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An Artificial Boundary Condition for a Class of Quasi-Newtonian Stokes Flows

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Abstract. An artificial boundary condition method, derived in terms of infinite Fourier series, is applied to solve a class of quasi-Newtonian Stokes flows. Based on the natural boundary reduction involving an artificial condition on the artificial boundary, the coupled variational problem and its numerical solution are obtained. The unique solvability of the continuous and discrete formulations are discussed, and the error analysis for the problem is also considered. Finally, an *a posteriori* error estimate for the corresponding problem is provided.

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1. Introduction

Interior and exterior nonlinear transmission problems often arise in elasticity [4, 10] and fluid mechanics [11]. The coupled finite element method (FEM) and artificial boundary condition method [8, 9], often called the natural boundary element method [5, 20] or DtN method [6, 13], can be one of the most effective methods to solve exterior nonlinear-linear transmission problems — cf. [2–4, 7, 10, 12, 15, 19] and references therein for more details.

There are several investigations for incompressible materials on bounded domains using finite or mixed finite element methods (e.g. [1, 2, 14, 16–18]), and some on unbounded domains (e.g. [3, 10, 12]), using coupling methods. The purpose of this work is to investigate a class of quasi-Newtonian Stokes flows where the kinematic viscosity is a nonlinear monotone function of the fluid velocity gradient in the plane.

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We consider the following configuration. Let Ω_0 be a bounded and simply connected domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega_0 = \Gamma_0$; and let Ω_1 be the annular region with the boundaries Γ_0 and Γ_1, where Γ_1 is another sufficiently smooth boundary with an interior region that contains Ω_0, and $\Omega^c = \mathbb{R}^2 \backslash (\overline{\Omega}_0 \cup \overline{\Omega}_1)$. In what follows, $\mathbb{R}^{2\times 2}$ denotes the space of square matrices of order 2 with real entries, $\mathbf{I} \triangleq (\delta_{ij})$ is the identity matrix of $\mathbb{R}^{2\times 2}$, and given $\boldsymbol{\tau} \triangleq (\tau_{ij})$, $\boldsymbol{\sigma} \triangleq (\boldsymbol{\sigma}_{ij}) \in \mathbb{R}^{2\times 2}$ we write

$$\mathrm{tr}(\boldsymbol{\tau}) \triangleq \sum_{i=1}^{2} \tau_{ii}\,, \qquad \boldsymbol{\sigma} : \boldsymbol{\tau} \triangleq \sum_{i,j=1}^{2} \boldsymbol{\sigma}_{ij}\tau_{ij}\,,$$

where $\boldsymbol{\sigma}(\boldsymbol{u},p) \triangleq (\boldsymbol{\sigma}_{ij}(\boldsymbol{u},p)) \in \mathbb{R}^{2\times 2}$ is the Cauchy stress tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u}) \triangleq (\varepsilon_{ij}(\boldsymbol{u})) \in \mathbb{R}^{2\times 2}$ denotes the strain tensor of small deformations with representation $\varepsilon_{ij}(\boldsymbol{u}) \triangleq \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. The constitutive equation in Ω_1 is then given by

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = \psi(|\nabla \boldsymbol{u}|)\nabla \boldsymbol{u} - p\mathbf{I}\,, \tag{1.1}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is the nonlinear kinematic viscosity function of the fluid that satisfies the Carreau law for viscoelastic flows $\psi(t) \triangleq \kappa_0 + \kappa_1(1+t^2)^{(\beta-2)/2}$, $\forall\ t, \kappa_0 \in \mathbb{R}^+$, $\beta \in [1,2]$ — cf. [18]. In passing, we note that Eq. (1.1) reduces to the usual linear model when $\beta = 2$, and that the extension of our approach to kinematic viscosity functions not satisfying Eq. (1.2) or Eq. (1.3) below (which includes the Carreau law with $\kappa_0 = 0$ or $\beta > 2$) will be reported elsewhere.

Let $\psi_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be the mapping defined by $\psi_{ij}(\boldsymbol{r}) \triangleq \psi(|\boldsymbol{r}|)r_{ij}$ for all $\boldsymbol{r} \triangleq (r_{ij}) \in \mathbb{R}^{2\times 2}$ with $i,j \in \{1,2\}$, and let the mapping $\boldsymbol{\Phi} : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $\boldsymbol{\Phi}(\boldsymbol{r}) \triangleq (\psi_{ij}(\boldsymbol{r}))$ for all $\boldsymbol{r} \in \mathbb{R}^{2\times 2}$. Then it is easy to check that ψ is of class C^1, and there exists $C_1, C_2 > 0$ such that for all $\boldsymbol{r} \triangleq (r_{ij})$, $\boldsymbol{s} \triangleq (s_{ij}) \in \mathbb{R}^{2\times 2}$ we have

$$|\psi_{ij}(\boldsymbol{r})| \le C_1 \|\boldsymbol{r}\|_{\mathbb{R}^{2\times 2}}\,, \quad \left|\frac{\partial}{\partial r_{kl}}\psi_{ij}(\boldsymbol{r})\right| \le C_1\,, \quad \forall\ i,j,k,l \in \{1,2\} \tag{1.2}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}}\psi_{ij}(\boldsymbol{r}) s_{ij}s_{kl} \ge C_2 \|\boldsymbol{s}\|^2_{\mathbb{R}^{2\times 2}}\,. \tag{1.3}$$

Furthermore, Eq. (1.1) can be rewritten as

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = \boldsymbol{\Phi}(\nabla \boldsymbol{u}) - p\mathbf{I}\,; \tag{1.4}$$

and for a linear elastic material in Ω^c this reduces to

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbf{I}\,, \tag{1.5}$$

where μ is the familiar Lamé constant.

We now take $[H^1(\Omega_1)]^2 \cap [H^1_{loc}(\Omega^c)]^2$ as the space of functions $\boldsymbol{v} \triangleq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ defined in $\Omega_1 \cup \Gamma_1 \cup \Omega^c$ such that $\boldsymbol{v}|_{\Omega_1} \in [H^1(\Omega_1)]^2$ and $\boldsymbol{v}|_{\Omega^c} \in [H^1_{loc}(\Omega^c)]^2$. For given $\boldsymbol{f} \in [L^2(\Omega_1)]^2$,

$\boldsymbol{u}_0 \in [H^{1/2}(\Gamma_1)]^2$ and $\boldsymbol{t}_0 \in [H^{-1/2}(\Gamma_1)]^2$ the nonlinear-linear exterior transmission problem can be described as: *Find a vector field* $\boldsymbol{u} \in [H^1(\Omega_1)]^2 \cap [H^1_{loc}(\Omega^c)]^2$ *and a scalar field* $p \in L^2(\mathbb{R}^2 \backslash \overline{\Omega}_0)$ *such that*

$$\begin{cases} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{f}\,, & \text{in } \Omega_1\,, \\ -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) = 0\,, & \text{in } \Omega^c\,, \\ \boldsymbol{u} = 0\,, & \text{on } \Gamma_0\,, \\ \nabla \cdot \boldsymbol{u} = 0\,, & \text{in } \Omega_1 \cup \Omega^c\,, \end{cases} \tag{1.6}$$

with the transmission conditions

$$\boldsymbol{u}^- = \boldsymbol{u}^+ + \boldsymbol{u}_0,\ \boldsymbol{\sigma}(\boldsymbol{u}^-, p^-)\boldsymbol{\nu} = \boldsymbol{\sigma}(\boldsymbol{u}^+, p^+)\boldsymbol{\nu} + \boldsymbol{t}_0 \quad \text{on } \Gamma_1\,, \tag{1.7}$$

where $\boldsymbol{u}^-, p^-$ and $\boldsymbol{u}^+, p^+$ respectively refer to the approximation to Γ_1 in Ω_1 and Ω^c, $\boldsymbol{\nu}$ denotes the unit outward normal to Γ_1, and the radiation condition at infinity is

$$\begin{cases} \boldsymbol{u} = \mathcal{O}(1) \text{ and } p = \mathcal{O}\left(\dfrac{1}{|\boldsymbol{x}|}\right), & \text{as } |\boldsymbol{x}| \to \infty,\ \boldsymbol{x} \in \mathbb{R}^2, \\ \boldsymbol{u} = \mathcal{O}\left(\dfrac{1}{|\boldsymbol{x}|}\right) \text{ and } p = \mathcal{O}\left(\dfrac{1}{|\boldsymbol{x}|}\right), & \text{as } |\boldsymbol{x}| \to \infty,\ \boldsymbol{x} \in \mathbb{R}^3. \end{cases} \tag{1.8}$$

The rest of our presentation is as follows. In Section 2, we obtain the natural integral equation for the unbounded domain cases, the coupled problem, and its well-posedness. In Section 3, we consider the finite element approximations of the equivalent variational problem and give an error estimate. Finally, in Section 4 we provide our *a posteriori* error estimate for the corresponding problem.

