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Uniformly Stable Explicitly Solvable Finite Difference Method for Fractional Diffusion Equations

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Abstract. A finite difference scheme for the one-dimensional space fractional diffusion equation is presented and analysed. The scheme is constructed by modifying the shifted Grünwald approximation to the spatial fractional derivative and using an asymmetric discretisation technique. By calculating the unknowns in differential nodal point sequences at the odd and even time levels, the discrete solution of the scheme can be obtained explicitly. We prove that the scheme is uniformly stable. The error between the discrete solution and the analytical solution in the discrete l^2 norm is optimal in some cases. Numerical results for several examples are consistent with the theoretical analysis.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Finite difference scheme, fractional diffusion equation, uniformly stable, explicitly solvable method, asymmetric technique, error estimate.

1. Introduction

Fractional differential equations (FDE) have extensive application in areas of physics [2, 4, 5, 16, 35, 41], chemistry [18], hydrology [1, 3, 33, 34] and in finance [29, 30, 32]. In particular, FDE describe anomalous phenomena that cannot be modelled accurately by second-order diffusion equations. Thus in contaminant transport in groundwater flow for example, the solutes moving through aquifers generally do not follow a second-order diffusion equation because of large deviations due to Brownian motion, so a governing equation with fractional-order anomalous diffusion provides a more adequate description [3].

Analytical methods invoking Fourier or Laplace transforms have been developed for FDE in a few cases [28, 39], but numerical methods are usually needed. Numerical solutions have been obtained via finite difference methods [6, 8, 9, 19, 22–25, 36–38], finite element methods [10, 11, 15], the DG method [14] and spectral methods [20, 21, 40]. Discretisation

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procedures and corresponding convergence analysis have been investigated, and in particular a shifted Grünwald discretisation with implicit time-stepping has been shown to be stable, convergent and first-order accurate in the space mesh size [26,27].

Unlike operators in integer-order diffusion equations, fractional diffusion operators are nonlocal and so raise subtle stability issues for corresponding numerical approximations. Numerical methods for FDE tend to yield full coefficient matrices with $\mathcal{O}(K^3)$ computational and $\mathcal{O}(K^2)$ storage costs where K is the number of unknowns, in contrast to numerical methods for second-order diffusion equations that usually generate banded coefficient matrices with $\mathcal{O}(K)$ nonzero entries.

In this article, we present a finite difference scheme to solve the FDE that is constructed by modifying the shifted Grünwald's method [26] with an asymmetric technique [31] and adopting different nodal point stencils at odd and even time levels. We prove that the scheme is uniformly stable. Formally, the scheme is implicit. However, the solution can be obtained explicitly by sequencing the nodal points from one side to the other, and then calculating the unknowns according to the sequences at the odd time levels and calculating the unknowns according to the opposite sequences at even time levels. The error between the numerical and analytical solutions in the discrete l^2 norm is $\mathcal{O}(\Delta t^2 h^{-2(\alpha-1)} + \Delta t + h)$, where $\alpha \in (1,2)$ is the order of the spatial fractional derivative, and h and Δt are the respective space and time mesh sizes. The error estimate is thus optimal, with the same order as the implicit shifted Grünwald finite difference scheme under the condition $\Delta t = \mathcal{O}(h^{\alpha-0.5})$. For $\alpha \leq 1.5$, the condition $\Delta t = \mathcal{O}(h)$ needed to balance the error due to the time and space discretisation is sufficient to verify the optimal error estimate. The asymmetric technique has previously been used to construct parallel algorithms by other researchers — e.g. see [12,13,42,43]. Earlier authors have investigated the stability and shown that the truncation error is $\mathcal{O}(\Delta t h^{-1} + \Delta t + h)$ for parabolic problems [12,13], or exploited the asymmetric technique in real calculations [42,43]. To the best of our knowledge, this article is the first to show that the error between the discrete and the analytical solutions is $\mathcal{O}(\Delta t^2 h^{-2(\alpha-1)})$.

In Section 2, we present our numerical scheme, and show that the discrete solution can be obtained explicitly by sequencing the nodal points apppropriately. In Section 3, we prove that the scheme is uniformly stable, and derive the error estimate in Section 4. In Section 5, numerical experiments are presented to verify the theoretical results. Throughout, C denotes a generic constant that may take different values in different contexts.

2. The Asymmetric Finite Difference Scheme

We consider the following initial-boundary value problem involving a one-dimensional FDE of order α, where $1 < \alpha < 2$ [26,27,33]:

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t)\,, \qquad x \in (L,R)\,,\ t \in (0,T]\,, \tag{2.1}$$

$$u(x=L,t) = 0\,,\ \ u(x=R,t) = b_R(t)\,, \qquad t \in (0,T]\,, \tag{2.2}$$

$$u(x,0) = \phi(x)\,, \qquad x \in (L,R)\,. \tag{2.3}$$

Let $h=(R-L)/K$ denote the spatial mesh size and $\Delta t = T/(2N)$ the time increment, where K and N are positive integers (the time interval is divided into $2N$ steps). We also write

$$t_n = n\Delta t\,, \quad x_l = L + lh\,, \quad u_l^n = u(x_l,t_n)\,, \quad d_l^n = d(x_l,t_n)\,, \quad f_l^n = f(x_l,t_n)\,,$$

for $n=0,1,\cdots,\le T/\Delta t$ and $l=0,\cdots,K$. It is well known that the fractional derivative can be approximated by the shifted Grünwald formula [26]:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{h^\alpha}\sum_{k=0}^{l} g_k u\big(x-(k-1)h,t\big) + \mathcal{O}(h)\,, \quad h\to 0\,, \tag{2.4}$$

where

$$g_k = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = (-1)^k\binom{\alpha}{k}\,. \tag{2.5}$$

Moreover, the coefficients $\{g_k\}$ satisfy the properties [26,27]

$$\begin{cases} g_0 = 1\,, \\ g_1 = -\alpha < 0\,, \\ 1 \ge g_2 \ge g_3 \ge \cdots \ge 0\,, \\ \displaystyle\sum_{k=0}^{\infty} g_k = 0\,. \end{cases} \tag{2.6}$$

Combining Eq. (2.4) with the implicit Euler discretisation of the time derivative yields the following finite difference equation corresponding to Eq. (2.1):

$$\frac{u_l^{n+1}-u_l^n}{\Delta t} = \frac{d_l^{n+1}}{h^\alpha}\sum_{k=0}^{l} g_k u_{l-k+1}^{n+1} + f_l^{n+1} + \mathcal{O}(\Delta t + h)\,. \tag{2.7}$$

The shifted Grünwald method based on Eq. (2.7) is first-order accurate in both the space mesh size h and time step Δt, and unconditionally stable [26]. The resulting linear system involves a full matrix, with the computational cost $\mathcal{O}(K^3)$ and storage $\mathcal{O}(K^2)$.

