East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.260816.051216a

Vol. **7**, No. 1, pp. 143-155
February 2017

# Accelerated GPMHSS Method for Solving Complex Systems of Linear Equations

Jing Wang[1], Xue-Ping Guo[2,*] and Hong-Xiu Zhong[3]

[1] *Shandong Computer Science Center, Jinan 250014, Shandong, P.R. China.*
[2] *Department of Mathematics, East China Normal University, Shanghai 200241, P.R. China.*
[3] *School of Science, Jiangnan University, Wuxi 214122, P.R. China.*

*Received 26 August 2016; Accepted (in revised version) 5 December 2016.*

**Abstract.** Preconditioned modified Hermitian and skew-Hermitian splitting method (PMHSS) is an unconditionally convergent iteration method for solving large sparse complex symmetric systems of linear equations, and uses one parameter $\alpha$. Adding another parameter $\beta$, the generalized PMHSS method (GPMHSS) is essentially a two-parameter iteration method. In order to accelerate the GPMHSS method, using an unexpected way, we propose an accelerated GPMHSS method (AGPMHSS) for large complex symmetric linear systems. Numerical experiments show the numerical behavior of our new method.

**AMS subject classifications**: 65F10, 65W50

**Key words**: Preconditioning, complex systems of linear equations, PMHSS method, GPMHSS method, AGPMHSS method.

## 1. Introduction

Many applications in scientific computing and engineering can be transformed into solving the following large sparse and complex symmetric linear equations

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n}, \qquad x, b \in \mathbb{C}^n, \tag{1.1}$$

where $A = W + iT$, $W$, $T \in \mathbb{R}^{n\times n}$ are symmetric matrices, with $W$ positive definite and $T$ positive semi-definite. Here and in the sequel, $i$ denotes the imaginary unit. Such applications arise in quantum mechanics [23], diffuse optimal tomography [1], structural dynamics [15], FFT-based solution of certain time-dependent PDEs [12], molecular scattering [21], and lattice quantum chromodynamics [16], etc.

Generally, direct methods and iteration methods are two main classes of methods for solving systems of linear equations. Direct solution methods, such as Gaussian elimination,

*Corresponding author. *Email addresses:* `island0609@hotmail.com` (J. Wang), `xpguo@math.ecnu.edu.cn` (X.-P. Guo), `zhonghongxiu@126.com` (H.-X. Zhong)

 

LU-decomposition, are often preferred to iterative methods because of their robustness and predictable behavior. However, when coefficient matrix $A$ is very large and sparse, direct solvers may cost too much time and storage. Iterative methods, such as Krylov subspace methods, are easier to keep and exploit the sparsity of $A$, thereby require much less computer storage than direct methods, and implement efficiently on high-performance computers than direct methods. Thus, iteration methods have been widely concerned by scholars all the time, see [17, 19, 22] and references therein.

Based on the Hermitian and skew-Hermitian splittings, Bai, Golub and Ng [8] have proposed the Hermitian and skew-Hermitian splitting (HSS) method for non-Hermitian positive-definite linear systems. They have also proved that this method converges unconditionally to the exact solution of the system, and if it is used to solve the system of linear equations with Hermitian positive-definite coefficient matrix, the convergence speed is same as that of the conjugate gradient method. Owing to the effectiveness and robustness of the HSS method, it has received attentions from many scholars, eg. see [5–7, 10, 11]. Even some scholars used HSS-type methods as the inner iterative solver, and Newton-type methods as the outer iterative solver, proposed several effective methods for solving nonlinear equations, eg. [11, 13, 18, 20, 24, 26, 27].

Nevertheless, when $A$ is complex, the convergence rate of each method referred above, reduces significantly since the resolution of the linear system (1.1) needs a complex algorithm. In order to overcome this deficiency, Bai *et al.* [2–4] proposed the modified HSS (MHSS) iteration and preconditioned modified HSS (PMHSS) to solve complex symmetric linear systems. Based on the PMHSS method, Xu [25] proposed its generalization for complex symmetric indefinite linear systems, while Mehdi *et al.* [14] presented the generalized preconditioned MHSS method (GPMHSS) for complex symmetric linear systems with two parameters. When the parameters satisfy some ordinary conditions, the GPMHSS iteration method can converge unconditionally with any initial vector.

