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General Solutions for a Class of Inverse Quadratic Eigenvalue Problems

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Abstract. Based on various matrix decompositions, we compare different techniques for solving the inverse quadratic eigenvalue problem, where $n \times n$ real symmetric matrices M, C and K are constructed so that the quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ yields good approximations for the given k eigenpairs. We discuss the case where M is positive definite for $1 \le k \le n$, and a general solution to this problem for $n+1 \le k \le 2n$. The efficiency of our methods is illustrated by some numerical experiments.

AMS subject classifications: 65F18

Key words: Quadratic eigenvalue problem, inverse quadratic eigenvalue problem, partially prescribed spectral information.

1. Introduction

For $n \times n$ complex matrices M, C and K, the quadratic eigenvalue problem (QEP) involves finding the eigenpairs $(\lambda,\ x)$ such that $Q(\lambda)x = 0$, where

$$Q(\lambda) = Q(\lambda; M, C, K) = \lambda^2 M + \lambda C + K \tag{1.1}$$

is a so-called quadratic pencil defined by M, C and K. The scalars λ and the corresponding nonzero vectors x are the eigenvalues and eigenvectors of the pencil, respectively. It is known that the QEP has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M is nonsingular. The "direct" problem is of course to find the eigenvalues and eigenvectors when the coefficient matrices M, C and K are given (cf. [5] and references therein), while the inverse quadratic eigenvalue problem (IQEP) is to determine the matrix coefficients M, C and K from a prescribed set of eigenvalues and eigenvectors (cf. [16] and references therein).

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The IQEP has received much attention because of the wide variety of its applications — including structural design [9], control design for second-order systems [6, 16], finite element model updating for damped or gyroscopic systems [7], system identification [1] and inverse problems for damped vibration systems [12]. Some general reviews and extensive bibliographies in this regard can be found in Refs. [3] and [4].

The formulation of an IQEP depends upon the type of eigen-information available, the conditions imposed upon the matrix coefficients, and the techniques used to decompose the matrix constituted by the given eigenvectors. The IQEP studied by Ram & Elhay [17] is for symmetric tridiagonal coefficients where instead of prescribed eigenpairs, two sets of eigenvalues are given. Based on the spectral theory of matrix polynomials, Lancaster *et al.* [8, 11, 13] considered the IQEP with: (1) Hermitian matrices M, C and K, (2) real symmetric matrices M, C and K, and (3) real symmetric positive definite or semi-definite matrices M, C and K, so that the quadratic pencil $Q(\lambda)$ has complete information on the eigenvalues and eigenvectors. We deal with the inverse problem with k given eigenpairs, where M is required to be real symmetric positive definite, and C and K are $n \times n$ real symmetric matrices. For $1 \leq k \leq n$, Yuan *et al.* [18] gave a detailed discussion involving QR decomposition, while for $n+1 \leq k \leq 2n$ Kuo *et al.* [10] studied the general solution to this problem with QR decomposition.

Our main concern is as follows: for a given eigen-information pair (Λ, X), find real symmetric matrices M, C and K where M is positive definite such that

$$MX\Lambda^2 + CX\Lambda + KX = 0 \tag{1.2}$$

is satisfied. Our motivation is to find a more efficient method to solve this problem, and the techniques we investigate below are the Rank Revealing QR (*RRQR*), SVD and UTV factorizations where U and V are orthogonal matrices, while T is an upper-two-diagonal matrix.

Since M, C and K are in $\mathbb{R}^{n\times n}$, we can transform the given complex eigenpairs into real eigenpairs. To facilitate the discussion, let the real eigenpairs constitute the pair of matrices $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ such that

$$\Lambda = diag\left\{\lambda_1^{[2]}, \cdots, \lambda_l^{[2]}, \lambda_{2l+1}, \cdots, \lambda_k\right\}, \tag{1.3}$$

with

$$\lambda_j^{[2]} = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad \beta_j \neq 0 \quad \text{for } j = 1, 2, \cdots, l \tag{1.4}$$

and

$$X = \left\{x_{1R}, x_{1I}, \cdots, x_{lR}, x_{lI}, x_{2l+1}, \cdots, x_k\right\}, \tag{1.5}$$

where x_{iR} and x_{iI} denote the real and imaginary parts of the corresponding eigenvector, respectively. Then the original eigenpairs can be described by the matrices

$$\tilde{\Lambda} = R^H \Lambda R = diag\left\{\lambda_1, \lambda_2, \cdots, \lambda_{2l-1}, \lambda_{2l}, \lambda_{2l+1}, \cdots, \lambda_k\right\}$$

and

$$\tilde{X} = XR = diag\,\{x_1, x_2, \cdots, x_{2l-1}, x_{2l}, x_{2l+1}, \cdots, x_k\} \in \mathbb{C}^{n\times k}\,, \tag{1.6}$$

where

$$R = diag\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \cdots, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, I_{k-2l}\right\} \text{ with } i^2 = -1\,,$$

$$x_{2j-1} = \frac{1}{\sqrt{2}}x_{jR} + \frac{i}{\sqrt{2}}x_{jI}\,, \qquad x_{2j} = \frac{1}{\sqrt{2}}x_{jR} - \frac{i}{\sqrt{2}}x_{jI}\,,$$

$$\lambda_{2j-1} = \alpha_j + i\beta_j\,, \qquad \lambda_{2j} = \alpha_j - i\beta_j\,, \quad \text{for} \quad j = 1, 2, \cdots, l\,.$$