Let us now construct our finite difference schemes using the asymmetric technique [31], under the initial and boundary boundary conditions

$$\begin{cases} v_l^0 = \phi(x_l)\,, & l=0,1,\cdots,K\,, \\ v_0^n = 0\,, \quad v_K^n = b_R(R,t^n), & n \le T/\Delta t\,. \end{cases} \tag{2.8}$$

Scheme I: For $1\le l\le K-1$ and $n\le N = T/(2\Delta t)$, find $\{v_l^{2n+2}\}$ such that

$$\begin{cases} a)\dfrac{v_l^{2n+1}-v_l^{2n}}{\Delta t} = \dfrac{d_l^{2n+1}}{h^\alpha}\left(\displaystyle\sum_{k=2}^{l} g_k v_{l-k+1}^{2n+1} - (\alpha-1)v_l^{2n+1} - v_l^{2n} + v_{l+1}^{2n}\right) + f_l^{2n+1}, \\ b)\dfrac{v_l^{2n+2}-v_l^{2n+1}}{\Delta t} = \dfrac{d_l^{2n+2}}{h^\alpha}\left(\displaystyle\sum_{k=2}^{l} g_k v_{l-k+1}^{2n+1} - (\alpha-1)v_l^{2n+1} - v_l^{2n+2} + v_{l+1}^{2n+2}\right) + f_l^{2n+2}. \end{cases} \tag{2.9}$$

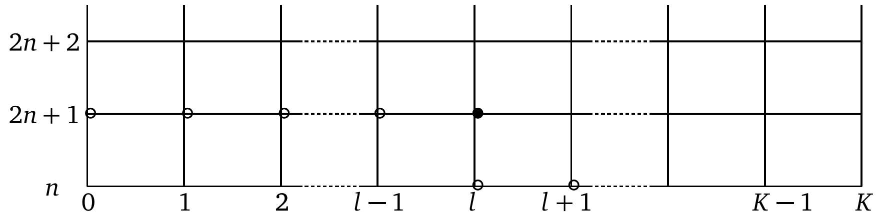

Figure 1: The finite difference stencil for point (l,2n+1).

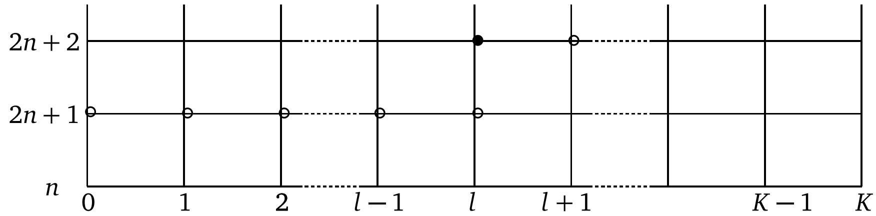

Figure 2: The finite difference stencil for point (l,2n+2).

We use different nodal point stencils at levels $2n+1$ and $2n+2$ (cf. Figs. 1 and 2).

Although formally the scheme is implicit, as previously mentioned the solution can be obtained directly by computing the unknowns according to an appropriate nodal point sequencing. Letting

$$r_l^n = d_l^n \Delta t / h^\alpha , \tag{2.10}$$

from Eq. (2.9) we have

$$\begin{cases} \big(1 + r_l^{n+1}(\alpha - 1)\big) v_l^{2n+1} \\ \qquad = \displaystyle\sum_{k=2}^{l} r_l^{n+1} g_k v_{l-k+1}^{2n+1} + (1 - r_l^{n+1}) v_l^{2n} + r_l^{n+1} v_{l+1}^{2n} + \Delta t f_l^{2n+1}, \\ (1 + r_l^{n+2}) v_l^{2n+2} \\ \qquad = \displaystyle\sum_{k=2}^{l} r_l^{n+2} g_k v_{l-k+1}^{2n+1} + \big(1 - r_l^{n+2}(\alpha - 1)\big) v_l^{2n+1} + r_l^{n+2} v_{l+1}^{2n+2} + \Delta t f_l^{2n+2}. \end{cases} \tag{2.11}$$

1) At time-step t_{2n+1}, we solve the first equation in (2.11) according to the sequence $(2n+1,1) \to (2n+1,2) \to \cdots \to (2n+1,l-1) \to (2n+1,l) \to \cdots$ (cf. Fig. 3). Then with known v_{l-k+1}^{2n+1} for $2 \le k \le l-1$, on combining with v_l^{2n}, v_{l+1}^{2n} we have v_l^{2n+1}.
2) At time-step t_{2n+2}, we solve the second equation in (2.11) according to the sequence $(2n+2,K-1) \to (2n+2,K-2) \to \cdots \to (2n+2,l) \to (2n+2,l-1) \to \cdots$ (cf. Fig. 4).

Thus similarly we have v_l^{2n+2}.

In summary, we obtain the unknowns explicitly one by one, and need not solve the linear algebraic system.

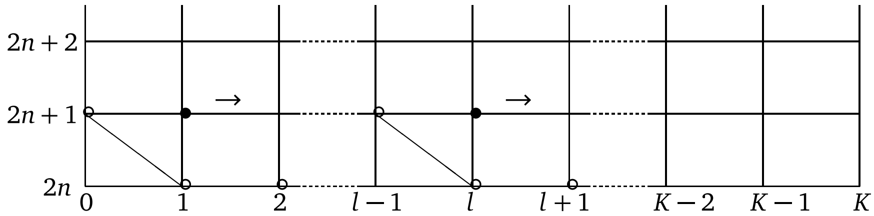

Figure 3: Time-step t_{2n+1}.