2. The Coupled Problem and its Well-Posedness

In this section, we derive the artificial boundary condition for the nonlinear-linear transmission problem in $\mathbb{R}^2$ and $\mathbb{R}^3$, and discuss the coupled problem and its well-posedness.

2.1. Natural integral equation

We introduce an artificial boundary Γ_R, a circle in $\mathbb{R}^2$ and a sphere in $\mathbb{R}^3$, such that the interior region of Γ_R contains $\overline{\Omega}_0 \cup \overline{\Omega}_1$; then we let Ω_2 be the annular domain bounded by Γ_1 and Γ_R and $\Omega_e \triangleq \mathbb{R}^2 \backslash (\overline{\Omega}_0 \cup \overline{\Omega}_i)$, with $\Omega_i \triangleq \Omega_1 \cup \Gamma_1 \cup \Omega_2$. Thus the original problem (1.6)–(1.8) can be separated into a problem in Ω_i and a problem in Ω_e, together with continuous conditions on the boundary Γ_R — i.e.

$$\boldsymbol{u}^{+-} = \boldsymbol{u}^{++}, \quad \boldsymbol{\sigma}(\boldsymbol{u}^{+-}, p^{+-})\boldsymbol{\nu} = \boldsymbol{\sigma}(\boldsymbol{u}^{++}, p^{++})\boldsymbol{\nu} \quad \text{on } \Gamma_R\,, \tag{2.1}$$

where $\boldsymbol{u}^{+-}$, p^{+-} and $\boldsymbol{u}^{++}$, p^{++} respectively refer to the approximation to Γ_R in Ω_2 and Ω_e and $\boldsymbol{\nu}$ denotes the unit outward normal to Γ_R.

The original problem confining in Ω_e can be represented as

$$\begin{cases} -\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p)=0\,, & \text{in } \Omega_e\,,\\ \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{\nu}=\mathscr{K}_\infty(\boldsymbol{u},p)\,, & \text{on } \Gamma_R\,,\\ \nabla\cdot\boldsymbol{u}=0\,, & \text{in } \Omega_e\,,\\ \boldsymbol{u}=\mathscr{O}(1) \text{ and } p=\mathscr{O}\left(\dfrac{1}{|\boldsymbol{x}|}\right)\,, & \text{as } |\boldsymbol{x}|\to\infty,\ x\in\mathbb{R}^2\,,\\ \boldsymbol{u}=\mathscr{O}\left(\dfrac{1}{|\boldsymbol{x}|}\right) \text{ and } p=\mathscr{O}\left(\dfrac{1}{|\boldsymbol{x}|}\right)\,, & \text{as } |\boldsymbol{x}|\to\infty,\ x\in\mathbb{R}^3\,. \end{cases} \tag{2.2}$$

Then following Refs. [8,9,20,21], using Fourier analysis we can obtain the exact artificial boundary condition in terms of $\boldsymbol{u}|_{\Gamma_R}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

2.1.1. Natural integral equation in $\mathbb{R}^2$

The exact artificial boundary condition on Γ_R in $\mathbb{R}^2$ is

$$\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{\nu}=\mathscr{K}_\infty(\boldsymbol{u},p)=T(\boldsymbol{u})\triangleq\big(T_1(\boldsymbol{u}),T_2(\boldsymbol{u})\big)^T\,, \tag{2.3}$$

where T is defined by

$$T_i(\boldsymbol{u})(R,\theta)=\frac{2\mu}{\pi R}\sum_{n=1}^{+\infty}\frac{\partial}{\partial\theta}\int_0^{2\pi}\frac{\cos n(\varphi-\theta)}{n}\frac{\partial u_i(R,\varphi)}{\partial\varphi}d\varphi\,,\quad \forall\, i\in\{1,2\},\ \theta\in[0,2\pi]\,.$$

Combining (2.3) with (1.6), we have

$$\begin{cases} -\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p)=\boldsymbol{f}\,, & \text{in } \Omega_1\subseteq\mathbb{R}^2\,,\\ -\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p)=0\,, & \text{in } \Omega_2\subseteq\mathbb{R}^2\,,\\ \boldsymbol{u}=0\,, & \text{on } \Gamma_0\,,\\ \nabla\cdot\boldsymbol{u}=0\,, & \text{in } \Omega_i\backslash\Gamma_1\subseteq\mathbb{R}^2\,,\\ \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{\nu}=\mathscr{K}_\infty(\boldsymbol{u},p)\,, & \text{on } \Gamma_R\,. \end{cases} \tag{2.4}$$

The solution of problem (2.4) and (1.7) is the restriction of the original problem (1.6)–(1.8) on the bounded domain $\Omega_i\subseteq\mathbb{R}^2$.

2.1.2. Natural integral equation in $\mathbb{R}^3$

The exact artificial boundary condition on Γ_R in $\mathbb{R}^3$ is

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{\nu}=-\frac{\mu}{R}\Bigg\{&\sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1}\frac{2l^2+4l+3}{l+2}A_l^m\mathbb{I}_l^m+\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}(l+2)B_l^m\mathbb{T}_l^m\\ &+\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(2l+2)C_l^m\mathbb{N}_l^m\Bigg\}=\mathscr{K}_\infty(\boldsymbol{u},p)\,,\end{aligned} \tag{2.5}$$

where

$$A_l^m = \frac{1}{(l+1)(2l+3)} \int_S u(R,\theta,\phi)\, \overline{\mathbb{I}}_l^m ds\,,$$
$$B_l^m = \frac{1}{l(l+1)} \int_S u(R,\theta,\phi)\, \overline{\mathbb{T}}_l^m ds\,,$$
$$C_l^m = \frac{1}{(l+1)(2l-1)} \int_S u(R,\theta,\phi)\, \overline{\mathbb{N}}_l^m ds\,,$$

and S denotes the unit spherical surface, such that the families $(\mathbb{I}_l^m, \mathbb{T}_l^m, \mathbb{N}_l^m)$ defined as in Ref. [21] form an orthogonal basis of $L^2(S)$.

Combining (2.5) with (1.6), we have

$$\begin{cases} -\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p) = \boldsymbol{f}, & \text{in } \Omega_1 \subseteq \mathbb{R}^3, \\ -\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p) = 0\,, & \text{in } \Omega_2 \subseteq \mathbb{R}^3, \\ \boldsymbol{u} = 0\,, & \text{on } \Gamma_0\,, \\ \nabla\cdot\boldsymbol{u} = 0\,, & \text{in } \Omega_i\backslash\Gamma_1 \subseteq \mathbb{R}^3, \\ \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{\nu} = \mathscr{K}_\infty(\boldsymbol{u},p)\,, & \text{on } \Gamma_R\,. \end{cases} \tag{2.6}$$

The solution of problem (2.5) and (1.7) is the restriction of the original problem (1.6)–(1.8) on the bounded domain $\Omega_i \subseteq \mathbb{R}^3$.

2.2. The equivalent variational problem and its well-posedness

2.2.1. The equivalent variational problems

Let us now focus on the problem (2.4) and (1.7) in $\mathbb{R}^2$. We use $W^{m,p}$ to denote the standard Sobolev spaces, with $\|\cdot\|$ and $|\cdot|$ referring to the corresponding norms and semi-norms. In particular, we define $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_{m,\Omega} = \|\cdot\|_{m,2,\Omega}$ and $|\cdot|_{m,\Omega} = |\cdot|_{m,2,\Omega}$. Let us also introduce the space

$$V_\Omega = \{v \mid v \in H^1(\Omega),\ v|_{\Gamma_0} = 0\}\,, \tag{2.7}$$

and the corresponding norms

$$\|v\|_{0,\Omega}^2 = \int_\Omega |v|^2 d\boldsymbol{x},$$
$$\|v\|_{V_\Omega} \triangleq \|v\|_{1,\Omega}^2 = \int_\Omega \left(|v|^2 + |\nabla v|^2\right) d\boldsymbol{x},$$
$$|v|_{V_\Omega}^2 \triangleq |v|_{1,\Omega}^2 = \int_\Omega \left(|\nabla v|^2\right) d\boldsymbol{x}.$$

Furthermore we introduce the spaces $X_\Omega = V_\Omega \times V_\Omega$ and let

$$V = \left\{(\boldsymbol{u}^-,\boldsymbol{u}^+) \mid (\boldsymbol{u}^-,\boldsymbol{u}^+) \in X_{\Omega_1}\times X_{\Omega_2}\,,\ \ \boldsymbol{u}^- = \boldsymbol{u}^+ + \boldsymbol{u}_0,\ \text{on } \Gamma_1\right\},$$
$$V^* = \left\{(\boldsymbol{u}^-,\boldsymbol{u}^+) \mid (\boldsymbol{u}^-,\boldsymbol{u}^+) \in X_{\Omega_1}\times X_{\Omega_2}\,,\ \ \boldsymbol{u}^- = \boldsymbol{u}^+,\ \text{on } \Gamma_1\right\},$$

with the norm $\|\boldsymbol{v}\|_V^2 = \|\boldsymbol{v}^-\|_{X_{\Omega_1}}^2 + \|\boldsymbol{v}^+\|_{X_{\Omega_2}}^2$ and $M = L^2(\Omega_1) \times L^2(\Omega_2)$.