Here x_j and λ_j are real-valued for $j = 2l+1, \cdots, k$. Thus our IQEP involves finding a real-valued quadratic pencil $Q(\lambda)$ with matrix coefficients possessing a certain specified structure so that $Q(\lambda_j)x_j = 0$ for all $j = 1, 2, \cdots, k$.

For convenience, let us denote the set of diagonal elements of $\tilde{\Lambda}$ (the spectrum of Λ) by $\sigma(\Lambda)$, and write (Λ, X) for an eigen-information pair of the quadratic pencil $Q(\lambda)$. In addition, we make the following assumptions:

(1) the eigenvalue matrix Λ in (1.3) has simple eigenvalues;

(2) the eigenvector matrix X in (1.5) has full rank, and the matrix $\begin{bmatrix} X \\ X\Lambda \end{bmatrix}$ is of full column rank. In Section 2, we prove that the above ISQEP is always solvable with our techniques, and representations of the solution sets are then produced. In Section 3, we present some numerical results to support our main results and for comparison with existing methods.

2. Main Results

2.1. Results for $1 \le k \le n$

In this subsection, we solve the ISQEP for a given matrix pair $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ $(k \le n)$ defined by (1.3), (1.4) and (1.5).

Theorem 2.1. *Given a matrix pair* $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k} (k \le n)$ *as in* (1.3), (1.4) *and* (1.5), *let*

$$X = Q\begin{pmatrix} R \\ 0 \end{pmatrix}P^T = Q_1 R P^T \tag{2.1}$$

be the RRQR decomposition of X*, where* $Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \in O\mathbb{R}^{n\times n}$ *(a set of orthogonal* $n\times n$ *real matrices) with* $Q_1 \in \mathbb{R}^{n\times k}, P \in O\mathbb{R}^{k\times k}$ *(a set of orthogonal* $k \times k$ *real matrices) and* R *an upper triangular matrix. Let* $S = RP^T\Lambda P R^{-1}$*. The general solution to the ISQEP is then*

$$M = Q\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}Q^T, \qquad C = Q\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}Q^T,$$

$$K = Q\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}Q^T, \tag{2.2}$$

where:

(i) $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{R}^{n\times n}$ *is an arbitrary symmetric positive definite matrix;*

(ii) $C_{22} = C_{22}^T$ *and* $K_{22} = K_{22}^T \in \mathbb{R}^{(n-k)\times(n-k)}$ *are arbitrary symmetric matrices;*

(iii) $C_{21} = C_{12}^T \in \mathbb{R}^{(n-k)\times k}$, *where* C_{21} *is arbitrary;*

(iv)
$$C_{11} = C_{11}^T = -(M_{11}S + S^T M_{11} + R^{-T}P^T DPR^{-1}) \in \mathbb{R}^{k\times k};$$

(v)
$$K_{11} = K_{11}^T = S^T M_{11} S + R^{-T}P^T D\Lambda PR^{-1} \in \mathbb{R}^{k\times k}; \text{ and} \tag{2.3}$$

(vi)
$$K_{21} = K_{12}^T = -(M_{21}S^2 + C_{21}S) \in \mathbb{R}^{(n-k)\times k}. \tag{2.4}$$

Here
$$D = diag\left\{ \begin{pmatrix} \varepsilon_1 & \eta_1 \\ \eta_1 & -\varepsilon_1 \end{pmatrix}, \cdots, \begin{pmatrix} \varepsilon_l & \eta_l \\ \eta_l & -\varepsilon_l \end{pmatrix}, \varepsilon_{2l+1}, \cdots, \varepsilon_k \right\}, \tag{2.5}$$

where ε_i *and* η_i *are arbitrary real numbers.*

Proof. Substituting (2.1) and (2.2) into (1.2) gives

$$M_{11}RP^T\Lambda^2 + C_{11}RP^T\Lambda + K_{11}RP^T = 0\,,$$
$$M_{21}RP^T\Lambda^2 + C_{21}RP^T\Lambda + K_{21}RP^T = 0\,.$$