In this paper, based on the GPMHSS method, we establish its successive-overrelaxation scheme. This work is organized as follows. In Section 2, we introduce the GPMHSS method due to Mehdi, Marzieh and Masoud [14]. In Section 3, we first give the corresponding fixed point equations of the GPMHSS method, and illustrate the equivalence between the new equations and (1.1). Then we propose an accelerated GPMHSS method (AGPMHSS) for (1.1). The theoretical analysis is given in Section 4. Numerical experiments are made in Section 5, which illustrate the numerical behavior of our new method.

## 2. The GPMHSS Method

In this section, we introduce the GPMHSS method [14] for solving large sparse and complex symmetric linear system (1.1). The splitting iteration method can be described as follows.

**The GPMHSS iteration method [14]**
Let $x_0 \in \mathbb{C}^n$ be an arbitrary initial guess. Compute $x_{k+1}$ for $k = 0, 1, \cdots$ using the following

iteration scheme until $\{x_k\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges,

$$\begin{cases} (\alpha V + W)x_{k+\frac{1}{2}} = (\alpha V - iT)x_k + b, \\ (\beta V + T)x_{k+1} = (\beta V + iW)x_{k+\frac{1}{2}} - ib, \end{cases}$$

where $\alpha, \beta$ are given positive constants, and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

In matrix-vector form, the above GPMHSS iteration method can be equivalently rewritten as

$$x_{k+1} = M(V;\alpha,\beta)x_k + G(V;\alpha,\beta)b, \qquad k = 0,1,2,\cdots, \tag{2.1}$$

where

$$M(V;\alpha,\beta) = (\beta V + T)^{-1}(\beta V + iW)(\alpha V + W)^{-1}(\alpha V - iT),$$
$$G(V;\alpha,\beta) = (\beta V + T)^{-1}((\beta V + iW)(\alpha V + W)^{-1} - iI).$$

Here, $M(V;\alpha,\beta)$ is the iteration matrix of the GPMHSS method, and

$$\rho(M(V;\alpha,\beta)) \le \max_{\widetilde{\lambda}_j \in sp(V^{-1}W)} \frac{\sqrt{\beta^2 + \widetilde{\lambda}_j^2}}{\alpha + \widetilde{\lambda}_j} \cdot \max_{\widetilde{\mu}_j \in sp(V^{-1}T)} \frac{\sqrt{\alpha^2 + \widetilde{\mu}_j^2}}{\beta + \widetilde{\mu}_j} = \sigma(\alpha,\beta),$$

where $\widetilde{\lambda}_j$, $\widetilde{\mu}_j$, $j = 1,2,\cdots,n$, are the eigenvalues of $V^{-1}W$ and $V^{-1}T$, respectively. $sp(\cdot)$ represents the spectrum of the corresponding matrix.

Denote $\tilde{\lambda}_{min}$ and $\tilde{\mu}_{min}$ the minimums of the eigenvalues of $V^{-1}W$ and $V^{-1}T$, respectively. If $\alpha \ge 0$, $\beta > 0$, $\sqrt{\alpha^2 + \tilde{\mu}_{min}^2} - \tilde{\mu}_{min} \le \beta < \sqrt{\alpha^2 + 2\alpha\tilde{\lambda}_{min}}$, then $\sigma(\alpha,\beta) < 1$, and the GPMHSS iteration converges to the unique solution of the linear system (1.1), see [14].

## 3. The Accelerated GPMHSS Iteration Method

In this section, we present a successive-overrelaxation (SOR) acceleration scheme for the GPMHSS iteration, and denote the new method as the AGPMHSS method.

From Algorithm 2, we can obtain the corresponding fixed point equations of the GPMHSS iteration method

$$\begin{cases} (\alpha V + W)x = (\alpha V - iT)y + b, \\ (\beta V + T)y = (\beta V + iW)x - ib. \end{cases} \tag{3.1}$$

Then, we have the following theorem, which means equations (1.1) is equivalent to equations (3.1).

**Theorem 3.1.** *If $x_*$ is the exact solution of equation* (1.1)*, then it is also the exact solution of equations* (3.1)*, and vice versa.*

*Proof.* Multiplying the first equation of (3.1) by $i$, then adding the result to the second equation, we can get

$$(\beta - \alpha i)V(x - y) = 0.$$

Since both $\alpha$ and $\beta$ are positive, matrix $V$ is symmetric positive definite, then we can obtain $x = y$.