The problem (2.4) and (1.7) is equivalent to the following weak formulation: *Find $\boldsymbol{u} = (\boldsymbol{u}^-, \boldsymbol{u}^+) \in V$ and $(p^-, p^+) \in M$ such that*

$$\begin{cases} D(\boldsymbol{u}, \boldsymbol{v}) + \widehat{D}(\boldsymbol{u}, \boldsymbol{v}) + B(p, \boldsymbol{v}) = L(\boldsymbol{v}) & \forall \boldsymbol{v} \in V^*, \\ B(q, \boldsymbol{u}) = 0 & \forall q \in M, \end{cases} \tag{2.8}$$

where

$$D(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega_1} \boldsymbol{\Phi}(\nabla \boldsymbol{u}) : \nabla \boldsymbol{v} d\boldsymbol{x} + 2\mu \int_{\Omega_2} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) d\boldsymbol{x} \triangleq [D(\boldsymbol{u}), \boldsymbol{v}], \tag{2.9}$$

$$\begin{aligned} \widehat{D}(\boldsymbol{u}, \boldsymbol{v}) &= -\int_{\Gamma_R} T(\boldsymbol{u}) \ \boldsymbol{v} ds \triangleq \langle \widehat{D}(\boldsymbol{u}), \boldsymbol{v} \rangle \\ &= \frac{2\mu}{\pi} \sum_{i=1}^{2} \sum_{n=1}^{+\infty} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos n(\varphi - \theta)}{n} \frac{\partial u_i(R, \varphi)}{\partial \varphi} \frac{\partial v_i(R, \theta)}{\partial \theta} d\varphi d\theta, \end{aligned} \tag{2.10}$$

$$B(q, \boldsymbol{v}) = -\int_{\Omega_1 \cup \Omega_2} q \ \nabla \cdot \boldsymbol{v} d\boldsymbol{x} \triangleq [B(\boldsymbol{v}), q], \tag{2.11}$$

$$L(\boldsymbol{v}) = \int_{\Omega_1 \cup \Omega_2} \boldsymbol{f} \boldsymbol{v} d\boldsymbol{x} + \int_{\Gamma_1} \boldsymbol{t}_0 \boldsymbol{v} ds \triangleq [L, \boldsymbol{v}]. \tag{2.12}$$

In practice, we need to truncate the series in Eq. (2.10) for some nonnegative integer N — i.e.

$$\widehat{D}_N(\boldsymbol{u}, \boldsymbol{v}) = \frac{2\mu}{\pi} \sum_{i=1}^{2} \sum_{n=1}^{N} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos n(\varphi - \theta)}{n} \frac{\partial u_i(R, \varphi)}{\partial \varphi} \frac{\partial v_i(R, \theta)}{\partial \theta} d\varphi d\theta, \tag{2.13}$$

so (2.8) can be changed into the equivalent variational problem: *Find $\boldsymbol{u}^N = (\boldsymbol{u}^{N-}, \boldsymbol{u}^{N+}) \in V$ and $(p^{N-}, p^{N+}) \in M$ such that*

$$\begin{cases} D_N(\boldsymbol{u}^N, \boldsymbol{v}) + \widehat{D}_N(\boldsymbol{u}^N, \boldsymbol{v}) + B(p^N, \boldsymbol{v}) = L(\boldsymbol{v}) & \forall \boldsymbol{v} \in V^*, \\ B(q, \boldsymbol{u}^N) = 0 & \forall q \in M. \end{cases} \tag{2.14}$$

2.2.2. Unique solvability

We now focus on the unique solvability of the variational formulations (2.8) and (2.14). And for this purpose we introduce the following abstract theorem [12,22].

Theorem 2.1. *Let V, M and V^* be Hilbert spaces and $A : V \to V'$ and $B : V \to M'$ be nonlinear and linear operators, respectively; and let $W \triangleq \ker(B) = \{\boldsymbol{v} \in V : [B(\boldsymbol{v}), q] = 0, \ \forall q \in M\}$. Assume that A is Lipschitz-continuous on V and that $A(\overline{\boldsymbol{u}} + \cdot)$ is uniformly* *strongly monotone on W for all $\overline{\boldsymbol{u}} \in V$ — i.e. there exist two positive constants γ and α such that*

$$\begin{aligned} &\|A(\boldsymbol{w}) - A(\boldsymbol{u})\|_{V'} \le \gamma \|\boldsymbol{w} - \boldsymbol{u}\|_V && \forall\, \boldsymbol{w}, \boldsymbol{u} \in V\,, \\ &[A(\overline{\boldsymbol{u}} + \boldsymbol{w}) - A(\overline{\boldsymbol{u}} + \boldsymbol{u}), \boldsymbol{w} - \boldsymbol{u}] \ge \alpha \|\boldsymbol{w} - \boldsymbol{u}\|_V^2 && \forall\, \overline{\boldsymbol{u}} \in V,\ \ \boldsymbol{w}, \boldsymbol{u} \in W\,, \end{aligned}$$

and we also assume that there exists a positive constant β such that

$$\sup_{v \in V^*, v \neq 0} \frac{[B(\boldsymbol{v}), q]}{\|\boldsymbol{v}\|_{V^*}} \ge \beta \|q\|_M \quad \forall\, q \in M\,.$$

Then given $(F, G) \in V' \times M'$, there exists a unique $(\boldsymbol{u}, p) \in V \times M$ such that

$$\begin{aligned} &[A(\boldsymbol{u}), \boldsymbol{v}] + [B(\boldsymbol{v}), p] = F(\boldsymbol{v}) && \forall \boldsymbol{v} \in V^*\,, \\ &[B(\boldsymbol{u}), q] = G(q) && \forall q \in M\,, \end{aligned}$$

and the following estimates hold:

$$\|\boldsymbol{u}\|_V \le \frac{1}{\alpha}\|F\| + \frac{1}{\beta}\left(1 + \frac{\gamma}{\alpha}\right)\|G\|\,, \qquad \|p\|_M \le \frac{1}{\beta}\left(1 + \frac{\gamma}{\alpha}\right)\left(\|F\| + \frac{\gamma}{\beta}\|G\|\right)\,.$$

We separate the proof of unique solvability into several Lemmas as below.

Lemma 2.1. *The nonlinear operator $D : V \to V'$ defined by*

$$[D(\boldsymbol{u}), \boldsymbol{v}] = \int_{\Omega_1} \boldsymbol{\Phi}(\nabla \boldsymbol{u}_1) : \nabla \boldsymbol{v}_1 d\boldsymbol{x} + 2\mu \int_{\Omega_2} \boldsymbol{\varepsilon}(\boldsymbol{u}_2) : \boldsymbol{\varepsilon}(\boldsymbol{v}_2) d\boldsymbol{x}$$

for all $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2), \boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in V$ is Lipschitz-continuous and strongly monotone on V.