Post-multiplying the above two equations by PR^{-1} yields

$$M_{11}S^2 + C_{11}S + K_{11} = 0\,, \tag{2.6}$$
$$M_{21}S^2 + C_{21}S + K_{21} = 0\,, \tag{2.7}$$

where $S = RP^T\Lambda PR^{-1}$. Thus finding M, C and K satisfying (1.2) is equivalent to finding the submatrices $M_{11}, M_{21}, C_{11}, C_{21}$, K_{11} and K_{21} that satisfy (2.6) and (2.7). Clearly, it follows from (2.7) that K_{21} is determined by (2.4) where M_{21} and C_{21} are arbitrary. As M and K are required to be symmetric positive definite and symmetric, respectively, in (2.2) M_{11} is symmetric positive definite and K_{11} is symmetric. From (2.6) it follows that

$$K_{11} = -(M_{11}S^2 + C_{11}S)\,. \tag{2.8}$$

Let M_{11} be an arbitrary symmetric positive definite matrix. We need to find a symmetric matrix C_{11} such that K_{11} in (2.8) is symmetric — i.e.

$$(M_{11}S^2 + C_{11}S)^T = M_{11}S^2 + C_{11}S\,. \tag{2.9}$$

After rearrangement, (2.9) becomes

$$C_{11}S - S^T C_{11} = -M_{11}S^2 + (S^2)^T M_{11}\,, \tag{2.10}$$

which has a particular solution

$$C_{11}^0 = -(M_{11}S + S^T M_{11})\,. \tag{2.11}$$

Next we consider the homogeneous equation

$$C_{11}S - S^T C_{11} = 0\,. \tag{2.12}$$

Substituting $S = RP^T \Lambda P R^{-1}$ into (2.12) yields

$$(RP^T)^T C_{11} RP^T \Lambda - \Lambda^T (RP^T)^T C_{11} RP^T = 0\,. \tag{2.13}$$

Corresponding to the structure, we have $s = k - l$ and partition $(RP^T)^T C_{11} RP^T$ as

$$(RP^T)^T C_{11} RP^T = \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1l} \\ \vdots & \ddots & \vdots \\ \Gamma_{l1} & \cdots & \Gamma_{ll} \end{pmatrix}, \tag{2.14}$$

where Γ_{jj} is a 2×2 matrix for $1 \le j \le l$ and Γ_{jj} is a 1×1 matrix for $l + 1 \le j \le s$. Substituting (2.14) into (2.13), and using assumption (2) and the same technique as in Ref. [19], we obtain that $\Gamma_{ij} = 0$ for $j \neq i$,

$$\Gamma_{jj}\lambda_j^{[2]} - (\lambda_j^{[2]})^T \Gamma_{jj} = 0\,, \quad j = 1, 2, \cdots, l \tag{2.15}$$

and

$$\Gamma_{l+j,l+j}\lambda_{2l+j} - \lambda_{2l+j}\Gamma_{l+j,l+j} = 0\,, \quad j = 1, 2, \cdots, s - l\,. \tag{2.16}$$

Since $\lambda_j^{[2]}$ has the form in (1.4) with $\beta_j \neq 0$, it is easy to see that the general solution of (2.5) has the form

$$\Gamma_{jj} = \begin{pmatrix} \varepsilon_j & \eta_j \\ \eta_j & -\varepsilon_j \end{pmatrix}, \quad j = 1, 2, \cdots, l \tag{2.17}$$

where ε_j, η_j are arbitrary real numbers and (2.16) holds for any real numbers $\Gamma_{l+j,l+j} = \varepsilon_{l+j}$. Thus the general solution of the homogeneous equation (2.12) has the form

$$C_{11} = (RP^T)^{-T} D (RP^T)^{-1}\,,$$

where D is defined in (2.5). Together with (2.11), this produces the general solution of (2.10):

$$C_{11} = -(RP^T)^{-T} D (RP^T)^{-1} - M_{11}S - S^T M_{11}\,. \tag{2.18}$$

Substituting (2.18) into (2.8) yields (2.3). From (2.17) and related discussion, the matrix D is symmetric. This completes the proof. □

Theorem 2.1 shows the solution to the ISQEP is underdetermined, and using this theorem we can construct a solution to the ISQEP.

2.2. Results for $n+1 \le k \le 2n$

To solve the ISQEP ($n+1 \le k \le 2n$), we cite the following lemma [2], and then obtain the general solution of the ISQEP in a parameterized form.