Consequently, the fixed point equations (3.1) can be transformed to the following form

$$\begin{cases} (W + iT)x = b, \\ (W + iT)x = b. \end{cases}$$

Hence, the exact solutions of (1.1) and (3.1) are same. $\square$

Equations (3.1) can be rewritten as

$$\begin{pmatrix} \alpha V + W & -(\alpha V - iT) \\ -(\beta V + iW) & \beta V + T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -ib \end{pmatrix}. \tag{3.2}$$

Denote

$$\hat{A}(\alpha,\beta) = \begin{pmatrix} \alpha V + W & -(\alpha V - iT) \\ -(\beta V + iW) & \beta V + T \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = z, \qquad \begin{pmatrix} b \\ -ib \end{pmatrix} = f,$$

then equation (3.2) becomes

$$\hat{A}(\alpha,\beta)z = f.$$

Thus the process that uses the GPMHSS iteration to solve (1.1) is same as one that solves (3.2) directly. And we will give a theorem to illustrate why the two processes are same.

**Theorem 3.2.** *If matrices $W$, $T$ are symmetric positive definite and symmetric positive semi-definite, respectively, and $\alpha \geq 0$, $\beta > 0$, $\sqrt{\alpha^2 + \tilde{\mu}_{min}^2} - \tilde{\mu}_{min} \leq \beta < \sqrt{\alpha^2 + 2\alpha\tilde{\lambda}_{min}}$, then matrix $\hat{A}(\alpha,\beta)$ is nonsingular.*

*Proof.* From

$$\begin{aligned} \hat{A}(\alpha,\beta) &= \begin{pmatrix} \alpha V + W & -(\alpha V - iT) \\ -(\beta V + iW) & \beta V + T \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -(\beta V + iW)(\alpha V + W)^{-1} & I \end{pmatrix} \begin{pmatrix} \alpha V + W & -(\alpha V - iT) \\ 0 & S(V;\alpha,\beta) \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} S(V;\alpha,\beta) &= (\beta V + T) - (\beta V + iW)(\alpha V + W)^{-1}(\alpha V - iT) \\ &= (\beta V + T)\big[I - (\beta V + T)^{-1}(\beta V + iW)(\alpha V + W)^{-1}(\alpha V - iT)\big] \\ &= (\beta V + T)\big(I - M(V;\alpha,\beta)\big), \end{aligned}$$

and matrices $\alpha V + W$, $\beta V + T$ are both positive definite, $\rho(M(V;\alpha,\beta)) < 1$, then $\hat{A}(\alpha,\beta)$ is nonsingular.

**Remark 3.1.** From Theorem 3.2, we can see, if $\alpha \ge 0$, $\beta > 0$, $\sqrt{\alpha^2+\tilde{\mu}_{min}^2}-\tilde{\mu}_{min} \le \beta < \sqrt{\alpha^2+2\alpha\tilde{\lambda}_{min}}$, then equation (3.2) must have a unique solution, which implies the GPMHSS iteration converges unconditionally to the unique solution of the linear system (1.1).

Using Jacobi iteration to solve equations (3.2), we can get the following iteration form

$$\begin{cases} x_{k+1}=(\alpha V+W)^{-1}(\alpha V-iT)y_k+b_1, \\ y_{k+1}=(\beta V+T)^{-1}(\beta V+iW)x_k-ib_2, \end{cases} \tag{3.3}$$

where

$$b_1=(\alpha V+W)^{-1}b, \qquad b_2=(\beta V+T)^{-1}b.$$

The above form can be simplified as follows

$$z_{k+1}=J(V;\alpha,\beta)z_k+g(V;\alpha,\beta),$$

where

$$z_k=\begin{pmatrix} x_k \\ y_k \end{pmatrix}, \qquad g(V;\alpha,\beta)=\begin{pmatrix} b_1 \\ -ib_2 \end{pmatrix},$$
$$J(V;\alpha,\beta)=\begin{pmatrix} 0 & (\alpha V+W)^{-1}(\alpha V-iT) \\ (\beta V+T)^{-1}(\beta V+iW) & 0 \end{pmatrix}.$$