Proof. From the definition of $\boldsymbol{\Phi}$ and ψ, we have $\boldsymbol{\Phi}(\nabla \boldsymbol{u}) = \psi(|\nabla \boldsymbol{u}|)\nabla \boldsymbol{u} = (\psi_{ij}(\nabla \boldsymbol{u}))$ for $i, j \in \{1, 2\}$. With $\nabla_{ij}\boldsymbol{v} \triangleq \partial v_i / \partial x_j$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$ we can write

$$[D(\boldsymbol{u}), \boldsymbol{v}] = \sum_{i,j=1}^{2} \int_{\Omega_1} \psi_{ij}(\nabla \boldsymbol{u}_1) \nabla_{ij} \boldsymbol{v}_1 d\boldsymbol{x} + 2\mu \sum_{i,j=1}^{2} \int_{\Omega_2} \varepsilon_{ij}(\boldsymbol{u}_2) \varepsilon_{ij}(\boldsymbol{v}_2) d\boldsymbol{x}\,.$$

Since ψ is of class C^1,

$$\psi_{ij}(\nabla \boldsymbol{u}_1) - \psi_{ij}(\nabla \boldsymbol{v}_1) = \int_0^1 \Big\{ \sum_{k,l=1}^{2} \frac{\partial \psi_{ij}}{\partial \delta_{kl}} (\boldsymbol{\delta}(\boldsymbol{x}, t)) \nabla_{kl}(\boldsymbol{u}_1 - \boldsymbol{v}_1) \Big\} dt \tag{2.15}$$

with $\boldsymbol{\delta}(\boldsymbol{x}, t) = \nabla \boldsymbol{v}_1 + t \nabla(\boldsymbol{u}_1 - \boldsymbol{v}_1)$. Thus from (1.2) and (2.15) we obtain

$$\|\boldsymbol{\Phi}(\nabla \boldsymbol{u}_1) - \boldsymbol{\Phi}(\nabla \boldsymbol{v}_1)\|_{X'_{\Omega_1}} \le C_1 \|\nabla(\boldsymbol{u}_1 - \boldsymbol{v}_1)\|_{0,\Omega_1} \le C_1 \|\boldsymbol{u}_1 - \boldsymbol{v}_1\|_{X_{\Omega_1}}\,, \tag{2.16}$$

where C_1 is a positive constant. Then combining (1.3) with (2.15),

$$\begin{aligned}&\sum_{i,j=1}^{2}\int_{\Omega_1}\left\{\psi_{ij}(\nabla \boldsymbol{u}_1)-\psi_{ij}(\nabla \boldsymbol{v}_1)\right\}\left\{\nabla_{ij}(\boldsymbol{u}_1)-\nabla_{ij}(\boldsymbol{v}_1)\right\}d\boldsymbol{x}\\ \geq &C_2\|\nabla(\boldsymbol{u}_1-\boldsymbol{v}_1)\|_{0,\Omega_1}^2=C_2|\boldsymbol{u}_1-\boldsymbol{v}_1|_{X_{\Omega_1}}^2,\end{aligned} \tag{2.17}$$

where C_2 is a positive constant.

It is notable that the mapping $\boldsymbol{\varepsilon}:[H^1(\Omega_2)]^2\to[L^2(\Omega_2)]^4$ is defined by

$$\varepsilon_{ij}(\boldsymbol{u}_2)=\frac{1}{2}\left(\frac{\partial u_{2i}}{\partial x_j}+\frac{\partial u_{2j}}{\partial x_i}\right)$$

and satisfies

$$\begin{aligned}\sum_{i,j=1}^{2}\left\|\varepsilon_{ij}(\boldsymbol{v}_2)\right\|_{0,\Omega_2}^2\leq&\frac{1}{2}\sum_{i,j=1}^{2}\left\{\left\|\frac{\partial v_{2i}}{\partial x_j}\right\|_{0,\Omega_2}^2+\left\|\frac{\partial v_{2j}}{\partial x_i}\right\|_{0,\Omega_2}^2\right\}\\ \leq&\left\|\boldsymbol{v}_2\right\|_{X_{\Omega_2}}^2\quad\forall\ \boldsymbol{v}_2\in X_{\Omega_2}\end{aligned} \tag{2.18}$$

by Korn's inequality, so we also have

$$\sum_{i,j=1}^{2}\left\|\varepsilon_{ij}(\boldsymbol{v}_2)\right\|_{0,\Omega_2}^2\geq C_3\left\|\boldsymbol{v}_2\right\|_{X_{\Omega_2}}^2\quad\forall\ \boldsymbol{v}_2\in X_{\Omega_2}\,, \tag{2.19}$$

where C_3 is a positive constant. □

Now in virtue of the linearity of $\boldsymbol{\varepsilon}$, (2.16), (2.18) and (2.17), (2.19) we can get the desired results. From Refs. [8, 9, 20], for the natural integral equation we have:

Lemma 2.2. *The linear operator* $\widehat{D},\widehat{D}_N:V\to V'$ *given by* (2.10) *and* (2.13) *satisfies*

$$\left|\langle\widehat{D}(\boldsymbol{u}),\boldsymbol{v}\rangle\right|\leq C_4\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V\,,\quad\langle\widehat{D}(\boldsymbol{u}),\boldsymbol{u}\rangle\geq 0\,;$$

and for the linear operator B the inf-sup condition as

Lemma 2.3. *There exists a positive constant* β *such that*

$$\sup_{v\in V^*,v\neq 0}\frac{[B(\boldsymbol{v}),q]}{\|\boldsymbol{v}\|_{V^*}}\geq\beta\|q\|_M\,,\quad\forall\ q\in M\,.$$

On combining Theorem 2.1 with Lemmas 2.1 to 2.3, we reach the following main theorem on the unique solvability and error estimate of the variational problems (2.8) and (2.14).

Theorem 2.2. *There exists a unique solution* $(\boldsymbol{u},p)\in V\times M$ *of* (2.8) *and a unique solution* $(\boldsymbol{u}^N,p^N)\in V\times M$ *of problem* (2.14). *Furthermore, if* $\boldsymbol{u}|_{\Gamma_R}\in[H^{m+\frac{1}{2}}(\Gamma_R)]^2$ *with positive integer m, then the following error estimate holds:*

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{V^*}+\left\|p-p^N\right\|_M\le\frac{C}{(N+1)^m}\left\|\boldsymbol{u}|_{\Gamma_R}\right\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2}. \tag{2.20}$$

Proof. Although we only show the unique solvability of (2.8), the unique results for the problem (2.14) can be obtained similarly. Let $A,A_N:V\to V'$ be the operator given by

$$[A(\boldsymbol{u}),\boldsymbol{v}]=[D(\boldsymbol{u}),\boldsymbol{v}]+\left\langle\widehat{D}(\boldsymbol{u}),\boldsymbol{v}\right\rangle,\quad [A_N(\boldsymbol{u}),\boldsymbol{v}]=[D(\boldsymbol{u}),\boldsymbol{v}]+\left\langle\widehat{D}_N(\boldsymbol{u}),\boldsymbol{v}\right\rangle. \tag{2.21}$$

Then by Lemmas 2.1–2.3, one can verify that the conditions of Theorem 2.1 are satisfied.

It remains to show (2.20). We assume that $(\boldsymbol{u},p)\in V\times M$and $(\boldsymbol{u}^N,p^N)\in V\times M$ represent the unique solution of the variational problems (2.8) and (2.14), respectively. For $\boldsymbol{u}^e=\boldsymbol{u}-\boldsymbol{u}^N$ and $p^e=p-p^N$, we have $(\boldsymbol{u}^e,p^e)\in V^*\times M$; and from (2.8), (2.14) and (2.21) that

$$\begin{cases}[A(\boldsymbol{u}),\boldsymbol{v}]-[A_N(\boldsymbol{u}^N),\boldsymbol{v}]+[B(\boldsymbol{v}),p^e]=0 & \forall\boldsymbol{v}\in V^*,\\ [B(\boldsymbol{u}^e),q]=0 & \forall q\in M.\end{cases} \tag{2.22}$$

In particular, we take $\boldsymbol{v}=\boldsymbol{u}^e$ and $q=p^e$ in (2.22) such that

$$[A(\boldsymbol{u}),\boldsymbol{u}^e]-[A_N(\boldsymbol{u}^N),\boldsymbol{u}^e]=0. \tag{2.23}$$

Then combining Lemmas 2.1–2.2 with (2.8),(2.14) (2.21) and (2.23), we deduce

$$\begin{aligned}\alpha\|u^e\|_V^2&\le[A_N(\boldsymbol{u})-A_N(\boldsymbol{u}^N),u^e]\\&=[A_N(\boldsymbol{u}),u^e]-[A(\boldsymbol{u}),u^e]=\left\langle\widehat{D}_N(\boldsymbol{u}),u^e\right\rangle-\left\langle\widehat{D}(\boldsymbol{u}),u^e\right\rangle.\end{aligned} \tag{2.24}$$