Lemma 2.1. *(cf. Ref.* [2]*) There exist real symmetric matrices M, C and K satisfying the equation* (1.2) *if and only if*

$$X^T C X = -(\Lambda^T X^T M X + X^T M X \Lambda) + D\,, \tag{2.19}$$

$$X^T K X = \Lambda^T X^T M X \Lambda - \Lambda^T D \tag{2.20}$$

for some $D \in D_\Lambda$, where

$$D_\Lambda = \left\{ D \in \mathbb{R}^{k\times k} | D = D^T\ , D\Lambda = \Lambda^T D \right\}\ .$$

Assume that the singular value decomposition of X^T is

$$X^T = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} Q^T = U_1 \Sigma\, Q^T \tag{2.21}$$

where $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \in O\mathbb{R}^{k\times k}$ with $U_1 \in \mathbb{R}^{k\times n}$, $Q \in O\mathbb{R}^{n\times n}$ and

$$\Sigma = diag\,\{\sigma_1, \sigma_2, \cdots, \sigma_n\} > 0\,.$$

Then it follows from (2.21) that $X = Q\Sigma U_1^T = (U_1 \Sigma Q^T)^T$ and $XU_2 = 0$, and denoting

$$M_r = (\Sigma Q^T) M (\Sigma Q^T)^T, \quad C_r = (\Sigma Q^T) C (\Sigma Q^T)^T, \quad K_r = (\Sigma Q^T) K (\Sigma Q^T)^T \tag{2.22}$$

we have the following result.

Lemma 2.2. *Let M_r, C_r and K_r be defined as in* (2.22)*. Then there are real symmetric matrices M, C and K satisfying* (1.2) *if and only if*

$$C_r = -[(U_1^T \Lambda^T U_1) M_r + M_r (U_1^T \Lambda U_1)] + U_1^T D U_1\,, \tag{2.23}$$

$$K_r = (U_1^T \Lambda^T U_1) M_r (U_1^T \Lambda U_1) - U_1^T \Lambda^T D U_1\,, \tag{2.24}$$

$$M_r (U_1^T \Lambda U_2) = U_1^T D U_2 \tag{2.25}$$

for some $D \in D(\Lambda, X)$, where

$$D(\Lambda, X) = \left\{ D \in D_\Lambda | U_2^T D U_2 = 0 \right\}\ . \tag{2.26}$$

Proof. (Necessity) Suppose that the real symmetric matrices M, C and K satisfy (1.2). From Lemma 2.1, it follows that (2.19) and (2.20) hold for some matrix $D \in D(\Lambda, X)$. Then from (2.19) we have

$$\begin{aligned}
&\begin{pmatrix} U_1^T D U_1 & U_1^T D U_2 \\ U_2^T D U_1 & U_2^T D U_2 \end{pmatrix} \\
&= U^T D U \\
&= U^T (X^T C X + \Lambda^T X^T M X + X^T M X \Lambda) U \\
&= \begin{pmatrix} C_r + (U_1^T \Lambda^T U_1) M_r + M_r (U_1^T \Lambda U_1) & M_r (U_1^T \Lambda U_2) \\ (U_2^T \Lambda^T U_1) M_r & 0 \end{pmatrix},
\end{aligned}$$

from which we get

$$C_r = -\left[(U_1^T\Lambda^T U_1)M_r + M_r(U_1^T\Lambda U_1)\right] + U_1^T DU_1\,, \tag{2.27}$$
$$M_r(U_1^T\Lambda U_2) = U_1^T DU_2\,, \tag{2.28}$$
$$U_2^T DU_2 = 0\,. \tag{2.29}$$

Similarly, from (2.20) we get

$$K_r = (U_1^T\Lambda^T U_1)M_r(U_1^T\Lambda U_1) - U_1^T\Lambda^T DU_1\,, \tag{2.30}$$
$$U_1^T\Lambda^T DU_2 = (U_1^T\Lambda^T U_1)M_r(U_1^T\Lambda U_2)\,, \tag{2.31}$$
$$U_2^T\Lambda^T DU_2 = (U_2^T\Lambda^T U_1)M_r(U_1^T\Lambda U_2)\,. \tag{2.32}$$

This shows that (2.23), (2.24) and (2.25) hold for some $D \in D(\Lambda, X)$.

(Sufficiency) Suppose that the real symmetric matrices M, C and K satisfy (2.23), (2.24) and (2.25) for some $D \in D(\Lambda, X)$. Then

$$\begin{pmatrix} U_1^T DU_1 & U_1^T DU_2 \\ U_2^T DU_1 & U_2^T DU_2 \end{pmatrix} = \begin{pmatrix} C_r + (U_1^T\Lambda^T U_1)M_r + M_r(U_1^T\Lambda U_1) & M_r(U_1^T\Lambda U_2) \\ (U_2^T\Lambda^T U_1)M_r & 0 \end{pmatrix}.$$

The equality (2.19) is then established. From (2.25) and (2.26) it is easy to derive $(D - U_1 M_r U_1^T \Lambda)U_2 = 0$, which produces (2.31) and (2.32) immediately. Together with (2.24), we therefore have (2.30), (2.31) and (2.32), whence (2.20). Thus from Lemma 2.1, we obtain (1.2). □

Next we consider the solvability of the matrix equation (2.25). First we note the following result concerning its coefficient matrix $U_1^T\Lambda U_2$:

Lemma 2.3. *(cf. Ref. [2]) The matrix $U_1^T\Lambda U_2$ in (2.25) is of full column rank.*

The following result then gives the general solution of the matrix equation (2.25).

