*, respectively. Assume that all the conditions in* Theorem 4.1 *are valid and Ω is convex. Then*

$$|||u - G_h u_h|||_{l^2(L^2)} \le C\left(h^{\frac{3}{2}} + \Delta t\right). \tag{5.10}$$

Proof. It follows from Lemma 4.2 in Ref. [13] that

$$\begin{aligned}\left\|u^n - G_h u_h^n\right\| &\le \left\|u^n - G_h u^n\right\| + \left\|G_h u^n - G_h Q_h u^n\right\| + \left\|G_h Q_h u^n - G_h u_h^n\right\| \\ &\le Ch^{\frac{3}{2}} + \left\|G_h u^n - G_h Q_h u^n\right\| + \left\|G_h Q_h u^n - G_h u_h^n\right\| .\end{aligned} \tag{5.11}$$

From the definition of G_h, we have

$$G_h u^n = G_h Q_h u^n , \tag{5.12}$$

and

$$\left\|G_h Q_h u^n - G_h u_h^n\right\| \le C\left\|Q_h u^n - u_h^n\right\| . \tag{5.13}$$

From Theorem 4.1 and (5.11)–(5.13),

$$|||u - G_h u_h|||_{l^2(L^2)}^2 \le Ch^3 + C|||Q_h u - u_h|||_{l^2(L^2)}^2 \le C\left(h^3 + (\Delta t)^2\right) . \tag{5.14}$$

Consequently, inequality (5.10) follows from inequality (5.14). □

6. Numerical Experiments

For a constrained parabolic optimisation problem

$$\min_{u\in K} J(u)\,,$$

where $J(u)$ is a convex functional on X and K is a close convex subset of X, we have the iterative scheme $(n=0,1,2,\cdots)$

$$\begin{cases} b(u_{n+\frac{1}{2}},v) = b(u_n,v) - \rho_n \left(J'(u_n),v\right)\,, \quad \forall\, v\in X\,, \\ u_{n+1} = P_K^b(u_{n+\frac{1}{2}})\,, \end{cases} \tag{6.1}$$

where $b(\cdot,\cdot)=\int_0^T(\cdot,\cdot)$ is a symmetric and positive definite bilinear form, ρ_h is a step size of iteration, and the projection operator P_K^b can be computed as in Ref. [13]. For the piecewise constant element $K^h=\left\{\, v_h\in U^h \ :\ a\le v_h\le b \,\right\}$, then

$$P_k^b\left(p_h^n\right)\Big|_\tau = \Pi_{[a,b]}\left(\pi^c\left(p_h^n\right)\Big|_\tau\right), \qquad \forall\,\tau\in\mathcal{T}^h,\ n=0,1,\cdots,N\,.$$

For an acceptable error Tol (cf. also Ref. [13]), by applying (6.1) to the discretised optimal control problem (2.6) we arrive at the following projection gradient algorithm (for ease of exposition, we have omitted the subscript h).

Algorithm 6.1. Projection gradient algorithm

Step 1. Initialise u_0.
Step 2. Solve the following equations:

$$\begin{cases} b(u_{n+\frac{1}{2}},v) = b(u_n,v) - \rho_n \int_0^T \left(u_n+p_n,v\right)\,, \quad u_{n+\frac{1}{2}},u_n\in U^h, \forall\, v\in U^h\,, \\ \left(\frac{y_n^i-y_n^{i-1}}{\Delta t},w\right) + a\left(y_n^i,w\right) = \left(f^i+u_n^i,w\right)\,, \quad y_n^i,y_n^{i-1}\in W^h, \forall\, w\in W^h\,, \\ \left(\frac{p_n^{i-1}-p_n^i}{\Delta t},q\right) + a\left(q,p_n^{i-1}\right) = \left(y_n^i-y_d^i,q\right)\,, \quad p_n^i,p_n^{i-1}\in W^h, \forall\, q\in W^h\,. \\ u_{n+1} = P_K^b(u_{n+\frac{1}{2}})\,. \end{cases} \tag{6.2}$$

Step 3. Calculate the iterative error: $E_{n+1}=|||u_{n+1}-u_n|||_{l^2(L^2)}$.
Step 4. If $E_{n+1}\le Tol$, stop; else go to Step 2.