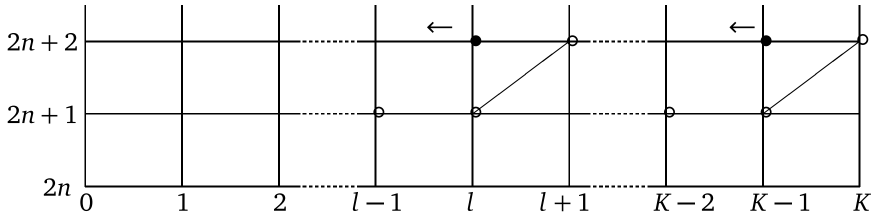

Figure 4: Time-step t_{2n+2}.

Remark 2.1. Although only the one-dimensional problem has been discussed here, one may proceed to two-dimensional problems by combining the proposed technique with an alternating-direction impllcit (ADI) method.

3. Stability of the Finite Difference Scheme

We now prove that the scheme is uniformly stable about the initial value and the term on the right-hand side. Assuming the coefficient $d(x,t) = d$ is a constant, we have that $r_l^n = r$ is a constant. From Eqs. (2.11),

$$\begin{cases} \displaystyle\sum_{k=2}^{l} -r g_k v_{l-k+1}^{2n+1} + \big(1 + r(\alpha-1)\big) v_l^{2n+1} = (1-r) v_l^{2n} + r v_{l+1}^{2n} + \Delta t f_l^{2n+1}\,, \\ \displaystyle (1+r) v_l^{2n+2} - r v_{l+1}^{2n+2} = \sum_{k=2}^{l} r g_k v_{i-k+1}^{2n+1} + \big(1 - r(\alpha-1)\big) v_l^{2n+1} + \Delta t f_l^{2n+2}\,. \end{cases} \tag{3.1}$$

We first use the Fourier method to analyse the stability about the initial value. When $f = 0$ and $b_R = 0$, Eqs. (3.1) become

$$\begin{cases} \displaystyle\sum_{k=2}^{l} -r g_k v_{i-k+1}^{2n+1} + \big(1 + r(\alpha-1)\big) v_l^{2n+1} = (1-r) v_l^{2n} + r v_{l+1}^{2n}\,, \\ \displaystyle (1+r) v_l^{2n+2} - r v_{l+1}^{2n+2} = \sum_{k=2}^{l} r g_k v_{l-k+1}^{2n+1} + \big(1 - r(\alpha-1)\big) v_l^{2n+1}\,. \end{cases} \tag{3.2}$$

Setting $v_l^{2n+1}=W^{2n+1}e^{i\beta lh}$ where β is a non-negative integer, we have

$$\begin{cases} \left\{\sum_{k=2}^{l} -rg_k e^{-i\beta(k-1)h}+1+r(\alpha-1)\right\} W^{2n+1}e^{i\beta lh}=(1-r+re^{i\beta h})W^{2n}e^{i\beta lh}\,, \\ (1+r-re^{i\beta h})W^{2n+2}e^{i\beta lh}=\left\{\sum_{k=2}^{l} rg_k e^{-i\beta(k-1)h}+1-r(\alpha-1)\right\} W^{2n+1}e^{i\beta lh}\,, \end{cases} \tag{3.3}$$

such that

$$W^{2n+2}=G^{2n+2}W^{2n} \tag{3.4}$$

involving the factor

$$\begin{aligned} G^{2n+2}=&\frac{(1-r+re^{i\beta h})}{(1+r-re^{i\beta h})}\frac{\sum_{k=2}^{l} rg_k e^{-i\beta(k-1)h}+1-r(\alpha-1)}{\sum_{k=2}^{l} -rg_k e^{-i\beta(k-1)h}+1+r(\alpha-1)} \\ =&\frac{(1-r+re^{i\beta h})}{(1+r-re^{i\beta h})}\frac{\sum_{k=1}^{l-1} rg_{k+1} e^{-i\beta kh}+1-r(\alpha-1)}{\sum_{k=1}^{l-1} -rg_{k+1} e^{-i\beta kh}+1+r(\alpha-1)} \\ \equiv&G_1G_2\,, \end{aligned} \tag{3.5}$$

where

$$G_1=\frac{a_1+i\sin\beta h}{a_2+i\sin\beta h},\qquad G_2=\frac{b_1-ic}{b_2-ic}\,, \tag{3.6}$$

with

$$\begin{cases} a_1=1-r+r\cos(\beta h)\,, \\ a_2=1+r-r\cos(\beta h)\,, \\ b_1=1-r(\alpha-1)+\sum_{k=1}^{l-1} rg_{k+1}\cos(\beta kh)\,, \\ b_2=1+r(\alpha-1)-\sum_{k=1}^{l-1} rg_{k+1}\cos(\beta kh)\,, \\ c=\sum_{k=1}^{l-1} rg_{k+1}\sin(\beta kh)\,. \end{cases}$$

On defining matrices A and B with the entries

$$A_{ij}=\begin{cases} 0 & j\geq i+1 \\ r(\alpha-1) & j=i \\ -rg_{i-j+1} & j\leq i-1 \end{cases}\,, \tag{3.7}$$

$$B_{ij} = \begin{cases} 0 & j \geq i+2 \\ -r & j = i+1 \\ r & j = i \\ 0 & j \leq i-1 \end{cases}, \tag{3.8}$$

the linear system (3.1) can be rewritten as

$$\begin{cases} (I+A)V^{2n+1} = (I-B)V^{2n} + \Delta t F^{2n+1}, \\ (I+B)V^{2n+2} = (I-A)V^{2n+1} + \Delta t F^{2n+2}, \end{cases} \tag{3.9}$$

where for non-negative integers m we have

$$V^m = (v_1^m, v_2^m, \cdots, v_{K-1}^m)^T, \qquad F^m = (f_1^m, f_2^m, \cdots, f_{K-1}^m)^T.$$