Following Refs. [8,9], we have the estimate

$$\left|\langle\widehat{D}(\boldsymbol{u}),\boldsymbol{v}\rangle-\langle\widehat{D}_N(\boldsymbol{u}),\boldsymbol{v}\rangle\right|\le\frac{C}{(N+1)^m}\left\|\boldsymbol{u}|_{\Gamma_R}\right\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2}\|\boldsymbol{v}\|_{V^*}, \tag{2.25}$$

where C is a positive constant, independent of N and m. From inequality (2.25), the inequality (2.24) yields

$$\alpha\|u^e\|_{V^*}^2\le\frac{C}{(N+1)^m}\left\|\boldsymbol{u}|_{\Gamma_R}\right\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2}\|u^e\|_{V^*}, \tag{2.26}$$

and hence

$$\|u^e\|_{V^*}\le\frac{C}{\alpha(N+1)^m}\left\|\boldsymbol{u}|_{\Gamma_R}\right\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2}. \tag{2.27}$$

From (2.8), (2.14) and (2.22), for all $\boldsymbol{v}\in V^*$ we obtain

$$\begin{aligned}[B(\boldsymbol{v}),p^e]&=[A_N(\boldsymbol{u}^N),\boldsymbol{v}]-[A(\boldsymbol{u}),\boldsymbol{v}]\\&=[A_N(\boldsymbol{u}^N),\boldsymbol{v}]-[A_N(\boldsymbol{u}),\boldsymbol{v}]+[A_N(\boldsymbol{u}),\boldsymbol{v}]-[A(\boldsymbol{u}),\boldsymbol{v}]\\&=[A_N(\boldsymbol{u}^N),\boldsymbol{v}]-[A_N(\boldsymbol{u}),\boldsymbol{v}]+\left\langle\widehat{D}_N(\boldsymbol{u}),\boldsymbol{v}\right\rangle-\left\langle\widehat{D}(\boldsymbol{u}),\boldsymbol{v}\right\rangle.\end{aligned}$$

Then from the Lipschitz-continuity of A_N and (2.25), we obtain

$$\frac{[B(\boldsymbol{v}),p^e]}{\|\boldsymbol{v}\|_{V^*}} \leq \left\{ M\,\|u^e\|_{V^*} + \frac{C}{(N+1)^m}\left\|\boldsymbol{u}|_{\Gamma_R}\right\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2} \right\}, \quad \forall \boldsymbol{v}\in V^*, \tag{2.28}$$

so combining Lemma 2.3 with (2.28) together with (2.27) we get the desired results. □

3. Finite Element Approximation

In this section, we consider the finite element approximation of problem (2.14). We first divide Ω_1 and Ω_2 into quasi-uniform triangulation meshes $\mathscr{T}_h$ with mesh size $h \in I$, where I is an at most numerable set of indexes, such that the nodes on Γ_1 are coincident. Let $\mathscr{N}_h$ denote the set of nodes in the domain $\Omega_1 \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_R$; and suppose that $\widetilde{V}_h$, $\widetilde{V}_h^*$ and $\widetilde{M}_h$ are finite-dimensional subspaces of V, V^* and M respectively, where

$$\widetilde{V}_h = \left\{ (\boldsymbol{u}_h^-,\boldsymbol{u}_h^+) \in X_{\Omega_1,h}\times X_{\Omega_2,h} \mid \forall \boldsymbol{b}\in\mathscr{N}_h\cap\Gamma_1,\ \boldsymbol{u}_h^-(\boldsymbol{b}) = \boldsymbol{u}_h^+(\boldsymbol{b}) + \boldsymbol{u}_0(\boldsymbol{b}) \right\},$$
$$\widetilde{V}_h^* = \left\{ (\boldsymbol{u}_h^-,\boldsymbol{u}_h^+) \in X_{\Omega_1,h}\times X_{\Omega_2,h} \mid \forall \boldsymbol{b}\in\mathscr{N}_h\cap\Gamma_1,\ \boldsymbol{u}_h^-(\boldsymbol{b}) = \boldsymbol{u}_h^+(\boldsymbol{b}) \right\},$$

and a discrete inf-sup condition is satisfied — viz. there exists a positive constant β^*, independent of h, such that

$$\sup_{\boldsymbol{v}_h\in\widetilde{V}_h^*,\boldsymbol{v}_h\neq 0} \frac{[B(\boldsymbol{v}_h),q_h]}{\|\boldsymbol{v}_h\|_{\widetilde{V}_h^*}} \geq \beta^*\left\|q_h\right\|_{\widetilde{M}_h} \quad \forall q_h\in\widetilde{M}_h\,. \tag{3.1}$$

The discrete problem corresponding to (2.14) is then: *Find* $\boldsymbol{u}_h^N = (\boldsymbol{u}_h^{N-},\boldsymbol{u}_h^{N+}) \in \widetilde{V}_h$ *and* $p_h^N = (p_h^{N-},p_h^{N+}) \in \widetilde{M}_h$ *such that*

$$\begin{cases} D_N(\boldsymbol{u}_h^N,\boldsymbol{v}_h) + \widehat{D}_N(\boldsymbol{u}_h^N,\boldsymbol{v}_h) + B(p_h^N,\boldsymbol{v}_h) = L(\boldsymbol{v}_h) & \forall \boldsymbol{v}_h\in\widetilde{V}_h^*\,, \\ B(q_h,\boldsymbol{u}_h^N) = 0 & \forall q_h\in\widetilde{M}_h\,. \end{cases} \tag{3.2}$$

For the errors $\|\boldsymbol{u}^N - \boldsymbol{u}_h^N\|_{V^*}$ and $\|p^N - p_h^N\|_M$, by a standard technique for mixed finite element methods [23] we have

$$\left\|\boldsymbol{u}^N - \boldsymbol{u}_h^N\right\|_{V^*} + \left\|p^N - p_h^N\right\|_M \leq C_0\left\{ \inf_{\boldsymbol{v}_h\in\widetilde{V}_h}\left\|\boldsymbol{u}^N - \boldsymbol{v}_h\right\|_{V^*} + \inf_{q_h\in\widetilde{M}_h}\left\|p^N - q_h\right\|_M \right\}, \tag{3.3}$$

and for the unique solvability of (3.2) one can refer to Ref. [22] for more details.

We now present the error estimate between the solutions of (2.8) and (3.2).

Theorem 3.1. *Suppose that* $(\boldsymbol{u},p)\in V\times M$ *and* $(\boldsymbol{u}_h^N,p_h^N)\in\widetilde{V}_h\times\widetilde{M}_h$ *are the solutions of problems* (2.8) *and* (2.14)*, respectively; and that* $\boldsymbol{u}|_{\Gamma_R}\in[H^{m+\frac{1}{2}}(\Gamma_R)]^2$ *for some positive integer m. Then there exists a positive constant C, independent of h, N and m, such that*

$$\begin{aligned}&\left\|\boldsymbol{u} - \boldsymbol{u}_h^N\right\|_{V^*} + \left\|p - p_h^N\right\|_M \\ &\leq C\left\{ \inf_{\boldsymbol{v}_h\in\widetilde{V}_h}\left\|\boldsymbol{u} - \boldsymbol{v}_h\right\|_{V^*} + \inf_{q_h\in\widetilde{M}_h}\left\|p - q_h\right\|_M + \frac{1}{(N+1)^m}\left\|\boldsymbol{u}|_{\Gamma_R}\right\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2} \right\}. \end{aligned} \tag{3.4}$$

Proof. From the triangle inequality and (3.3),

$$
\begin{aligned}
&\left\|\boldsymbol{u}-\boldsymbol{u}_h^N\right\|_{V^*}+\left\|p-p_h^N\right\|_M\\
\le\ &\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{V^*}+\left\|\boldsymbol{u}^N-\boldsymbol{u}_h^N\right\|_{V^*}+\left\|p-p^N\right\|_M+\left\|p^N-p_h^N\right\|_M\\
\le\ &\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{V^*}+\left\|p-p^N\right\|_M+C_0\left\{\inf_{\boldsymbol{v}_h\in\widetilde{V}_h}\left\|\boldsymbol{u}^N-\boldsymbol{v}_h\right\|_{V^*}+\inf_{q_h\in\widetilde{M}_h}\left\|p^N-q_h\right\|_M\right\}. \qquad (3.5)
\end{aligned}
$$