The following examples were solved numerically by the Algorithm 6.1 with codes developed based on AFEPack. (This package is freely available and the details can be found at http://www.acm.caltech.edu/ rli/AFEPack/.) The discretisation was described in Section 2 — i.e. the state and the adjoint state are approximated by piecewise linear functions and the control is approximated by piecewise constant functions. Let I be the 2×2 identity matrix and denote $|||\cdot|||_{l^2(L^2)}$ by $|||\cdot|||$. We choose the domain $\Omega=[0,1]\times[0,1]$ and $T=1$, and also assume that

$$K=\left\{\ v(x,t)\in L^2(J;L^2(\Omega)) : a\le v(x,t)\le b\ ,\quad (x,t)\in\Omega\times J\ \right\}\ .$$

We solve the following parabolic optimal control problems:

$$\begin{cases}\displaystyle\min_{u(x,t)\in K}\frac{1}{2}\int_0^T\left(\int_\Omega (y(x,t)-y_d(x,t))^2+\int_\Omega (u(x,t))^2\right)dt\ ,\\ y_t(x,t)-\mathrm{div}(A(x,t)\nabla y(x,t))=f(x,t)+u(x,t)\ ,\quad x\in\Omega,\ t\in J\ ,\\ y(x,t)=0\ ,\quad x\in\partial\Omega,\ t\in J,\\ y(x,0)=y_0(x)\ ,\quad x\in\Omega\ .\end{cases}$$

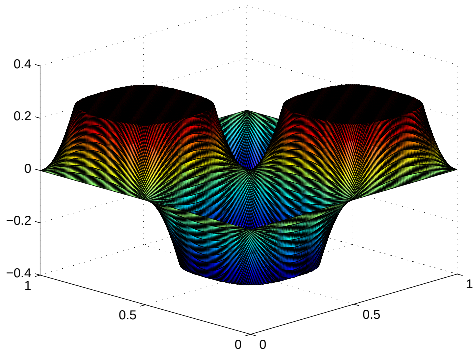
Example 6.1. The data are as follows:

$$\begin{aligned}&A(x,t)=\begin{pmatrix}\sin(\frac{\pi x_1}{2})(t+0.5) & 0\\ 0 & \sin(\frac{\pi x_2}{2})(1.5-t)\end{pmatrix}\\ &a=-0.25,\ b=0.25,\\ &p(x,t)=\sin(2\pi x_1)\sin(2\pi x_2)\sin(\pi t),\\ &y(x,t)=\sin(2\pi x_1)\sin(2\pi x_2)t,\\ &u(x,t)=\max(-0.25,\min(0.25,-p(x,t))),\\ &f(x,t)=y_t(x,t)-\mathrm{div}(A(x,t)\nabla y(x,t))-u(x,t),\\ &y_d(x,t)=y(x,t)+p_t(x,t)+\mathrm{div}(A^*(x,t)\nabla p(x,t)).\end{aligned}$$

Numerical results based on a sequence of uniformly refined meshes are shown in Table 1. In Fig. 1, we plot the profile of the numerical solution u_h at $t=0.5$ when $\Delta t=1/270$ and $h=1.25E-2$.

Table 1: The error of the control variable, Example 6.1.

h	Δt	$\|\|\|u-u_h\|\|\|$	Rate	$\|\|\|Q_hu-u_h\|\|\|$	Rate	$\|\|\|u-G_hu_h\|\|\|$	Rate
1.0E-1	1/10	5.01845E-2	—	2.38443E-2	—	3.81096E-2	—
5.0E-2	1/30	2.62036E-2	0.94	8.51186E-3	1.49	1.34949E-2	1.50
2.5E-2	1/90	1.29308E-2	1.02	2.98631E-3	1.51	4.97359E-3	1.51
1.25E-2	1/270	6.36698E-3	1.02	1.05551E-3	1.50	1.71985E-3	1.53



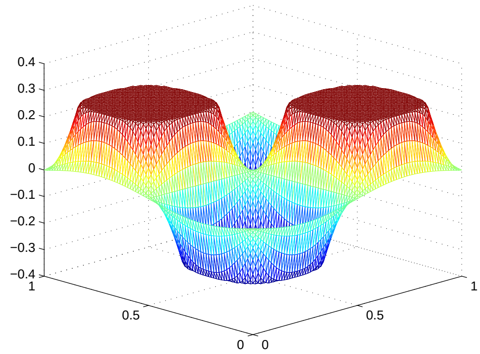


Figure 1: The exact solution u (left) and the numerical solution u_h (right) at $t = 0.5$.

