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The Mediating Morphism of the Multilinear Optimal Map

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Abstract. In this short note, we study a relation between the tensor product of matrices and a multilinear map defined by the optimal operator. In this particular case, the linear transform (mediating morphism) hidden in the abstract definition of the general tensor product can be determined explicitly.

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1. Introduction

Many preconditioners have been proposed in structured matrix computations since 1986 [13]. Among them, the most famous ones are Strang's circulant preconditioner [13], the optimal preconditioner [3] and the superoptimal preconditioner [14]. In this note, we concentrate on the study of a multilinear operator defined by the optimal preconditioner. Thus given a unitary matrix $U \in \mathbb{C}^{n\times n}$, let

$$\mathscr{M}_U \equiv \left\{U^*\Lambda U \mid \Lambda \text{ is any } n\times n \text{ diagonal matrix}\right\}. \tag{1.1}$$

For an arbitrary matrix $A \in \mathbb{C}^{n\times n}$, the optimal preconditioner $c_U(A)$ is defined to be the solution of

$$\min_{W\in\mathscr{M}_U} \|A-W\|_F\,, \tag{1.2}$$

where $\|\cdot\|_F$ is the Frobenius norm and W runs over $\mathscr{M}_U$ [1,3]. Computational and mathematical properties of the optimal preconditioner $c_U(A)$ have been studied extensively [1,2,4,5,11], and it has also been considered from an operator viewpoint [8,10].

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In this short note, we study a relation between a multilinear map f defined by the operator c_U and the tensor matrix product $\otimes$, and seek an exact form of the mediating morphism g such that $f = g \circ \otimes$. Matrices with tensor structure can be solved efficiently by the optimal preconditioner [9], and such matrices have many practical applications — e.g. see [6] for an application to the inverse heat problem. We believe the result in this paper may give some insights for designing preconditioners for these matrices. Some preliminaries related to the concepts involved are reviewed in the next section, and our main results are given in the subsequent section.

2. Preliminaries

Some important properties of the optimal preconditioner $c_U(A)$ defined by (1.2) are first summarised. We use $\delta(A)$ to denote the diagonal matrix with diagonal the same as the diagonal of the matrix A — i.e. if $A = (a_{pq})$, then

$$\delta(A) = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} .$$

The following result can be found in Refs. [1,8,10].

Theorem 2.1. *For arbitrary $A = (a_{pq}) \in \mathbb{C}^{n\times n}$, the optimal preconditioner $c_U(A)$ is uniquely determined by A and given by*

$$c_U(A) = U^* \delta(UAU^*)U . \tag{2.1}$$

Proof. For the completeness of this note, we include the following brief proof. Noting that the Frobenius norm is unitary invariant,

$$\|W - A\|_F = \|U^* \Lambda U - A\|_F = \|\Lambda - UAU^*\|_F .$$

Since Λ can only affect the diagonal entries of UAU^*, the minimizer of $\|\Lambda - UAU^*\|_F$ over all diagonal matrices is $\Lambda = \delta(UAU^*)$, so $c_U(A) = U^*\delta(UAU^*)U$. □

Suppose now that the Banach algebra of all $n \times n$ matrices over the complex field is equipped with a matrix norm $\|\cdot\|$ and denoted by $(\mathbb{C}^{n\times n}, \|\cdot\|)$; and let $(\mathscr{M}_U, \|\cdot\|)$ be the sub-algebra of $(\mathbb{C}^{n\times n}, \|\cdot\|)$, where $\mathscr{M}_U$ is defined by (1.1). Then obviously, c_U is a linear operator from $(\mathbb{C}^{n\times n}, \|\cdot\|)$ into $(\mathscr{M}_U, \|\cdot\|)$. We call c_U the optimal operator, and there is the following theorem on properties involving the operator norms of c_U — cf. Refs. [1,8]):

Theorem 2.2. *We have*

(i) $\|c_U\|_F \equiv \sup\limits_{\|A\|_F=1} \|c_U(A)\|_F = 1$; *and*

(ii) $\|c_U\|_2 \equiv \sup\limits_{\|A\|_2=1} \|c_U(A)\|_2 = 1$, *where $\|\cdot\|_2$ is the l_2 norm of the matrix.*

We now turn to review some basic concepts for the tensor product. Suppose that $\mathscr{V}_1, \cdots, \mathscr{V}_k$ and $\mathscr{W}$ are vector spaces. A function $f : \mathscr{V}_1 \times \cdots \times \mathscr{V}_k \to \mathscr{W}$ is called multilinear if it is linear in each coordinate separately — i.e. if

$$
\begin{aligned}
&f(v_1, \cdots, v_{j-1}, \alpha u_1 + \beta u_2, v_{j+1}, \cdots, v_k) \\
&= \alpha f(v_1, \cdots, v_{j-1}, u_1, v_{j+1}, \cdots, v_k) + \beta f(v_1, \cdots, v_{j-1}, u_2, v_{j+1}, \cdots, v_k)
\end{aligned}
$$

for all $j = 1, \cdots, k$. The general definition of the tensor product is given via its universal property.