Lemma 2.4. *(cf. Refs. [14, 15]) Let $B = U_1^T\Lambda U_2$. Then for any $D \in D(\Lambda, X)$, the matrix equation $M_r B = U_1^T DU_2$ for M_r is solvable, and moreover, M_r is given by*

$$M_r = V \begin{pmatrix} B^T U_1^T DU_2 & U_2^T DU_1 Z \\ Z^T U_1^T DU_2 & W \end{pmatrix} V^T \tag{2.33}$$

where $W^T = W \in \mathbb{R}^{(2n-k)\times(2n-k)}$ is arbitrary, and

$$V = \begin{pmatrix} B(B^T B)^{-1} & Z \end{pmatrix}$$

with $Z \in \mathbb{R}^{n\times(2n-k)}$ satisfying $B^T Z = 0$ and $Z^T Z = I_{2n-k}$.

With Lemmas 2.2 and 2.4, we have the main result that completely characterises the ISQEP $(n+1 \le k \le 2n)$ in the following theorem.

Theorem 2.2. *Let $R = U_1(\Sigma Q^T)^{-T} = U_1\Sigma^{-T}Q^T$ and V be defined as in Lemma 2.4. Then the general solution of the ISQEP can be represented in terms of W and D in the following parameterized form:*

$$\begin{aligned}
&M = (\Sigma Q^T)^{-1}V \begin{pmatrix} B^T U_1^T D U_2 & U_2^T D U_1 Z \\ Z^T U_1^T D U_2 & W \end{pmatrix} V^T(\Sigma Q^T)^{-T} , \\
&C = R^T D R - R^T \Lambda^T X^T M - M X \Lambda R , \\
&K = R^T \Lambda^T X^T M X \Lambda R - R^T \Lambda^T D R ,
\end{aligned}$$

where $W^T = W \in \mathbb{R}^{(2n-k)\times(2n-k)}$ and $D \in D(\Lambda, X)$ are arbitrary.

Let the UTV decomposition of X^T be

$$X^T = U \begin{pmatrix} T \\ 0 \end{pmatrix} Q^T = U_1 T Q^T , \tag{2.34}$$

where $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \in O\mathbb{R}^{k\times k}$, with $U_1 \in \mathbb{R}^{k\times n}$, $Q \in O\mathbb{R}^{n\times n}$ and

$$T = \begin{pmatrix} * & * & 0 & \cdots & 0 & 0 \\ 0 & * & * & \cdots & 0 & 0 \\ 0 & 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & * \end{pmatrix} . \tag{2.35}$$

Then from (2.34) it follows that $X = QT^T U_1^T = (U_1 T Q^T)^T$ and $XU_2 = 0$. Denoting

$$M_r = (TQ^T)M(TQ^T)^T, \quad C_r = (TQ^T)C(TQ^T)^T, \quad K_r = (TQ^T)K(TQ^T)^T ,$$

from Lemma 2.2 and Lemma 2.4 we similarly get the solution of ISQEP ($n + 1 \le k \le 2n$) as follows.

Theorem 2.3. *Let $R = U_1(TQ^T)^{-T} = U_1 T^{-T} Q^T$ and V be defined as in Lemma 2.4. Then the general solution of the ISQEP can be represented in terms of W and D in the following parameterized form:*

$$\begin{aligned}
&M' = (TQ^T)^{-1}V \begin{pmatrix} B^T U_1^T D U_2 & U_2^T D U_1 Z \\ Z^T U_1^T D U_2 & W \end{pmatrix} V^T(TQ^T)^{-T} , \\
&C' = R^T D R - R^T \Lambda^T X^T M - M X \Lambda R , \\
&K' = R^T \Lambda^T X^T M X \Lambda R - R^T \Lambda^T D R ,
\end{aligned}$$

where $W^T = W \in \mathbb{R}^{(2n-k)\times(2n-k)}$ and $D \in D(\Lambda, X)$ are arbitrary.

Let the *RRQR* decomposition of X^T be

$$X^T = Q \begin{pmatrix} T \\ 0 \end{pmatrix} P^T = Q_1 T P^T \tag{2.36}$$

where $Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \in O\mathbb{R}^{k\times k}$, with $Q_1 \in \mathbb{R}^{k\times n}$, $P \in O\mathbb{R}^{n\times n}$ and T an $n \times n$ upper triangular matrix. From (2.36), it follows that $X = PT^T Q_1^T = (Q_1 T P^T)^T$ and $XQ_2 = 0$. Finally, denoting

$$M_r = (TP^T)M(TP^T)^T\,, \quad C_r = (TP^T)C(TP^T)^T, \quad K_r = (TP^T)K(TP^T)^T\,, \tag{2.37}$$

from Lemma 2.2 and Lemma 2.4 we similarly get the solution of ISQEP ($n+1 \le k \le 2n$) as follows.