For iteration (3.3), employing the successive-overrelaxation acceleration, thus we get the following accelerated GPMHSS (AGPMHSS) iteration

$$\begin{cases} x_{k+1}=(1-\delta)x_k+\delta(\alpha V+W)^{-1}\big[(\alpha V-iT)y_k+b\big], \\ y_{k+1}=(1-\delta)y_k+\delta(\beta V+T)^{-1}\big[(\beta V+iW)x_{k+1}-ib\big], \end{cases} \tag{3.4}$$

here, $\delta$ is the relaxation factor. Similarly, the above form can be simplified as follow

$$z_{k+1}=N(V;\alpha,\beta,\delta)z_k+\tilde{g}(V;\alpha,\beta,\delta),$$

where

$$N(V;\alpha,\beta,\delta)=\begin{pmatrix} (1-\delta)I & \delta(\alpha V+W)^{-1}(\alpha V-iT) \\ (1-\delta)\delta(\beta V+T)^{-1}(\beta V+iW) & (1-\delta)I+\delta^2M(V;\alpha,\beta) \end{pmatrix},$$
$$\tilde{g}(V;\alpha,\beta,\delta)=\begin{pmatrix} \delta(\alpha V+W)^{-1}b \\ \delta(\beta V+T)^{-1}[\delta(\beta V+iW)(\alpha V+W)^{-1}-iI]b \end{pmatrix},$$

$M(V;\alpha,\beta)$ is the iteration matrix of the GPMHSS iteration method. Obviously, $N(V;\alpha,\beta,\delta)$ is the iteration matrix of the AGPMHSS iteration.

If $\delta=1$,

$$N_1(V;\alpha,\beta,\delta)=\begin{pmatrix} 0 & (\alpha V+W)^{-1}(\alpha V-iT) \\ 0 & M(V;\alpha,\beta) \end{pmatrix},$$

is the iteration matrix of the GPMHSS method, and is also the iteration matrix of Gauss-Seidel method for equations (3.1). If $\alpha = 0$, iteration (3.4) is the SOR scheme of the biased PMHSS iteration, and the iteration matrix is

$$N_2(V;\alpha,\beta,\delta) = \begin{pmatrix} (1-\delta)I & -i\delta W^{-1}T \\ (1-\delta)\delta(\beta V+T)^{-1}(\beta V+iW) & (1-\delta)+\delta^2 M(V;\alpha,\beta) \end{pmatrix}.$$

## 4. Convergence Analysis of the AGPMHSS Method

In this section, we will discuss the convergence properties of the AGPMHSS iteration. First we give some lemmas that are useful to our main theorem.

**Lemma 4.1.** *Under the assumptions of Theorem* 3.2*, then for equations* (3.1)*, both Jacobi method and Gauss-Seidel method converge with any initial guess, and*

$$|\lambda_M| = |\lambda_{N_1}| = |\lambda_J|^2, \tag{4.1}$$

*where* $\lambda_M$*,* $\lambda_{N_1}$*,* $\lambda_J$ *are eigenvalues of matrices* $M(V;\alpha,\beta)$*,* $N_1(V;\alpha,\beta,\delta)$ *and* $J(V;\alpha,\beta)$*, respectively.*

*Proof.* If $\lambda_{N_1}$ is a nonzero eigenvalue of $N_1(V;\alpha,\beta,\delta)$, from the definition of eigenvalue, we can obtain

$$\left|\lambda_{N_1} I - N_1(V;\alpha,\beta,\delta)\right| = 0.$$

Thus,

$$\begin{aligned}\left|\lambda_{N_1} I - N_1(V;\alpha,\beta,\delta)\right| &= \begin{vmatrix} \lambda_{N_1} I & -(\alpha V+W)^{-1}(\alpha V - iT) \\ 0 & \lambda_{N_1} I - M(V;\alpha,\beta) \end{vmatrix} \\ &= \lambda^n \left|\lambda I - M(V;\alpha,\beta)\right| \\ &= 0.\end{aligned}$$