Next, we observe that

$$
\inf_{\boldsymbol{v}_h\in\widetilde{V}_h}\left\|\boldsymbol{u}^N-\boldsymbol{v}_h\right\|_{V^*}\le\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{V^*}+\inf_{\boldsymbol{v}_h\in\widetilde{V}_h}\left\|\boldsymbol{u}-\boldsymbol{v}_h\right\|_{V^*}, \qquad (3.6)
$$

and

$$
\inf_{q_h\in\widetilde{M}_h}\left\|p^N-q_h\right\|_M\le\left\|p-p^N\right\|_M+\inf_{q_h\in\widetilde{M}_h}\left\|p-q_h\right\|_M. \qquad (3.7)
$$

Then combining (3.5)–(3.7) with (2.20), we obtain the desired results. □

Furthermore, let us assume the solution of (2.8) is $(\boldsymbol{u},p)$ with $\boldsymbol{u}\in X\cap([H^{m+1}(\Omega_1)]^2\times[H^{m+1}(\Omega_2)]^2)$ and $p\in M\cap(H^m(\Omega_1)\times H^m(\Omega_2))$, where m is a positive integer; and also that the subspaces $\widetilde{V}_h$ and $\widetilde{M}_h$ satisfy the following approximate properties:

$$
\inf_{\boldsymbol{v}_h\in\widetilde{V}_h}\left\|\boldsymbol{u}-\boldsymbol{v}_h\right\|_{V^*}\le Ch^m|\boldsymbol{u}|_{k+1,V}, \quad \inf_{q_h\in\widetilde{M}_h}\left\|p-q_h\right\|_M\le Ch^m|p|_{k+1,M}. \qquad (3.8)
$$

Then from (3.8) the inequality (3.4) can be changed into

$$
\begin{aligned}
&\left\|\boldsymbol{u}-\boldsymbol{u}_h^N\right\|_{V^*}+\left\|p-p_h^N\right\|_M\\
\le\ &C\left\{h^m(|\boldsymbol{u}|_{k+1,V}+|p|_{k,M})+\frac{1}{(N+1)^m}\left\|\boldsymbol{u}|_{\Gamma_R}\right\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2}\right\}, \qquad (3.9)
\end{aligned}
$$

and if we take $N=\mathscr{O}(h^{-1})$ then (3.9) can be rewritten as

$$
\left\|\boldsymbol{u}-\boldsymbol{u}_h^N\right\|_{V^*}+\left\|p-p_h^N\right\|_M\le Ch^m\left\{|\boldsymbol{u}|_{k+1,V}+|p|_{k,M}+\left\|\boldsymbol{u}|_{\Gamma_R}\right\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2}\right\}. \qquad (3.10)
$$

From the error estimate (3.9), it follows that the error can be affected not only by the order of the artificial boundary condition but also by the finite element approximation.

4. The *a posteriori* error estimator

4.1. Preliminaries

First, in order to specify V_h, V_h^* and M_h we let $P_1(T)$ and $P_2(T)$ be defined as in Ref. [12] for each $T\in\mathscr{T}_h$, so we set

$$
V_h\triangleq\left\{(\boldsymbol{v}_h^-,\boldsymbol{v}_h^+)\in\widetilde{V}_h \mid \boldsymbol{v}_h^+|_T,\boldsymbol{v}_h^-|_T\in[P_2(T)]^2,\ \forall T\in\mathscr{T}_h\right\}, \qquad (4.1)
$$

$$
V_h^*\triangleq\left\{(\boldsymbol{v}_h^-,\boldsymbol{v}_h^+)\in\widetilde{V}_h^* \mid \boldsymbol{v}_h^+|_T,\boldsymbol{v}_h^-|_T\in[P_2(T)]^2,\ \forall T\in\mathscr{T}_h\right\}, \qquad (4.2)
$$

$$
M_h\triangleq\left\{(q_h^-,q_h^+)\in L^2(\Omega_1)\times L^2(\Omega_2) \mid q_h^+|_T,\ q_h^-|_T\in P_1(T),\ \forall T\in\mathscr{T}_h\right\}. \qquad (4.3)
$$

Then V_h and M_h constitute the simplest Hood and Taylor finite element subspaces satisfying the inf-sup condition (3.1), and the approximation properties (3.8) hold.

For simplicity, $\boldsymbol{\nu}_T$ denotes the unit outward normal, and $\mathscr{E}(T)$ refers to the set of its edges. With $\mathscr{E}_h \triangleq \cup\{\mathscr{E}(T) \mid T \in \mathscr{T}_h\}$, this set can be split into the form $\mathscr{E} \triangleq \mathscr{E}_h(\Omega_i) \cup \mathscr{E}_h(\Gamma_0) \cup \mathscr{E}_h(\Gamma_R)$, where $\mathscr{E}_h(\Omega_i) \triangleq \{S \mid S \in \mathscr{E}_h,\ S \subseteq \Omega_i\}$, $\mathscr{E}_h(\Gamma_0) \triangleq \{S \mid S \in \mathscr{E}_h,\ S \subseteq \Gamma_0\}$ and $\mathscr{E}_h(\Gamma_R) \triangleq \{S \mid S \in \mathscr{E}_h,\ S \subseteq \Gamma_R\}$. In addition, from Ref. [12] $X_{\Omega,h}$ possesses the local properties stated below.

Lemma 4.1. *Let $\mathscr{I} : V \to V_h$ be the Clément interpolation operator defined as in Ref. [12]. Then there exist two positive constants C_1 and C_2, independent of h, such that*

$$\left\|\boldsymbol{v} - \mathscr{I}_h(\boldsymbol{v})\right\|_{[L^2(T)]^2\times[L^2(T)]^2} \le C_1 h_T \|\boldsymbol{v}\|_{[H^1(\Delta(T))]^2\times[H^1(\Delta(T))]^2}$$

and

$$\left\|\boldsymbol{v} - \mathscr{I}_h(\boldsymbol{v})\right\|_{[L^2(S)]^2} \le C_2 h_S^{1/2} \|\boldsymbol{v}\|_{[H^1(\Delta(S))]^2}$$

$\forall \boldsymbol{v} \in V$, $T \in \mathscr{T}_h$ and $S \in \mathscr{E}_h$, where $h_T(h_S)$ denotes the diameter of T (S), $\Delta(T) \triangleq \cup\{\widetilde{T} \mid \widetilde{T} \in \mathscr{T}_h,\ \widetilde{T} \cap T \neq \emptyset\}$ and $\Delta(S) \triangleq \cup\{\widetilde{T} \mid \widetilde{T} \in \mathscr{T}_h,\ \widetilde{T} \cap S \neq \emptyset\}$.

It is notable that the family of triangulations $\{\mathscr{T}_h\}_{h\in I}$ is regular, so the numbers of triangles in $\Delta(T)$ and $\Delta(S)$ are bounded, independent of h.

4.2. Main estimates

We may now provide a reliable *a posteriori* error estimate. For this purpose, we introduce a functional depending on the mapping $\boldsymbol{\Phi}$ and artificial boundary condition $\mathscr{K}_\infty$. More precisely, let $\mathscr{H} : V \to \mathbb{R}$ be defined by

$$\mathscr{H}(\boldsymbol{v}) \triangleq \int_{\Omega_1} \boldsymbol{\Phi}(\nabla \boldsymbol{v}_1) d\boldsymbol{x} + \mu \int_{\Omega_2} \varepsilon(\boldsymbol{v}_2) : \varepsilon(\boldsymbol{v}_2) + \frac{1}{2}\langle \mathscr{K}_\infty(\boldsymbol{v}_2), \boldsymbol{v}_2\rangle_{\Gamma_R},$$

where $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in V$. Then we have the following result.