Theorem 2.4. *Let $R = Q_1(TP^T)^{-T} = Q_1 T^{-T} P^T$ and V be defined as those in Lemma 2.4. Then the general solution of the ISQEP can be represented in the following parameterized forms in terms of W and D:*

$$M^{''} = (TP^T)^{-1} V \begin{pmatrix} B^T U_1^T D U_2 & U_2^T D U_1 Z \\ Z^T U_1^T D U_2 & W \end{pmatrix} V^T (TP^T)^{-T}, \tag{2.38}$$

$$C^{''} = R^T D R - R^T \Lambda^T X^T M - M X \Lambda R, \tag{2.39}$$

$$K^{''} = R^T \Lambda^T X^T M X \Lambda R - R^T \Lambda^T D R, \tag{2.40}$$

where $W^T = W \in \mathbb{R}^{(2n-k)\times(2n-k)}$ and $D \in D(\Lambda, X)$ are arbitrary.

3. Numerical Experiments

In this section, we present some numerical examples to illustrate the solutions constructed in Sections 2. We report all the numerical results in five significant digits using MATLAB with full precision on a PC, where $\tilde{\lambda}_i$ are the computed eigenvalues of $Q(\lambda)$.

Example 3.1. Consider the ISQEP where the partial eigen-structure $(\Lambda, X) \in \mathbb{C}^{5\times5} \times \mathbb{C}^{5\times5}$ is as in (1.3) and (1.5), with $\lambda_1 = -0.31828 - 0.86754i = \bar{\lambda}_2$, $\lambda_3 = -0.95669 + 0.17379i = \bar{\lambda}_4$, $\lambda_5 = -4.4955$, and the corresponding eigenvectors

$$x_1 = \bar{x}_2 = \begin{pmatrix} 15.159 - 11.123i \\ -77.470 - 14.809i \\ 2.1930 - 10.275i \\ 0.3821 + 16.329i \\ 57.042 + 18.419i \end{pmatrix}, \quad x_3 = \bar{x}_4 = \begin{pmatrix} 65.621 + 34.379i \\ 22.625 - 24.189i \\ -37.062 - 15.825i \\ -9.6496 - 14.401i \\ -0.61893 - 25.609i \end{pmatrix}, \quad x_5 = \begin{pmatrix} 2.2245 \\ 1.5893 \\ 2.1455 \\ 2.1752 \\ 1.6586 \end{pmatrix}.$$

It is easy to check that the matrix pair $(\Lambda, X) \in \mathbb{C}^{5\times5} \times \mathbb{C}^{5\times5}$ satisfies the assumptions (1) and (2).

According to Theorem 2.1, we get the solution with $1 \le k \le n$, with the accuracy of the approximated eigenvalues shown in Table 1.

Table 1: Absolute errors for ISQEP($1 \leq k \leq n$) with decomposition of X.

$Eigenvalues$	$\mid \lambda_i - \tilde{\lambda}_i \mid (RRQR)$	$\mid \lambda_i - \tilde{\lambda}_i \mid (QR)$
$\lambda_1 = \lambda_2$	$9.6273e-11$	$2.1615e-10$
$\lambda_3 = \lambda_4$	$7.3066e-11$	$1.6542e-10$
λ_5	$9.5035e-14$	$1.6858e-12$

Example 3.2. Consider the ISQEP with the partial eigen-information $(\Lambda, X) \in \mathbb{C}^{6\times 6} \times \mathbb{C}^{5\times 6}$ as in (1.3) and (1.5), $\lambda_1 = -0.31828 - 0.86754i = \bar{\lambda}_2$, $\lambda_3 = -0.95669 + 0.17379i = \bar{\lambda}_4$, $\lambda_5 = -4.4955$, $\lambda_6 = 1.5135$, and the corresponding eigenvectors

$$x_1 = \bar{x}_2 = \begin{pmatrix} 15.159 - 11.123i \\ -77.470 - 14.809i \\ 2.1930 - 10.275i \\ 0.3821 + 16.329i \\ 57.042 + 18.419i \end{pmatrix}, \quad x_3 = \bar{x}_4 = \begin{pmatrix} 65.621 + 34.379i \\ 22.625 - 24.189i \\ -37.062 - 15.825i \\ -9.6496 - 14.401i \\ -0.61893 - 25.609i \end{pmatrix}, \quad x_5 = \begin{pmatrix} 2.2245 \\ 1.5893 \\ 2.1455 \\ 2.1752 \\ 1.6586 \end{pmatrix}, \quad x_6 = \begin{pmatrix} 34.675 \\ -5.8995 \\ 37.801 \\ -66.071 \\ -6.6174 \end{pmatrix}.$$

It is easy to check that the matrix pair $(\Lambda, X) \in \mathbb{C}^{6\times 6} \times \mathbb{C}^{5\times 6}$ satisfies the assumptions (1) and (2).