Lemma 3.1. *If $f = 0$ and $b_R = 0$ in Eq.* (3.1)*, then*

$$\|V^{2n+2}\| \leq \|V^{2n}\|.$$

Proof. We first prove that the norm of G_2 is less that 1. Indeed,

$$|G_2|^2 - 1 = \frac{b_1^2 - b_2^2}{b_2^2 + c^2} = \frac{-4r\left(\alpha - 1 - \sum_{k=1}^{l-1} g_{k+1}\cos(\beta k h)\right)}{b_2^2 + c^2}.$$

From Eq. (2.6) and $1 < \alpha < 2$ we have that

$$\left|\sum_{k=1}^{l-1} g_{k+1}\cos(\beta k h)\right| \leq \sum_{k=1}^{l-1} g_{k+1} \leq \alpha - 1$$

such that $|G_2|^2 < 1$, therefore

$$|G_2| \leq 1. \tag{3.10}$$

Similarly, we can prove $|G_1| \leq 1$. On combining with Eq. (3.10) and Eq. (3.5), we have that

$$|G^{2n+2}| \leq 1, \tag{3.11}$$

which complete the proof. □

Let us now divide V^{2n+1} and V^{2n+2} into

$$V^{2n+1} = \hat{V}^{2n+1} + \tilde{V}^{2n+1} \quad \text{and} \quad V^{2n+2} = \hat{V}^{2n+2} + \tilde{V}^{2n+2} \tag{3.12}$$

according to the following rules:

$$\begin{cases} (I+A)\hat{V}^{2n+1} = (I-B)V^{2n}, \\ (I+A)\tilde{V}^{2n+1} = \Delta t F^{2n+1}, \\ (I+B)\hat{V}^{2n+2} = (I-A)\hat{V}^{2n+1}, \\ (I+B)\tilde{V}^{2n+2} = (I-A)\tilde{V}^{2n+1} + \Delta t F^{2n+2}. \end{cases} \tag{3.13}$$

Then we have

$$\begin{cases} (I+A)\hat{V}^{2n+1}=(I-B)V^{2n}\,, \\ (I+B)\hat{V}^{2n+2}=(I-A)\hat{V}^{2n+1}\,, \end{cases} \tag{3.14}$$

such that

$$(I+B)\tilde{V}^{2n+2}=(I-A)(I+A)^{-1}\Delta tF^{2n+1}+\Delta tF^{2n+2}\,. \tag{3.15}$$

Lemma 3.2. *The eigenvalues of matrices A and B are positive.*

Proof. We first prove that any eigenvalue λ of the matrix A is positive. From the well known Gerschgorin theorem [17], for every eigenvalue λ we have

$$A_{ll}-R\le\lambda\le A_{ll}+R\,, \tag{3.16}$$

where R is the radius

$$R=\sum_{j=1,j\neq l}^{K-1}A_{lj}=\sum_{j=1}^{l-1}rg_{l-j+1}=r\sum_{j=2}^{l}g_j\,.$$

It is clear that

$$A_{ll}-R_l=-rg_1-rg_0-r\sum_{j=2}^{l}g_j=-r\sum_{j=0}^{l}g_j\,, \tag{3.17}$$

so from (2.6) and (3.16)

$$\lambda\ge-r\sum_{j=0}^{i}g_j=r\sum_{j=l+1}^{\infty}g_j>0\,. \tag{3.18}$$

Similarly, we can prove that the eigenvalues of matrix B are positive. □

The stability of our scheme about the initial value and right-hand side term is then ensured under the following Theorem.

Theorem 3.1. *Suppose $d(x,t)=d$ is a constant function. If $b_R=0$, then the scheme is uniformly stable about the initial value and right-hand side term — i.e.*

$$\|V^{2n+2}\|\le\|V^0\|+\sum_{m=1}^{2n+2}\Delta t\|F^m\|\,.$$

Proof. Using Lemma 3.1. we immediately obtain the inequalities

$$\|I+B\|\ge\max\lambda_{I+B}\ge1\,,$$

$$\|(I+B)^{-1}\|=\|I+B\|^{-1}\le1\,, \tag{3.19}$$

$$\|(I-A)(I+A)^{-1}\|\le1\,, \tag{3.20}$$

hence from Eq. (3.15)

$$\begin{aligned}\|\tilde{V}^{2n+2}\|\le&\|(I+B)^{-1}\|\|(I-A)(I+A)^{-1}\|\Delta t\|F^{2n+1}\|+\Delta t\|F^{2n+2}\|\\ \le&\Delta t\|F^{2n+1}\|+\Delta t\|F^{2n+2}\|\,.\end{aligned} \tag{3.21}$$

From Eqs. (3.14) and Lemma 3.1,

$$\|\hat{V}^{2n+2}\| \le \|V^{2n}\| \,. \tag{3.22}$$

On combining inequalities (3.21) and (3.22) with Eq. (3.12), we thus obtain

$$\|V^{2n+2}\| = \|\hat{V}^{2n+2} + \tilde{V}^{2n+2}\| \le \|V^{2n}\| + \Delta t\|F^{2n+1}\| + \Delta t\|F^{2n+2}\| \,, \tag{3.23}$$

and invoking this inequality successively completes the proof. □

4. Error Estimate

In this section, we present an error estimate of our scheme, again assuming that $d(x,t) = d$ is a constant. Letting u_l^m denote the exact solution of $Eq.$ (2.1) $\sim$ $Eq.$ (2.3) at (x_l, t_m), we have

$$\begin{aligned}
&\frac{u_l^{2n+1}-u_l^{2n}}{\Delta t} - \frac{d}{h^\alpha}\left\{\sum_{k=2}^{l} g_k u_{l-k+1}^{2n+1} - (\alpha-1)u_l^{2n+1} - u_l^{2n} + u_{l+1}^{2n}\right\} - f_l^{2n+1} \\
=&\frac{u_l^{2n+1}-u_l^{2n}}{\Delta t} - \left(\frac{\partial u}{\partial t}\right)_l^{2n+1} - \frac{d}{h^\alpha}\sum_{k=0}^{l} g_k u_{l-k+1}^{2n+1} + \left(d\frac{\partial^\alpha u}{\partial x^\alpha}\right)_l^{2n+1} \\
&+\frac{d}{h^\alpha}\left(u_{l+1}^{2n+1} - u_l^{2n+1} - (u_{l+1}^{2n} - u_l^{2n})\right) \,.
\end{aligned} \tag{4.1}$$