Definition 2.1. (cf. Refs. [7, 12]) A pair $(\mathscr{X}, h : \mathscr{V}_1 \times \cdots \times \mathscr{V}_k \to \mathscr{X})$ is universal for multilinearity if for every multilinear map $f : V_1 \times \cdots \times \mathscr{V}_k \to \mathscr{W}$ there is a unique linear transformation $g : \mathscr{X} \to \mathscr{W}$ for which $f = g \circ h$. The map g is called the mediating morphism for f. If $(\mathscr{X}, h)$ is universal for multilinearity, then $\mathscr{X}$ is called the tensor product or Kronecker product of $\mathscr{V}_1, \cdots, \mathscr{V}_k$ and denoted by

$$\mathscr{X} \equiv \mathscr{V}_1 \otimes \cdots \otimes \mathscr{V}_k .$$

The map h is called the tensor map — cf. the following graph:

$$
\begin{tikzcd}
\mathscr{V}_1 \times \cdots \times \mathscr{V}_k \arrow[r, "h"] \arrow[d, "f"'] & \mathscr{X} = \mathscr{V}_1 \otimes \cdots \otimes \mathscr{V}_k \arrow[dl, "g"] \\
\mathscr{W} &
\end{tikzcd}
$$

When the vector spaces under consideration are spaces of matrices, the tensor product can be obtained more concretely. In fact, if $A = (a_{ij}) \in \mathbb{C}^{p \times q}$ and $B \in \mathbb{C}^{p' \times q'}$ then

$$
A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & a_{22}B & \cdots & a_{2q}B \\ \vdots & \vdots & & \vdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{pmatrix}, \tag{2.2}
$$

which is a $pp' \times qq'$ matrix.

In the next section, we consider a multilinear map defined by the optimal operator c_U on the Cartesian product of $\mathbb{C}^{n \times n}$, and explicitly determine the corresponding mediating morphism that is guaranteed in Definition 2.1.

3. The Mediating Morphism of the Multilinear Optimal Map

In this section, we consider the tensor product on $\mathscr{V} = \mathbb{C}^{n\times n}$. For simplicity, we first consider the tensor product of two matrix spaces. Let f denote a bilinear map from $\mathscr{V}\times\mathscr{V}$ to $\mathscr{M}_U$ defined by

$$f(A,B) \equiv c_U(A)c_U(B)\ . \tag{3.1}$$

We call f the bilinear optimal map, and now seek the mediating morphism (a linear map) g from $\mathscr{V}\otimes\mathscr{V}$ to $\mathscr{M}_U$ such that $f = g\circ\otimes$ — cf. the following graph:

$$\begin{tikzcd}
\mathscr{V}\times\mathscr{V} \arrow[r, "\otimes"] \arrow[d, "f"'] & \mathscr{V}\otimes\mathscr{V} \arrow[dl, "g"] \\
\mathscr{M}_U &
\end{tikzcd}$$

To construct this corresponding mediating morphism g, we start with a general standard approach. For $i,j \in \{1,\cdots,n\}$ let $E_{ij} = \mathbf{e}_i\mathbf{e}_j^T$, where $\{\mathbf{e}_j \mid j \in \{1,\cdots,n\}\}$ is the standard basis of $\mathbb{C}^{n\times 1} = \mathbb{C}^n$ (column vector space). Then $\{E_{ij} \mid i,j \in \{1,\cdots,n\}\}$ is a basis of $\mathscr{V}$ and $\{E_{ij}\otimes E_{st} \mid i,j,s,t \in \{1,\cdots,n\}\}$ a basis of $\mathscr{V}\otimes\mathscr{V}$, respectively. In this setting, for arbitrary $A = (a_{ij}), B = (b_{st}) \in \mathscr{V}$ one has

$$\begin{aligned}
f(A,B) =& c_U\left(\sum_{i=1}^n\sum_{j=1}^n a_{ij}E_{ij}\right)\cdot c_U\left(\sum_{s=1}^n\sum_{t=1}^n b_{st}E_{s,t}\right)\\
=& \sum_{i=1}^n\sum_{j=1}^n\sum_{s=1}^n\sum_{t=1}^n a_{ij}b_{st}c_U(E_{ij})\cdot c_U(E_{s,t})
\end{aligned}$$