Similarly, if $\lambda_J$ is an eigenvalue of $J(V;\alpha,\beta)$, we have

$$\begin{aligned}|\lambda_J I - J(V;\alpha,\beta)| &= \begin{vmatrix} \lambda_J I & -(\alpha V+W)^{-1}(\alpha V - iT) \\ -(\beta V+T)^{-1}(\beta V+iW) & \lambda_J I \end{vmatrix} \\ &= \left|\lambda_J^2 I - M(V;\alpha,\beta)\right| \\ &= 0.\end{aligned}$$

Consequently, if $\lambda_M$ is an eigenvalue of matrix $M(V;\alpha,\beta)$, we can easily get $|\lambda_M| = |\lambda_{N_1}| = |\lambda_J|^2$. $\square$

It is easy to know that the above two iteration are both convergent, since

$$\rho\big(M(V;\alpha,\beta)\big) = \rho\big(N_1(V;\alpha,\beta,\delta)\big) = \big(\rho(J(V;\alpha,\beta))\big)^2 < 1.$$

**Lemma 4.2.** *Assume $\delta \neq 0$, $\lambda$ is the nonzero eigenvalue of $N(V;\alpha,\beta,\delta)$, if $\mu$ satisfies*

$$(\lambda+\delta-1)^2 = \lambda\delta^2\mu^2, \tag{4.2}$$

*then $\mu$ is the eigenvalue of $J(V,\alpha,\beta)$. Conversely, if $\mu$ is the eigenvalue of $J(V,\alpha,\beta)$, and satisfies* (4.2)*, then $\lambda$ is the eigenvalue of $N(V;\alpha,\beta,\delta)$.*

*Proof.* If $\lambda$ is the nonzero eigenvalue of $N(V;\alpha,\beta,\delta)$, from the definition of eigenvalue, we can obtain

$$\big|\lambda I - N(V;\alpha,\beta,\delta)\big| = 0.$$

While

$$\begin{aligned}\big|\lambda I - N(V;\alpha,\beta,\delta)\big| &= \begin{vmatrix} (\lambda+\delta-1)I & -\delta(\alpha V+W)^{-1}(\alpha V - iT) \\ -(1-\delta)\delta(\beta V+T)^{-1}(\beta V+iW) & (\lambda+\delta-1)I-\delta^2 M(V;\alpha,\beta)\end{vmatrix} \\ &= \big|(\lambda+\delta-1)((\lambda+\delta-1)I-\delta^2M(V;\alpha,\beta))-\delta^2(1-\delta)M(V;\alpha,\beta)\big| \\ &= \big|(\lambda+\delta-1)^2 I - \lambda\delta^2 M(V;\alpha,\beta)\big|. \end{aligned} \tag{4.3}$$

If $\lambda$ and $\mu$ satisfy (4.2), thus, together with (4.3) and Lemma 4.1, then $\mu$ is the eigenvalue of iteration matrix $J(V,\alpha,\beta)$. And vice versa. □

The following theorem is the convergence theorem of the AGPMHSS iteration, we discuss two cases for the convergence.

**Theorem 4.1.** *Assume $A = W + iT$, matrix $W$ is real symmetric positive definite, $T$ is real symmetric semi-positive definite, and $\alpha$, $\beta$ satisfy $\alpha \geq 0$, $\beta > 0$, $\sqrt{\alpha^2+\tilde{\mu}_{min}^2}-\tilde{\mu}_{min} \leq \beta < \sqrt{\alpha^2+2\alpha\tilde{\lambda}_{min}}$, then*
*(I) when all eigenvalues of matrix $J(V;\alpha,\beta)$ are real numbers or purely imaginary numbers, then the AGPMHSS iteration converges if and only if*

$$0 < \delta < 2.$$

*(II) when matrix $J(V;\alpha,\beta)$ has complex eigenvalues (i.e., the real part is not zero), then the AGPMHSS iteration converges if and only if*

$$0 < \delta < 2\sqrt{2}-2 \ \ and \ \ \tau < 1,$$

*where $\tau$ is the module of $\lambda$.*

*Proof.* Because we use successive over-relaxation acceleration for iteration (3.3), and SOR iteration converges if relaxation factor $0 < \delta < 2$, thus we only consider the case of $0 < \delta < 2$. From the above discussion, we know when $\delta = 1$, the AGPMHSS iteration is the GPMHSS iteration, and converges under the assumption of the theorem [14]. Thus, we consider two cases, i.e., $0 < \delta < 2$ and $\delta \neq 1$.