From Theorems 2.2, 2.3 and 2.4, we get the solutions for $n+1 \leq k \leq 2n$, with the accuracy of the approximated eigenvalues as shown in Table 2.

Table 2: Absolute errors for ISQEP ($n+1 \leq k \leq 2n$) with decomposition of X.

$Eigenvalues$	$\mid \lambda_i - \tilde{\lambda}_i \mid (RRQR)$	$\mid \lambda_i - \tilde{\lambda}_i \mid (QR)$
$\lambda_1 = \lambda_2$	$8.3090e-3$	$5.3768e-2$
$\lambda_3 = \lambda_4$	$5.0347e-3$	$6.9076e-3$
λ_5	$3.1807e-1$	1.8448
λ_6	$4.4399e-3$	$9.1065e-2$
$Eigenvalues$	$\mid \lambda_i - \tilde{\lambda}_i \mid (SVD)$	$\mid \lambda_i - \tilde{\lambda}_i \mid (UTV)$
$\lambda_1 = \lambda_2$	$3.6599e-4$	$5.0891e-3$
$\lambda_3 = \lambda_4$	$7.7766e-5$	$1.1036e-3$
λ_5	$3.7208e-2$	$3.5517e-1$
λ_6	$1.2133e-4$	$1.7605e-1$

From the above numerical results, in terms of the eigenvalues we observe that the RRQR and SVD methods are superior to the QR method for both $1 \leq k \leq n$ and for $n+1 \leq k \leq 2n$, as is the UTV method for most of the eigenvalues.

Example 3.3. Consider a 20×20 triplet $(M_0,\ C_0,\ K_0)$ with M_0 the identity matrix, C_0 and K_0 five-diagonal matrices with $C_0(i,i) = 5$, $C_0(i,j) = 2$ if $|i-j| = 1$, $C_0(i,j) = -1$ if $|i-j| = 2$, and $K_0(i,i) = 3$, $K_0(i,j) = 1$ if $|i-j| = 1$, $K_0(i,j) = -2$ if $|i-j| = 2$. We first compute all 40 eigenpairs of $Q_0(\lambda) = \lambda^2 I + \lambda C_0 + K_0$ and $(\Lambda, X) \in \mathbb{R}^{12\times 12} \times \mathbb{R}^{20\times 12}$, chosen from those 40 computed eigenpairs of $Q_0(\lambda)$, where the selected eigenvalues are $\lambda_1 = -0.9505 + 0.4397i = \overline{\lambda}_2$, $\lambda_3 = -1.4268 + 0.6214i = \overline{\lambda}_4$, $\lambda_5 = -5.9454$, $\lambda_6 = -6.4673$, $\lambda_7 = -6.8169$, $\lambda_8 = -7.1928$, $\lambda_9 = -7.1919$, $\lambda_{10} = -7.0824$, $\lambda_{11} = -7.0993$, $\lambda_{12} = -6.8732$, and the corresponding eigenvectors are as follows:

$$V_1=\begin{pmatrix}-0.5819-0.0011i\\0.7117+0.2883i\\-0.3770-0.2476i\\-0.0252+0.0584i\\0.4376+0.0956i\\-0.6585-0.2201i\\0.5302+0.2225i\\-0.1440-0.0860i\\-0.3065-0.0927i\\0.6102+0.2148i\\-0.6102-0.2148i\\0.3065+0.0927i\\0.1440+0.0860i\\-0.5302-0.2225i\\0.6585+0.2201i\\-0.4376-0.0956i\\0.0252-0.0584i\\0.3770+0.2476i\\-0.7117-0.2883i\\0.5819+0.0011i\end{pmatrix},\quad V_2=\begin{pmatrix}-0.5819+0.0011i\\0.7117-0.2883i\\-0.3770+0.2476i\\-0.0252-0.0584i\\0.4376-0.0956i\\-0.6585+0.2201i\\0.5302-0.2225i\\-0.1440+0.0860i\\-0.3065+0.0927i\\0.6102-0.2148i\\-0.6102+0.2148i\\0.3065-0.0927i\\0.1440-0.0860i\\-0.5302+0.2225i\\0.6585-0.2201i\\-0.4376+0.0956i\\0.0252+0.0584i\\0.3770-0.2476i\\-0.7117+0.2883i\\0.5819-0.0011i\end{pmatrix},\quad V_3=\begin{pmatrix}0.4900+0.0810i\\-0.5539-0.2377i\\0.1750+0.1668i\\0.3123+0.0431i\\-0.5691-0.2056i\\0.4157+0.1985i\\0.0373-0.0327i\\-0.4621-0.1560i\\0.5517+0.2153i\\-0.2423-0.0985i\\-0.2423-0.0985i\\0.5517+0.2153i\\-0.4621-0.1560i\\0.0373-0.0327i\\0.4157+0.1985i\\-0.5691-0.2056i\\0.3123+0.0431i\\0.1750+0.1668i\\-0.5539-0.2377i\\0.4900+0.0810i\end{pmatrix},$$