A Taylor expansion yields

$$\begin{aligned}
&\frac{1}{h}\left(u_{l+1}^{2n+1} - u_l^{2n+1} - (u_{l+1}^{2n} - u_l^{2n})\right) \\
=&\left(\frac{\partial u}{\partial x}\right)_l^{2n+1} - \left(\frac{\partial u}{\partial x}\right)_l^{2n} + \frac{h}{2}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_l^{2n+1} - \left(\frac{\partial^2 u}{\partial x^2}\right)_l^{2n}\right] + \mathscr{O}(h^2) \\
=&\Delta t\left(\frac{\partial^2 u}{\partial x \partial t}\right)_l^{2n+1} + \mathscr{O}(\Delta t^2 + \Delta t h + h^2)
\end{aligned}$$

at the time-step t^{2n+1}. so Eq. (4.1) becomes

$$\begin{aligned}
&\frac{u_l^{2n+1}-u_l^{2n}}{\Delta t} - \frac{d}{h^\alpha}\left\{\sum_{k=2}^{l} g_k u_{l-k+1}^{2n+1} - (\alpha-1)u_l^{2n+1} - u_l^{2n} + u_{l+1}^{2n}\right\} - f_l^{2n+1} \\
=&\mathscr{O}(\Delta t + h) + \frac{d\Delta t}{h^{\alpha-1}}\left(\frac{\partial^2 u}{\partial x \partial t}\right)_l^{2n+1} + \mathscr{O}(\Delta t^2 h^{1-\alpha} + \Delta t h^{2-\alpha} + h^{3-\alpha}) \,.
\end{aligned} \tag{4.2}$$

Since $1 < \alpha < 2$, we obtain

$$\begin{aligned}
&\frac{u_l^{2n+1}-u_l^{2n}}{\Delta t} - \frac{d}{h^\alpha}\left\{\sum_{k=2}^{l} g_k u_{l-k+1}^{2n+1} - (\alpha-1)u_l^{2n+1} - u_l^{2n} + u_{l+1}^{2n}\right\} - f_l^{2n+1} \\
=&\frac{d\Delta t}{h^{\alpha-1}}\left(\frac{\partial^2 u}{\partial x \partial t}\right)_l^{2n+1} + \mathscr{O}(\Delta t^2 h^{1-\alpha} + \Delta t + h) \,.
\end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned}&\frac{u_l^{2n+2}-u_l^{2n+1}}{\Delta t}-\frac{d}{h^\alpha}\left\{\sum_{k=2}^{l}g_k u_{l-k+1}^{2n+1}-(\alpha-1)u_l^{2n+1}-u_l^{2n+2}+u_{l+1}^{2n+2}\right\}-f_l^{2n+2}\\ =&-\frac{d\Delta t}{h^{\alpha-1}}\left(\frac{\partial^2 u}{\partial x\partial t}\right)_l^{2n+1}+\mathscr{O}(\Delta t^2 h^{1-\alpha}+\Delta t+h)\,,\end{aligned}\tag{4.4}$$

and we set $U_l^m=(u_1^m,u_2^m,\cdots,u_{K-1}^m)$.

Theorem 4.1. *Let $u(x_l,t_n)$ denote the exact solution of Eqs. (2.1)–(2.3), and v_l^n the solution of Eq. (2.9). Suppose $d(x)$ is a constant function. When h and Δt are sufficiently small, there exists a positive constant C independent of Δt and h such that for $m\le T/\Delta t$*

$$||V^m-U^m||\le C\left(\Delta t^2 h^{-2(\alpha-1)}+\Delta t+h\right).$$

Proof. At the odd time-step t^{2n+1}, define e_l^{2n+1} as

$$\begin{cases}e_0^{2n+1}=v_0^{2n+1}-u_0^{2n+1}=0\,,\\ e_l^{2n+1}=v_l^{2n+1}-u_l^{2n+1}+\dfrac{d\Delta t^2}{h^{\alpha-1}}\left(\dfrac{\partial^2 u}{\partial x\partial t}\right)_l^{2n+1}, & 0\le l\le K-1\,,\\ e_K^{2n+1}=v_l^{2n+1}-u_0^{2n+1}=0\,,\end{cases}\tag{4.5}$$

where we suppose $(\frac{\partial^2 u}{\partial x\partial t})_0^{2n+1}=0$; and at the even time-step t^{2n+2}, define e_l^{2n+2} as

$$\begin{cases}e_0^{2n+1}=v_0^{2n+1}-u_0^{2n+1}=0\,,\\ e_l^{2n+1}=v_l^{2n+1}-u_l^{2n+1}, & 0\le l\le K-1\,,\\ e_K^{2n+1}=v_l^{2n+1}-u_0^{2n+1}=0\,.\end{cases}\tag{4.6}$$

From Eqs. (4.3), (4.4) and (2.9), for $1\le l\le K-1$ and $n\le N=T/(2\Delta t)$ we have