and

$$A\otimes B = \sum_{i=1}^n\sum_{j=1}^n\sum_{s=1}^n\sum_{t=1}^n a_{ij}b_{st}E_{ij}\otimes E_{st}\ ,$$

so if we define the linear map $\tilde{g}$ from $\mathscr{V}\otimes\mathscr{V}$ to $\mathscr{M}_U$ by

$$\tilde{g}(A\otimes B) \equiv \sum_{i=1}^n\sum_{j=1}^n\sum_{s=1}^n\sum_{t=1}^n a_{ij}b_{st}\left[c_U(E_{ij})\cdot c_U(E_{st})\right] \tag{3.2}$$

then we immediately get $f(A,B) = \tilde{g}(A\otimes B)$.

Theorem 3.1. *For arbitrary matrices $A,B \in \mathscr{V}$, consider the bilinear optimal map f defined by* (3.1). *Then the mediating morphism $\tilde{g}$ for f is given by* (3.2).

Although Theorem 3.1 has provided the mediating morphism $\tilde{g}$ of f in terms of the basis of $\mathscr{V} \otimes \mathscr{V}$, it is more desirable to find an *exact* form of this linear map using matrix operations, because the linear space under consideration consists of matrices. Instead of deducing the exact form of the linear map from (3.2), we construct the linear map exactly using matrix notation in the following theorem.

Theorem 3.2. *For arbitrary $A, B \in \mathscr{V}$, consider the bilinear optimal map f defined by* (3.1). *Then the mediating morphism g for f is given exactly by*

$$g(A \otimes B) \equiv U^* \left(\sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T \right)^T \delta\big((U \otimes U)(A \otimes B)(U^* \otimes U^*)\big) \left(\sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T \right) U \,.$$

Proof. For $B_1, B_2 \in \mathscr{V}$, from (2.2) it is easy to see that

$$\delta(B_1 \otimes B_2) = \delta(B_1) \otimes \delta(B_2)\,.$$

We can use this property to obtain

$$\begin{aligned}
g(A \otimes B) =& U^* \left(\sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T \right)^T \delta\big((UAU^*) \otimes (UBU^*)\big) \left(\sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T \right) U \\
=& U^* \left(\sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T \right)^T \delta(UAU^*) \otimes \delta(UBU^*) \left(\sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T \right) U \\
=& U^* \left(\sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T \right)^T \left[\sum_{j=1}^{n} \delta(UAU^*) \otimes \delta(UBU^*) \cdot \mathbf{e}_j \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T \right] U \\
=& U^* \left(\sum_{\ell=1}^{n} \mathbf{e}_\ell \cdot \mathbf{e}_\ell^T \otimes \mathbf{e}_\ell^T \right) \left[\sum_{j=1}^{n} (UAU^*)_{jj} \mathbf{e}_j \otimes \big((UBU^*)_{jj} \mathbf{e}_j\big) \cdot \mathbf{e}_j^T \right] U \\
=& U^* \left[\sum_{j=1}^{n} (UAU^*)_{jj} (UBU^*)_{jj} \mathbf{e}_j \cdot \mathbf{e}_j^T \right] U \\
=& U^* \delta(UAU^*) \delta(UBU^*) U \\
=& U^* \delta(UAU^*) U \cdot U^* \delta(UBU^*) U \\
=& c_U(A) c_U(B) = f(A, B)\,,
\end{aligned}$$

where we have used (2.1) and (3.1) in the last line. This yields the given result. □

For a tensor product involving more than two matrix spaces, with slight modifications of the proof for Theorem 3.2 one can show that the following theorem holds:

Theorem 3.3. *For arbitrary matrices $A_1, \cdots, A_k \in \mathscr{V}$, suppose the multilinear optimal map is defined by*

$$\mathbf{f}(A_1, \cdots, A_k) \equiv c_U(A_1) \cdots c_U(A_k)\,.$$

Then the mediating morphism $\mathbf{g}$ *for* $\mathbf{f}$ *is given by*

$$\mathbf{g}(A_1 \otimes \cdots \otimes A_k) = R^* \delta\big((U \otimes \cdots \otimes U)(A_1 \otimes \cdots \otimes A_k)(U^* \otimes \cdots \otimes U^*)\big)R\,,$$

where

$$R = \left(\sum_{j=1}^{n} \mathbf{e}_j \otimes \cdots \otimes \mathbf{e}_j \cdot \mathbf{e}_j^T\right) U\,.$$

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