$$V_4=\begin{pmatrix}0.4900-0.0810i\\-0.5539+0.2377i\\0.1750-0.1668i\\0.3123-0.0431i\\-0.5691+0.2056i\\0.4157-0.1985i\\0.0373+0.0327i\\-0.4621+0.1560i\\0.5517-0.2153i\\-0.2423+0.0985i\\-0.2423+0.0985i\\0.5517-0.2153i\\-0.4621+0.1560i\\0.0373+0.0327i\\0.4157-0.1985i\\-0.5691+0.2056i\\0.3123-0.0431i\\0.1750-0.1668i\\-0.5539+0.2377i\\0.4900-0.0810i\end{pmatrix},\quad V_5=\begin{pmatrix}-0.1682\\-0.0864\\0.1289\\0.0805\\-0.1300\\-0.0923\\0.1223\\0.1002\\-0.1157\\-0.1082\\0.1082\\0.1157\\-0.1002\\-0.1223\\0.0923\\0.1300\\-0.0805\\-0.1289\\0.0864\\0.1682\end{pmatrix},\quad V_6=\begin{pmatrix}0.1546\\0.1405\\-0.0569\\-0.1317\\0.0279\\0.1503\\0.0286\\-0.1376\\-0.0739\\0.1135\\0.1135\\-0.0739\\-0.1376\\0.0286\\0.1503\\0.0279\\-0.1317\\-0.0569\\0.1405\\0.1546\end{pmatrix},\quad V_7=\begin{pmatrix}0.1044\\0.1467\\0.0514\\-0.0600\\-0.0412\\0.0718\\0.1110\\0.0140\\-0.0886\\-0.0576\\0.0576\\0.0886\\-0.0140\\-0.1110\\-0.0718\\0.0412\\0.0600\\-0.1467\\-0.1044\end{pmatrix},$$

$$V_8=\begin{pmatrix}-0.0294\\-0.0478\\-0.0170\\0.0532\\0.1048\\0.0818\\-0.0139\\-0.1146\\-0.1390\\-0.0618\\0.0618\\0.1390\\0.1146\\0.0139\\-0.0818\\-0.1048\\-0.0532\\0.0170\\0.0478\\0.0294\end{pmatrix},\quad V_9=\begin{pmatrix}0.0119\\0.0457\\0.0748\\0.0583\\-0.0140\\-0.0999\\-0.1312\\-0.0717\\0.0459\\0.1390\\0.1390\\0.0459\\-0.0717\\-0.1312\\-0.0999\\-0.0140\\0.0583\\0.0457\\0.0748\\0.0119\end{pmatrix},\quad V_{10}=\begin{pmatrix}0.0454\\0.0585\\-0.0072\\-0.1051\\-0.1412\\-0.0757\\0.0319\\0.0864\\0.0574\\0.0055\\0.0055\\0.0574\\0.0864\\0.0319\\-0.0757\\-0.1412\\-0.1051\\-0.0072\\0.0585\\0.0454\end{pmatrix},\quad V_{11}=\begin{pmatrix}0.0367\\0.1043\\0.1409\\0.0936\\-0.0191\\-0.1178\\-0.1364\\-0.0825\\-0.0205\\0.0015\\-0.0015\\0.0205\\0.0825\\0.1364\\0.1178\\0.0191\\-0.0936\\-0.1409\\-0.1043\\-0.0367\end{pmatrix},\quad V_{12}=\begin{pmatrix}0.0204\\0.0584\\0.0862\\0.0887\\0.0834\\0.0964\\0.1262\\0.1455\\0.1383\\0.1214\\0.1214\\0.1383\\0.1455\\0.1262\\0.0964\\0.0834\\0.0887\\0.0862\\0.0584\\0.0204\end{pmatrix}.$$

Here (Λ, X) satisfies the assumptions (1) and (2) in Section 1. Then according to Theorem 2.1, by choosing $M = M_0$, $D = diag((\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}),(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}), 1, \cdots, 1)$ and then randomly generating $C_{22} = C_{22}^T$, $K_{22} = K_{22}^T$, and $C_{12} = C_{21}^T$, we get M_i, C_i, K_i, $i = 1, 2$ for QR and RRQR decompositions of matrix X, respectively. The residuals are estimated by

$$\left\|M_1 X\Lambda^2 + C_1 X\Lambda + K_1 X\right\|_2 = 108.5162,$$
$$\left\|M_2 X\Lambda^2 + C_2 X\Lambda + K_2 X\right\|_2 = 2.6976 \times 10^{-14}.$$

These results again show that the RRQR method can be much superior to the QR method.

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