$$\begin{cases}\dfrac{e_l^{2n+1}-e_l^{2n}}{\Delta t}=&\dfrac{d}{h^\alpha}\left\{\displaystyle\sum_{k=2}^{l}g_k e_{l-k+1}^{2n+1}-(\alpha-1)e_l^{2n+1}-e_l^{2n}+e_{l+1}^{2n}\right\}\\ &-\dfrac{d\Delta t^2}{h^{2\alpha-1}}\displaystyle\sum_{k=2}^{l}g_k\left(\left(d\frac{\partial^2 u}{\partial x\partial t}\right)_{l-k+1}^{2n+1}-\left(d\frac{\partial^2 u}{\partial x\partial t}\right)_l^{2n+1}\right)\\ &+\mathscr{O}(\Delta t^2 h^{1-\alpha}+\Delta t+h)\,,\\ \dfrac{e_l^{2n+2}-e_l^{2n+1}}{\Delta t}=&\dfrac{d}{h^\alpha}\left\{\displaystyle\sum_{k=2}^{l}g_k e_{l-k+1}^{2n+1}-(\alpha-1)e_l^{2n+1}-e_l^{2n+2}+e_{l+1}^{2n+2}\right\}\\ &-\dfrac{d\Delta t^2}{h^{2\alpha-1}}\displaystyle\sum_{k=2}^{l}g_k\left(\left(d\frac{\partial^2 u}{\partial x\partial t}\right)_{l-k+1}^{2n+1}-\left(d\frac{\partial^2 u}{\partial x\partial t}\right)_l^{2n+1}\right)\\ &+\mathscr{O}(\Delta t^2 h^{1-\alpha}+\Delta t+h)\,.\end{cases}\tag{4.7}$$

A Taylor expansion yields

$$\left(d\frac{\partial^2 u}{\partial x \partial t}\right)^{2n+1}_{l-k+1} - \left(d\frac{\partial^2 u}{\partial x \partial t}\right)^{2n+1}_{l} = \mathcal{O}\left((k-1)h\right). \tag{4.8}$$

Using the estimates for $\{g_k\}$ in Ref. [27], we know there exist two constants C_1 and C_2 such that

$$\frac{C_1}{k^{\alpha+1}} \le g_k \le \frac{C_2}{k^{\alpha+1}}, \tag{4.9}$$

so again using a Taylor expansion there exist two values for a constant C_3 such that

$$\left| g_k \left(\left(d\frac{\partial^2 u}{\partial x \partial t}\right)^{2n+1}_{l-k+1} - \left(d\frac{\partial^2 u}{\partial x \partial t}\right)^{2n+1}_{l} \right) \right| \le \frac{1}{k^{\alpha+1}} C(k-1)h \le \frac{C_3}{k^{\alpha}} h. \tag{4.10}$$

Since $\alpha > 1$, there consequently exists a positive constant C independent of h such that

$$\left| \sum_{k=2}^{l+1} g_k \left(\left(d\frac{\partial^2 u}{\partial x \partial t}\right)^{2n+1}_{l-k+1} - \left(d\frac{\partial^2 u}{\partial x \partial t}\right)^{2n+1}_{l} \right) \right| \le \sum_{k=2}^{l+1} \frac{C_3}{k^{\alpha}} h \le Ch, \tag{4.11}$$

so from Eqs. (4.7)

$$\begin{cases} \dfrac{e_l^{2n+1} - e_l^{2n}}{\Delta t} = \dfrac{d}{h^\alpha}\left\{ \displaystyle\sum_{k=2}^{l} g_k e_{l-k+1}^{2n+1} - (\alpha-1)e_l^{2n+1} - v_l^{2n} + e_{l+1}^{2n} \right\} \\ \qquad\qquad + \mathcal{O}\left(\Delta t^2 h^{-2(1-\alpha)} + \Delta t + h\right), \\ \dfrac{e_l^{2n+2} - e_l^{2n+1}}{\Delta t} = \dfrac{d}{h^\alpha}\left\{ \displaystyle\sum_{k=2}^{l} g_k e_{l-k+1}^{2n+1} - (\alpha-1)e_l^{2n+1} - e_l^{2n+2} + e_{l+1}^{2n+2} \right\} \\ \qquad\qquad + \mathcal{O}\left(\Delta t^2 h^{-2(1-\alpha)} + \Delta t + h\right). \end{cases} \tag{4.12}$$

Clearly, $e_l^0 = 0$ for $0 \le l \le K$. Setting $E^m = (e_1^m, e_2^m, \cdots, e_{K-1}^m)$, from Theorem 3.1 we get the error estimate $||E^m|| \le C(\Delta t^2 h^{-2(1-\alpha)} + \Delta t + h)$. Noting $e_l^{2n} = v_l^{2n} - u_l^{2n}$ and $e_l^{2n+1} = v_l^{2n+1} - u_l^{2n+1} + \frac{d\Delta t^2}{h^{\alpha-1}}(\frac{\partial^2 u}{\partial x \partial t})_l^{2n+1}$ completes the estimate of $V^m - U^m$. $\square$

Remark 4.1. From Eq. (4.3), the truncation error is $\mathcal{O}(\Delta t h^{-(\alpha-1)} + \Delta t + h)$. From the above, a property of the asymmetric discretisation technique is that the difference between the discrete and the analytic solutions is bounded by $C(\Delta t^2 h^{-2(\alpha-1)} + \Delta t + h)$.

5. Numerical Experiments

Using our scheme, we now present some numerical examples for

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t) \tag{5.1}$$

with $0 < x < 1$ and $0 < t \leq 1$, on assuming the diffusion coefficient is of the form

$$d(x) = d(x,\alpha) = \Gamma(4-\alpha)x^{\alpha}/6$$

and adopting various values for α. The source function $f(x,t)$ and the initial and boundary value conditions are selected according to the assumed $d(x)$, α and the analytical solution.

For the different values of α, in the tables below we list errors in discrete l^2-norms and the convergence rate — i.e. errors

$$\| e^n \|_{l^2} = \| e_h^n \|_{l^2} = \left(\sum_{j=1}^{K-1} \left| u(x_j, t_n) - v_j^n \right|^2 h \right)^{\frac{1}{2}},$$

for two mesh sizes h_1 and h_2, and the convergence rate for the space mesh sizes

$$\frac{\log\left(\|e_{h_1}^n\|_{l^2} / \|e_{h_2}^n\|_{l^2} \right)}{\log(h_1/h_2)}.$$

We choose the discretising mesh sizes according to one of the following rules.

1. **Case One**. Set $\Delta t = h$. According to the theoretical analysis, the convergence rate is $\mathcal{O}(h^{\min\{1,2(2-\alpha)\}})$ and is optimal for $\alpha \in (1,1,5]$.
2. **Case Two**. Set $\Delta t = h^{\alpha-0.5}$. According to the theoretical analysis, the convergence rate is $\mathcal{O}(h)$ and is optimal for $\alpha \in (1,2)$.