Example 6.2. The data are as follows:

$$A(x,t) = 2(t+0.5)I,\ a = -0.5,\ b = 0.5,$$
$$p(x,t) = 2x_1x_2\sin(2\pi x_1)\sin(2\pi x_2)(1-t),$$
$$y(x,t) = 2x_1x_2\sin(2\pi x_1)\sin(2\pi x_2)t,$$
$$u(x,t) = \max(-0.5, \min(0.5, -p(x,t))),$$
$$f(x,t) = y_t(x,t) - \mathrm{div}(A(x,t)\nabla y(x,t)) - u(x,t),$$
$$y_d(x,t) = y(x,t) + p_t(x,t) + \mathrm{div}(A^*(x,t)\nabla p(x,t)).$$

Numerical results are listed in Table 2. In Fig. 2, we show the profile of the numerical solution u_h at $t = 0.5$ when $\Delta t = 1/270$ and $h = 1.25E-2$.

The convergence rate is computed from the formula

$$\text{Rate} = \frac{log(e_{i+1}) - log(e_i)}{log(h_{i+1}) - log(h_i)},$$

where e_i (e_{i+1}) denotes the error when the spatial partition size is h_i (h_{i+1}). From Table 1 and Table 2, it easy to see that $|||u-u_h||| = O(h+\Delta t)$, $|||Q_hu-u_h||| = O(h^{3/2}+\Delta t)$ and $|||u-G_hu_h||| = O(h^{3/2}+\Delta t)$, confirming our theoretical results.

Table 2: The error of the control variable, Example 6.2.

h	Δt	$\|\|\|u-u_h\|\|\|$	Rate	$\|\|\|Q_hu-u_h\|\|\|$	Rate	$\|\|\|u-G_hu_h\|\|\|$	Rate
1.0E-1	1/10	3.66180E-2	—	1.26204E-2	—	2.23411E-2	—
5.0E-2	1/30	1.82111E-2	1.01	4.47999E-3	1.49	8.24907E-3	1.50
2.5E-2	1/90	9.13557E-3	1.00	1.58406E-3	1.50	2.87023E-3	1.52
1.25E-2	1/270	4.57745E-3	1.00	5.54272E-4	1.52	9.98743E-4	1.52

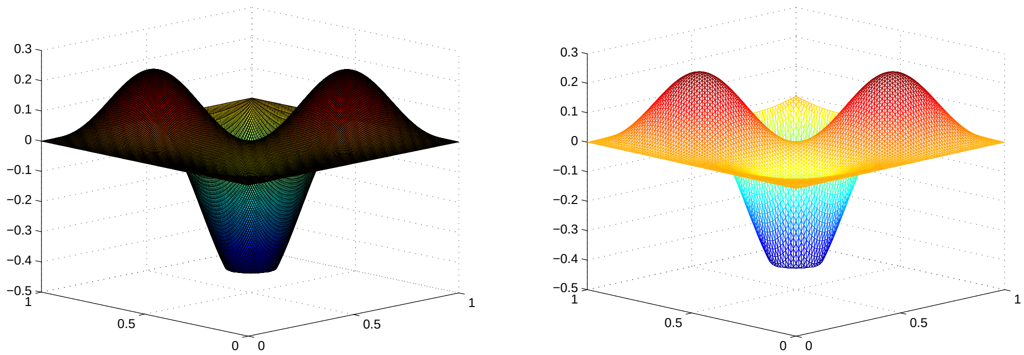


Figure 2: The exact solution u (left) and the numerical solution u_h (right) at $t = 0.5$.

7. Conclusion

We discussed the superconvergence analysis of fully discrete FE approximation for parabolic optimal control problems with time-dependent coefficients. Our superconvergence results for FE discrete parabolic optimal control problems seem to be new, and can be extended to general convex optimal control problems. In future, we will investigate superconvergence for hyperbolic optimal control problems, and in particular the superconvergence of these problems by the mixed FE method.

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