Denote $\mu$ the eigenvalue of matrix $J$, from Lemma 4.1, we have $|\mu| < 1$.

(I) When $\mu$ is a real number or pure imaginary number, simplifying equation (4.2), we can get

$$\lambda^2+\big[2(\delta-1)-\delta^2\mu^2\big]\lambda+(\delta-1)^2=0. \tag{4.4}$$

(i) If $\lambda$ is complex, since the complex eigenvalues of matrix appear in pairs, thus

$$|\lambda|^2=\lambda\overline{\lambda}=(\delta-1)^2,$$

but $0<\delta<2$, hence $|\lambda|^2<1$, and the AGPMHSS iteration converges.
(ii) If $\lambda$ is real, (4.4) is a quadratic equation with real coefficients, thus, using the formula for extracting roots, we have

$$\lambda=\frac{\delta^2\mu^2-2(\delta-1)\pm\sqrt{(2(\delta-1)-\delta^2\mu^2)^2-4(\delta-1)^2}}{2},$$

here the discriminant part

$$\begin{aligned}\Delta&=\sqrt{(2(\delta-1)-\delta^2\mu^2)^2-4(\delta-1)^2}\\&=\sqrt{4(\delta-1)^2-4\delta^2\mu^2(\delta-1)+\delta^4\mu^4-4(\delta-1)^2}\\&=\sqrt{\delta^2\mu^2(\delta^2\mu^2-4(\delta-1))}.\end{aligned}$$

Since $|\mu|<1$, and $\mu$ is a real or pure imaginary number, thus $\mu^2<1$. Together with $0<\delta<2$, hence

$$\Delta<\sqrt{\delta^2(\delta-2)^2}=\big|\delta(\delta-2)\big|=\delta(2-\delta),$$

consequently,

$$\begin{aligned}\lambda&<\frac{\delta^2-2(\delta-1)+\delta(\delta-2)}{2}\\&=1,\end{aligned}$$

and the AGPMHSS iteration converges.

(II) When $\mu$ is a complex number with nonzero real part, i.e., $\mu=u_1+iu_2, u_1u_2\neq 0$, then $\lambda$ is complex, denote $\lambda=\tau e^{i\theta}(\tau\in\mathbb{R}^+,\theta\in[0,2\pi])$. In order to let the AGPMHSS iteration converge, $\tau$ must satisfy $\tau<1$.

Substituting $\lambda=\tau e^{i\theta}$ into (4.4), we have

$$(\tau e^{i\theta}+\delta-1)^2=\tau e^{i\theta}\delta^2\mu^2,$$

i.e.,

$$\mu^2=\left(\frac{\tau e^{i\theta}+\delta-1}{\tau^{\frac{1}{2}}e^{\frac{i\theta}{2}}\delta}\right)^2=\left(\frac{1}{\delta}\tau^{\frac{1}{2}}e^{\frac{i\theta}{2}}+\frac{\delta-1}{\delta}\tau^{-\frac{1}{2}}e^{-\frac{i\theta}{2}}\right)^2.$$

Denote

$$\tau_1=\frac{1}{\delta}\tau^{\frac{1}{2}},\qquad \tau_2=\frac{\delta-1}{\delta}\tau^{-\frac{1}{2}}.$$

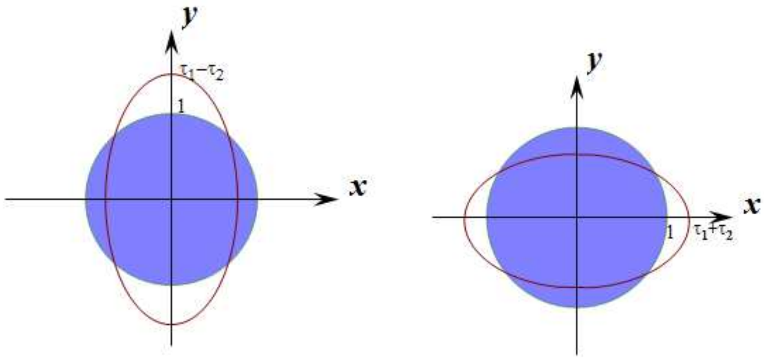


Figure 1: The intersection part of the disc and ellipse.