Example 5.1. Analytical solution and source coefficient

$$u = e^t x^4, \qquad f = -\alpha x^4 e^t/(4-\alpha). \tag{5.2}$$

The numerical results are listed in Tables 1-3.

Example 5.2. Analytical solution and source coefficient

$$u = e^t x^2(1-x), \qquad f = \alpha x^2 e^t t/3. \tag{5.3}$$

The numerical results are listed in Tables 4-6.

Table 1: Error and convergence rates for Example 5.1.

	$\alpha = 1.4$		$\alpha = 1.5$	
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	Error	Rate
(10, 10)	0.0461	-	0.0397	-
(20, 20)	0.0236	0.966	0.0208	0.933
(40, 40)	0.0119	0.988	0.0109	0.932
(80, 80)	0.0060	0.988	0.0056	0.961
(160, 160)	0.0030	1.000	0.0029	0.949
(320, 320)	0.0015	1.000	0.0015	0.951
(640, 640)	7.328×10^{-4}	1.033	7.430×10^{-4}	1.013

Table 2: Error and convergence rates for Example 5.1.

$\alpha = 1.6$					
Case One			Case Two		
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate
(10, 10)	0.0351	-	(10, 14)	0.0296	-
(20, 20)	0.0196	0.841	(20, 28)	0.0158	0.906
(40, 40)	0.0110	0.833	(40, 58)	0.0083	0.929
(80, 80)	0.0062	0.827	(80, 124)	0.0043	0.949
(160, 160)	0.0035	0.825	(160, 266)	0.0022	0.967
(320, 320)	0.0019	0.881	(320, 570)	0.0011	1.000
(640, 640)	0.0011	0.788	(640, 1222)	5.608×10^{-4}	0.972

Table 3: Error and convergence rates for Example Example 5.1.

$\alpha = 1.7$					
Case One			Case Two		
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate
(10, 10)	0.0324	-	(10, 16)	0.0231	-
(20, 20)	0.0203	0.675	(20, 28)	0.0145	0.672
(40, 40)	0.0130	0.643	(40, 84)	0.0061	1.249
(80, 80)	0.0084	0.630	(80, 194)	0.0031	0.976
(160, 160)	0.0055	0.611	(160, 442)	0.0016	0.954
(320, 320)	0.0035	0.652	(320, 1016)	8.068×10^{-4}	0.988
(640, 640)	0.0023	0.606	(640, 2332)	4.066×10^{-4}	0.989

Table 4: Error and convergence Rates for Example 5.2.

	$\alpha = 1.4$		$\alpha = 1.5$	
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	Error	Rate
(10, 10)	0.0164	-	0.0136	-
(20, 20)	0.0085	0.948	0.0072	0.917
(40, 40)	0.0043	0.983	0.0037	0.961
(80, 80)	0.0022	0.967	0.0019	0.962
(160, 160)	0.0011	1.000	9.967×10^{-4}	0.931
(320, 320)	5.436×10^{-4}	1.017	5.084×10^{-4}	0.971
(640, 640)	2.705×10^{-4}	1.007	2.579×10^{-4}	0.979

Example 5.3. Analytical solution and source coefficient, and initial and boundary condi-

Table 5: Error and convergence rates for Example 5.2.

$\alpha=1.6$					
Case One			Case Two		
$\left(\frac{1}{h},\frac{1}{\Delta t}\right)$	Error	Rate	$\left(\frac{1}{h},\frac{1}{\Delta t}\right)$	Error	Rate
(10, 10)	0.0112	-	(10, 14)	0.0102	-
(20, 20)	0.0063	0.830	(20, 28)	0.0054	0.918
(40, 40)	0.0035	0.848	(40, 58)	0.0029	0.897
(80, 80)	0.0020	0.807	(80, 124)	0.0015	0.951
(160, 160)	0.0011	0.863	(160, 266)	7.560×10^{-4}	0.988
(320, 320)	6.053×10^{-4}	0.862	(320, 570)	3.841×10^{-4}	0.977
(640, 640)	3.349×10^{-4}	0.854	(640, 1222)	1.942×10^{-4}	0.984

Table 6: Error and convergence rates for Example 5.2.

$\alpha=1.7$					
Case One			Case Two		
$\left(\frac{1}{h},\frac{1}{\Delta t}\right)$	Error	Rate	$\left(\frac{1}{h},\frac{1}{\Delta t}\right)$	Error	Rate
(10, 10)	0.0095	-	(10, 16)	0.0076	-
(20, 20)	0.0059	0.687	(20, 28)	0.0046	0.724
(40, 40)	0.0038	0.635	(40, 84)	0.0021	1.131
(80, 80)	0.0024	0.663	(80, 194)	0.0011	0.933
(160, 160)	0.0016	0.585	(160, 442)	5.445×10^{-4}	1.014
(320, 320)	0.0010	0.678	(320, 1016)	2.756×10^{-4}	0.982
(640, 640)	6.524×10^{-4}	0.616	(640, 2332)	1.390×10^{-4}	0.987

tions

$$\begin{cases} u=e^{-t}\sin x\,, \\ f=e^{-t}\sin x-\dfrac{\Gamma(4-\alpha)}{6}x^{\alpha}\dfrac{d^2}{dx^2}\displaystyle\int_0^x \frac{e^{-t}\sin\varphi}{(x-\varphi)^{\alpha-1}}d\varphi\,, \\ u(0,t)=0\,,\quad u(1,t)=e^{-t}\sin 1\,. \end{cases} \tag{5.4}$$

The numerical results are listed in Tables 7-9.

Example 5.4. The analytical solution u, source coefficient f are as follows.

$$\begin{cases} u=e^{t}(1-\cos x), \\ f=e^{t}(1-\cos x)-\dfrac{\Gamma(4-\alpha)}{6}x^{\alpha}\dfrac{d^2}{dx^2}\displaystyle\int_0^x \frac{e^{t}(1-\cos\varphi)}{(x-\varphi)^{\alpha-1}}d\varphi, \\ u(0,t)=0,\quad u(1,t)=e^{t}(1-\cos 1). \end{cases} \tag{5.5}$$

The numerical results are listed in Tables 10-12.