Lemma 4.2. *The functional $\mathscr{H}$ has continuous second-order Gâteaux derivatives and there exist two positive constants C_1 and C_2 such that*

$$C_1\|\boldsymbol{v}\|_V^2 \le (\mathscr{D}^2\mathscr{H})(\boldsymbol{z})(\boldsymbol{v}, \boldsymbol{v}) \le C_2\|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{z}, \boldsymbol{v} \in V\,. \tag{4.4}$$

Proof. From the definition of the Gâteaux derivative, we know the first-order Gâteaux derivative applies V to its dual and the second-order Gâteaux derivative applies V to the dual of $V \times V$ — i.e.

$$\mathscr{D}\mathscr{H}(\boldsymbol{z})(\boldsymbol{v}) \triangleq \lim_{t\to 0} \frac{\mathscr{H}(\boldsymbol{z} + t\boldsymbol{v}) - \mathscr{H}(\boldsymbol{z})}{t} \quad \forall\, \boldsymbol{z}, \boldsymbol{v} \in V\,, \tag{4.5}$$

so that

$$\mathscr{D}\mathscr{H}(\boldsymbol{z})(\boldsymbol{v}) = \sum_{i,j=1}^{2}\int_{\Omega_1}\psi_{ij}(\nabla \boldsymbol{z}_1)\nabla_{ij}\boldsymbol{v}_1 d\boldsymbol{x} + 2\mu\sum_{i,j=1}^{2}\int_{\Omega_2}\varepsilon_{ij}(\boldsymbol{z}_2)\varepsilon_{ij}(\boldsymbol{v}_2)d\boldsymbol{x} + \left\langle \mathscr{K}_\infty(\boldsymbol{z}_2)\boldsymbol{v}_2\right\rangle \tag{4.6}$$

where ψ_{ij} and ∇_{ij} are defined in Lemma 2.1, and

$$\mathscr{D}^2\mathscr{H}(\boldsymbol{z})(\boldsymbol{v},\boldsymbol{w}) \triangleq \lim_{t\to 0}\frac{\mathscr{D}\mathscr{H}(\boldsymbol{z}+t\boldsymbol{v})(\boldsymbol{w}) - \mathscr{D}\mathscr{H}(\boldsymbol{z})(\boldsymbol{w})}{t} \quad \forall\, \boldsymbol{z},\boldsymbol{v},\boldsymbol{w}\in V\,. \tag{4.7}$$

Then it follows from (4.5)–(4.7) that

$$\begin{aligned}\mathscr{D}^2\mathscr{H}(\boldsymbol{z})(\boldsymbol{v},\boldsymbol{w}) = &\sum_{i,j=1}^{2}\sum_{k,l=1}^{2}\int_{\Omega_1}\frac{\partial\psi_{ij}(\nabla \boldsymbol{z}_1)}{\partial\delta_{kl}}\nabla_{kl}\boldsymbol{v}_1\nabla_{ij}\boldsymbol{w}_1 d\boldsymbol{x}\\ &+2\mu\sum_{i,j=1}^{2}\int_{\Omega_2}\varepsilon_{ij}(\boldsymbol{v}_2)\varepsilon_{ij}(\boldsymbol{w}_2)d\boldsymbol{x} + \left\langle \mathscr{K}_\infty(\boldsymbol{v}_2),\boldsymbol{w}_2\right\rangle,\end{aligned} \tag{4.8}$$

hence from (1.2)–(1.3), Lemmas (2.1) and (2.2) we have (4.4). The continuity of $\mathscr{D}^2\mathscr{H}$ follows from (4.8), as the function ψ is of class C^1. □

Now let $(\boldsymbol{u}^N,p^N)\in V\times M$ and $(\boldsymbol{u}_h^N,p_h^N)\in V_h\times M_h$ be the unique solution of the continuous and discrete formulations (2.14) and (3.2), respectively. Then we have the following result.

Lemma 4.3. *There exists a positive constant C_0, independent of h, such that*

$$\begin{aligned}&\left\|\boldsymbol{u}^N-\boldsymbol{u}_h^N\right\|_{V^*}+\left\|p^N-p_h^N\right\|_M\\ \le\, &C_0\sup_{\substack{(v,q)\in V^*\times M\\ \|(v,q)\|\le 1}}\Big\{[D_N(\boldsymbol{u}^N)-D_N(\boldsymbol{u}_h^N),\boldsymbol{v}]+\langle\widehat{D}_N(\boldsymbol{u}^N-\boldsymbol{u}_h^N),\boldsymbol{v}\rangle\\ &+\left[B(\boldsymbol{v}),p^N-p_h^N\right]+\left[B(\boldsymbol{u}^N-\boldsymbol{u}_h^N,q\right]\Big\}.\end{aligned} \tag{4.9}$$

Proof. From the continuity of $\mathscr{D}^2\mathscr{H}$ there exists $\boldsymbol{z}^N\in V$, $\boldsymbol{z}^N$. a convex linear combination of $\boldsymbol{u}^N$ and $\boldsymbol{u}_h^N$ such that

$$\mathscr{D}^2\mathscr{H}(\boldsymbol{z}^N)(\boldsymbol{u}^N-\boldsymbol{u}_h^N,\boldsymbol{v})=\mathscr{D}\mathscr{H}(\boldsymbol{u}^N)(\boldsymbol{v})-\mathscr{D}\mathscr{H}(\boldsymbol{u}_h^N)(\boldsymbol{v}),\quad \forall \boldsymbol{v}\in V^*. \tag{4.10}$$

From Lemma 2.2 and (4.4),

$$\mathscr{D}^2\mathscr{H}(\boldsymbol{z}^N)(\boldsymbol{v},\boldsymbol{v})\ge C_1\|\boldsymbol{v}\|^2_{V^*},\quad \forall \boldsymbol{v}\in V^*\,.$$

Moreover, from Lemma 2.3 the linear operator B satisfies the continuous inf-sup condition, so Brezzi's theory implies that there exists a positive constant C_0 such that

$$\|(\widetilde{\boldsymbol{u}},\widetilde{p})\|_{V^*\times M}\le \sup_{\substack{(v,q)\in V^*\times M\\ \|(v,q)\|\le 1}}\Big\{\mathscr{D}^2\mathscr{H}(\boldsymbol{z}^N)(\widetilde{\boldsymbol{u}},\boldsymbol{v})+[B(\boldsymbol{v}),\widetilde{p}]+[B(\widetilde{\boldsymbol{u}}),q]\Big\}, \tag{4.11}$$

for all $(\widetilde{\boldsymbol{u}},\widetilde{p})\in V^*\times M$. In particular, we take $(\widetilde{\boldsymbol{u}},\widetilde{p})=(\boldsymbol{u}^N-\boldsymbol{u}_h^N,p^N-p_h^N)$, when (4.11) becomes

$$\begin{aligned}&\left\|\boldsymbol{u}^N-\boldsymbol{u}_h^N\right\|_{V^*}+\left\|p^N-p_h^N\right\|_M\\ \leq&\sup_{\substack{(\boldsymbol{v},q)\in V^*\times M\\ \|(\boldsymbol{v},q)\|\leq 1}}\left\{\mathscr{D}^2\mathscr{H}(\boldsymbol{z}^N)(\boldsymbol{u}^N-\boldsymbol{u}_h^N,\boldsymbol{v})+\left[B(\boldsymbol{v}),p^N-p_h^N\right]+\left[B(\boldsymbol{u}^N-\boldsymbol{u}_h^N),q\right]\right\}. \quad (4.12)\end{aligned}$$

According to (2.14), (3.2) and (4.6), we obtain

$$D_N(\boldsymbol{u}^N,\boldsymbol{v})+\widehat{D}_N(\boldsymbol{u}^N,\boldsymbol{v})=\mathscr{D}\mathscr{H}(\boldsymbol{u}^N)(\boldsymbol{v}) \tag{4.13}$$

and

$$D_N(\boldsymbol{u}_h^N,\boldsymbol{v})+\widehat{D}_N(\boldsymbol{u}_h^N,\boldsymbol{v})=\mathscr{D}\mathscr{H}(\boldsymbol{u}_h^N)(\boldsymbol{v})\,, \tag{4.14}$$

hence from (4.10) and (4.12)–(4.14) we get the desired result. □

Let us now proceed to estimate the terms on the right-hand side of (4.9). For simplicity, let $\Theta\triangleq[D_N(\boldsymbol{u}^N)-D_N(\boldsymbol{u}_h^N),\boldsymbol{v}]+\langle\widehat{D}_N(\boldsymbol{u}^N-\boldsymbol{u}_h^N),\boldsymbol{v}\rangle+[B(\boldsymbol{v}),p^N-p_h^N]$, $\Omega\triangleq\Omega_1\cup\Omega_2$, $\boldsymbol{f}\triangleq\begin{cases}\boldsymbol{f}_1, & \text{in }\Omega_1,\\ 0, & \text{in }\Omega_2,\end{cases}$ and $[\![\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}_T]\!]$ denote the jump of the tractions $\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}_T$ across the edges on T. From (2.14) and (3.2),

$$0=\left[D_N(\boldsymbol{u}_h^N)-D_N(\boldsymbol{u}^N),\boldsymbol{v}_h\right]+\left\langle\widehat{D}_N(\boldsymbol{u}_h^N-\boldsymbol{u}^N),\boldsymbol{v}_h\right\rangle+\left[B(\boldsymbol{v}_h),p_h^N-p^N\right]\ ,$$

so Θ can be rewritten as

$$\begin{aligned}\Theta=&\left[D_N(\boldsymbol{u}^N)-D_N(\boldsymbol{u}_h^N),\boldsymbol{v}-\boldsymbol{v}_h\right]+\left\langle\widehat{D}_N(\boldsymbol{u}^N-\boldsymbol{u}_h^N),\boldsymbol{v}-\boldsymbol{v}_h\right\rangle\\&+\left[B(\boldsymbol{v}-\boldsymbol{v}_h),p^N-p_h^N\right]\ . \quad (4.15)\end{aligned}$$