We observe that the numerical results in these tables are consistent with the theoretical analysis, and that our scheme performs better for $\alpha\le 1.5$.

Table 7: Error and convergence rates for Example 5.3.

	$\alpha = 1.4$		$\alpha = 1.5$	
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	Error	Rate
(10, 10)	0.0016	-	0.0013	-
(20, 20)	9.017×10^{-4}	0.827	7.424×10^{-4}	0.808
(40, 40)	4.815×10^{-4}	0.905	4.084×10^{-4}	0.862
(80, 80)	2.489×10^{-4}	0.952	2.163×10^{-4}	0.917
(160, 160)	1.265×10^{-4}	0.997	1.121×10^{-4}	0.948
(320, 320)	6.361×10^{-5}	0.991	5.738×10^{-5}	0.967
(640, 640)	3.183×10^{-5}	0.999	2.913×10^{-5}	0.978

Table 8: Error and convergence rates for Example 5.3.

$\alpha = 1.6$					
Case One			Case Two		
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate
(10, 10)	9.738×10^{-4}	-	(10, 14)	9.812×10^{-4}	-
(20, 20)	6.301×10^{-4}	0.628	(20, 28)	5.683×10^{-4}	0.788
(40, 40)	3.750×10^{-4}	0.749	(40, 58)	3.117×10^{-4}	0.867
(80, 80)	2.143×10^{-4}	0.807	(80, 124)	1.637×10^{-4}	0.929
(160, 160)	1.199×10^{-4}	0.838	(160, 266)	8.417×10^{-4}	0.960
(320, 320)	6.640×10^{-5}	0.853	(320, 570)	4.282×10^{-5}	0.975
(640, 640)	3.659×10^{-5}	0.860	(640, 1222)	2.163×10^{-5}	0.985

Table 9: Error and convergence rates for Example 5.3.

$\alpha = 1.7$					
Case One			Case Two		
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate
(10, 10)	7.822×10^{-4}	-	(10, 16)	6.852×10^{-4}	-
(20, 20)	6.026×10^{-4}	0.376	(20, 28)	4.618×10^{-4}	0.569
(40, 40)	4.173×10^{-4}	0.530	(40, 84)	2.174×10^{-4}	1.087
(80, 80)	2.767×10^{-4}	0.593	(80, 194)	1.131×10^{-4}	0.943
(160, 160)	1.802×10^{-4}	0.619	(160, 442)	5.818×10^{-5}	0.959
(320, 320)	1.166×10^{-4}	0.628	(320, 1016)	2.948×10^{-5}	0.981
(640, 640)	7.536×10^{-5}	0.630	(640, 2332)	1.486×10^{-5}	0.988

6. Conclusion

We have constructed a finite difference scheme to solve the one-dimensional space FDE considered. The scheme is constructed by modifying the shifted Grünwald approximation using an asymmetric technique, with different nodal point stencils at odd and even time

Table 10: Error and convergence rates for Example 5.4.

	$\alpha = 1.4$		$\alpha = 1.5$	
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	Error	Rate
(10, 10)	0.0034	-	0.0032	-
(20, 20)	0.0017	1.000	0.0017	0.913
(40, 40)	8.344×10^{-4}	1.027	9.048×10^{-4}	0.910
(80, 80)	4.031×10^{-4}	1.049	4.675×10^{-4}	0.953
(160, 160)	1.937×10^{-4}	1.058	2.390×10^{-4}	0.968
(320, 320)	9.286×10^{-5}	1.060	1.213×10^{-4}	0.978
(640, 640)	4.454×10^{-5}	1.060	6.1129×10^{-5}	0.985

Table 11: Error and convergence Rates for Example 5.4.

$\alpha = 1.6$					
Case One			Case Two		
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate
(10, 10)	0.0035	-	(10, 14)	0.0019	-
(20, 20)	0.0021	0.737	(20, 28)	0.0012	0.663
(40, 40)	0.0013	0.692	(40, 58)	6.420×10^{-4}	0.902
(80, 80)	7.511×10^{-4}	0.791	(80, 124)	3.320×10^{-4}	0.951
(160, 160)	4.376×10^{-4}	0.779	(160, 266)	1.689×10^{-4}	0.975
(320, 320)	2.535×10^{-4}	0.788	(320, 570)	8.552×10^{-5}	0.982
(640, 640)	1.463×10^{-5}	0.793	(640, 1222)	4.309×10^{-5}	0.989

Table 12: Error and convergence rates for Example 5.4.

$\alpha = 1.7$					
Case One			Case Two		
$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate	$\left(\frac{1}{h}, \frac{1}{\Delta t}\right)$	Error	Rate
(10, 10)	0.0043	-	(10, 16)	0.0017	-
(20, 20)	0.0030	0.520	(20, 28)	0.00895	0.926
(40, 40)	0.0021	0.515	(40, 84)	4.626×10^{-4}	0.952
(80, 80)	0.0014	0.585	(80, 194)	2.344×10^{-4}	0.981
(160, 160)	9.390×10^{-4}	0.576	(160, 442)	1.209×10^{-4}	0.954
(320, 320)	6.262×10^{-4}	0.584	(320, 1016)	6.096×10^{-5}	0.988
(640, 640)	4.161×10^{-4}	0.590	(640, 2332)	3.071×10^{-5}	0.989

levels. The solution of the scheme can be obtained by computing the unknowns using appropriate nodal point sequences. We have proved that the scheme is uniformly stable and demonstrated that the error in the discrete l^2 norm is $\mathcal{O}(\Delta t^2 h^{-2(\alpha-1)} + \Delta t + h)$, where $1 < \alpha < 2$ is the order of the space fractional derivative and h and Δt are the space and time mesh sizes. This result shows that the error estimate is optimal when $\Delta t = \mathcal{O}(h^{\alpha-0.5})$, so it is more suitable for solving the FDE with $\alpha \leq 1.5$. The theoretical results have been verified by some numerical experiments.

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