Then using $\boldsymbol{v}=\boldsymbol{v}_h=0$ on Γ_0, $-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)=\boldsymbol{f}$ in Ω, $\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}=\mathscr{K}_N(\boldsymbol{u}^N,p^N)$ and the transmission condition (1.7) on Γ_1, similar to the Green's formula in Ref. [20] one has

$$\begin{aligned}\Theta=&\sum_{T\in\mathscr{T}_h}\int_T\left[\boldsymbol{f}+\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\right](\boldsymbol{v}-\boldsymbol{v}_h)d\boldsymbol{x}+\sum_{S\in\mathscr{E}_h(\Gamma_1)}\int_S\boldsymbol{t}_0(\boldsymbol{v}-\boldsymbol{v}_h)ds\\&+\sum_{S\in\mathscr{E}_h(\Omega)}\int_S\left[\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}_T-\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}_T\right](\boldsymbol{v}-\boldsymbol{v}_h)ds\\&+\sum_{S\in\mathscr{E}_h(\Gamma_R)}\int_S\left[\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}-\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}\right](\boldsymbol{v}-\boldsymbol{v}_h)ds\,. \quad (4.16)\end{aligned}$$

From the continuity of the tractions $\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}_T$ through the edges $S\in\mathscr{E}_h(\Omega)$,

$$\sum_{S\in\mathscr{E}_h(\Omega)}\int_S\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}_T(\boldsymbol{v}-\boldsymbol{v}_h)ds=0\,,$$

such that (4.16) becomes

$$\begin{aligned}\Theta =& \sum_{T\in\mathscr{T}_h}\int_T \left[\boldsymbol{f}+\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\right](\boldsymbol{v}-\boldsymbol{v}_h)d\boldsymbol{x}+\sum_{S\in\mathscr{E}_h(\Gamma_1)}\int_S \boldsymbol{t}_0(\boldsymbol{v}-\boldsymbol{v}_h)ds\\ &+\sum_{S\in\mathscr{E}_h(\Omega)}\int_S \boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}_T(\boldsymbol{v}-\boldsymbol{v}_h)ds\\ &+\sum_{S\in\mathscr{E}_h(\Gamma_R)}\int_S \left[\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}-\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}\right](\boldsymbol{v}-\boldsymbol{v}_h)ds\,. \end{aligned}\tag{4.17}$$

Then taking $\boldsymbol{v}_h=\mathscr{I}_h(\boldsymbol{v})\in V_h$ in (4.17), from Lemma 4.1 and the Cauchy-Schwarz's inequality we have

$$\begin{aligned}\Theta \le& C\sum_{T\in\mathscr{T}_h}\Big\{h_T\left\|\boldsymbol{f}+\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\right\|_{[L^2(T)]^2\times[L^2(T)]^2}\|\boldsymbol{v}\|_{[H^1(\Delta(T))]^2\times[H^1(\Delta(T))]^2}\\ &+\sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Gamma_1)} h_S^{1/2}\|\boldsymbol{t}_0\|_{[L^2(S)]^2\times[L^2(S)]^2}\|\boldsymbol{v}\|_{[H^1(\Delta(S))]^2\times[H^1(\Delta(S))]^2}\\ &+\sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Omega)} h_S^{1/2}\left\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}_T]\!]\right\|_{[L^2(S)]^2\times[L^2(S)]^2}\|\boldsymbol{v}\|_{[H^1(\Delta(S))]^2\times[H^1(\Delta(S))]^2}\\ &+\sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Gamma_R)} h_S^{1/2}\left\|\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}-\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}\right\|_{[L^2(S)]^2\times[L^2(S)]^2}\\ &\times\|\boldsymbol{v}\|_{[H^1(\Delta(T))]^2\times[H^1(\Delta(T))]^2}\Big\}\\ \le& C\widetilde{\eta}\sum_{T\in\mathscr{T}_h}\Big\{\|\boldsymbol{v}\|^2_{[H^1(\Delta(T))]^2\times[H^1(\Delta(T))]^2}+\|\boldsymbol{v}\|^2_{[H^1(\Delta(S))]^2\times[H^1(\Delta(S))]^2}\Big\}\end{aligned}$$

where C is a positive constant independent of h, and

$$\begin{aligned}\widetilde{\eta}^2 =& \sum_{T\in\mathscr{T}_h}\Big\{h_T^2\left\|\boldsymbol{f}+\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\right\|^2_{[L^2(T)]^2\times[L^2(T)]^2}+\sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Gamma_1)} h_S\|\boldsymbol{t}_0\|^2_{[L^2(S)]^2\times[L^2(S)]^2}\\ &+\sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Omega)} h_S\left\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}_T]\!]\right\|^2_{[L^2(S)]^2\times[L^2(S)]^2}\\ &+\sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Gamma_R)} h_S\left\|\boldsymbol{\sigma}(\boldsymbol{u}^N,p^N)\boldsymbol{\nu}-\boldsymbol{\sigma}(\boldsymbol{u}_h^N,p_h^N)\boldsymbol{\nu}\right\|^2_{[L^2(S)]^2\times[L^2(S)]^2}\Big\}.\end{aligned}$$

Since the numbers of triangles in $\Delta(T)$ and $\Delta(S)$ are bounded independently of $\mathscr{T}_h$,

$$\Theta \le C\widetilde{\eta},\quad \forall\ \boldsymbol{v}\in V,\ \text{with } \|\boldsymbol{v}\|\le 1\,. \tag{4.18}$$

On the other hand, from (2.14) one has $[B(\boldsymbol{u}^N),q]=0$ for all $q\in M$, so

$$\left[B(\boldsymbol{u}^N-\boldsymbol{u}_h^N),q\right]=-\left[B(\boldsymbol{u}_h^N),q\right]=\int_\Omega q\nabla\cdot(\boldsymbol{u}_h^N)d\boldsymbol{x}\,,$$

and hence

$$\begin{aligned}\left|[B(\boldsymbol{u}^N - \boldsymbol{u}_h^N), q]\right| &\le \left\|\nabla \cdot (\boldsymbol{u}_h^N))\right\|_{[L^2(\Omega)]^2} \|q\|_{[L^2(\Omega)]^2} \\ &\le \left\{ \sum_{T\in\mathscr{T}_h} \left\|\nabla \cdot (\boldsymbol{u}_h^N))\right\|^2_{[L^2(\Omega)]^2} \right\}^{1/2} \end{aligned} \tag{4.19}$$

for all $q \in M$ with $\|q\| \le 1$. Finally, the following result can be obtained by invoking (4.18) and (4.19) in (4.9).

Theorem 4.1. *There exists a positive constant C, independent of h, such that*

$$\left\|\boldsymbol{u}^N - \boldsymbol{u}_h^N\right\|_{V^*} + \left\|p^N - p_h^N\right\|_M \le C\left\{ \sum_{T\in\mathscr{T}_h} \eta_T^2 \right\}, \tag{4.20}$$

with $T \in \mathscr{T}_h$ *and*

$$\begin{aligned}\eta_T^2 =& h_T^2 \left\|\boldsymbol{f} + \nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h^N, p_h^N)\right\|^2_{[L^2(T)]^2\times[L^2(T)]^2} + \sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Gamma_1)} h_S \|\boldsymbol{t}_0\|^2_{[L^2(S)]^2\times[L^2(S)]^2} \\ &+ \sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Omega)} h_S \left\|[\![\boldsymbol{\sigma}(\boldsymbol{u}_h^N, p_h^N)\boldsymbol{\nu}_T]\!]\right\|^2_{[L^2(S)]^2\times[L^2(S)]^2} + \left\|\nabla\cdot(\boldsymbol{u}_h^N))\right\|^2_{[L^2(T)]^2} \\ &+ \sum_{S\in\mathscr{E}(T)\cap\mathscr{E}_h(\Gamma_R)} h_S \left\|\boldsymbol{\sigma}(\boldsymbol{u}^N, p^N)\boldsymbol{\nu} - \boldsymbol{\sigma}(\boldsymbol{u}_h^N, p_h^N)\boldsymbol{\nu}\right\|^2_{[L^2(S)]^2\times[L^2(S)]^2} .\end{aligned}$$

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