Then

$$\begin{aligned}\mu^2 &= \left(\tau_1 e^{\frac{i\theta}{2}} + \tau_2 e^{-\frac{i\theta}{2}}\right)^2\\ &= \left((\tau_1+\tau_2)\cos\frac{\theta}{2} + i(\tau_1-\tau_2)\sin\frac{\theta}{2}\right)^2\\ &= (u_1+iu_2)^2.\end{aligned}$$

Hence

$$u_1 = (\tau_1+\tau_2)\cos\frac{\theta}{2}, \qquad u_2 = (\tau_1-\tau_2)\sin\frac{\theta}{2},$$
$$\left(\frac{u_1}{\tau_1+\tau_2}\right)^2 + \left(\frac{u_2}{\tau_1-\tau_2}\right)^2 = 1.$$

Since $|\mu| < 1$, then

$$u_1^2 + u_2^2 < 1.$$

Consequently, $\delta$ must make the following equations have solutions

$$\begin{cases}\left(\dfrac{u_1}{\tau_1+\tau_2}\right)^2 + \left(\dfrac{u_2}{\tau_1-\tau_2}\right)^2 = 1,\\ u_1^2+u_2^2<1,\end{cases} \tag{4.5}$$

i.e., the disc and the ellipse have points of intersection.

When $0 < \delta < 1$, $\tau_1 - \tau_2 > \tau_1 + \tau_2$, equations (4.5) can be transformed to the case in Fig. 1. In order to make ellipse and disc intersect, the following inequalities must be satisfied

$$\tau_1 - \tau_2 > 1 \quad \text{and} \quad \tau_1 + \tau_2 < 1.$$

From

$$\tau_1 - \tau_2 \;=\; \frac{1}{\delta}\tau^{\frac{1}{2}} - \frac{\delta-1}{\delta}\tau^{-\frac{1}{2}} \geq\; 2\sqrt{\frac{1-\delta}{\delta^2}} > 1,$$

we have

$$\delta < 2\sqrt{2}-2.$$

Since

$$\tau_1+\tau_2 \;=\; \frac{1}{\delta}\tau^{\frac{1}{2}}+\frac{\delta-1}{\delta}\tau^{-\frac{1}{2}} \;<\; 1,$$

computing square of each side in the above inequality and simplifying, then we obtain

$$\tau+(1-\delta)^2\tau^{-1} \;<\; (\delta-1)^2+1.$$

From

$$2(1-\delta)\le \tau+(1-\delta)^2\tau^{-1},$$

we get

$$(\delta-1)^2 > 2(1-\delta) \;\Rightarrow\; \delta^2>0.$$

Hence, the value range of $\delta$ is $0<\delta<2\sqrt{2}-2$.

When $1<\delta<2$, $\tau_1+\tau_2>\tau_1-\tau_2$, equations (4.5) can be transformed into the second image in Figure 1. In order to make ellipse and disc intersect, the following inequalities must be satisfied

$$\tau_1+\tau_2>1 \quad \text{and} \quad \tau_1-\tau_2<1.$$

Solving them, then we get $\delta\ge 2$, which contradicts with the value range of $\delta$.

Hence the proof is complete. $\square$

## 5. Numerical Results

In this section, the validity and feasibility of numerical analysis for the AGPMHSS iteration will be given. We compare our AGPMHSS method with the PMHSS method [3] and GPMHSS method [14]. Consider the linear equations $(W+iT)x=b$ [2], where

$$\begin{aligned}
W &= 10(I\otimes V_c+V_c\otimes I)+9(e_1e_m^T+e_me_1^T)\otimes I,\\
T &= I\otimes V+V\otimes I,
\end{aligned}$$

here $V=tridiag(-1,2,-1)\in\mathbb{R}^{m\times m}$, $V_c=V-e_1e_m^T-e_me_1^T\in\mathbb{R}^{m\times m}$, $e_1$ and $e_m$ are the first and last columns of identity matrix $I$, respectively. $b$ is chosen as $b=(1+i)A\mathbf{1}$, where $\mathbf{1}$ is a vector with all elements being 1.

In our computations, we choose $x_0=0$ as the initial vector, the stopping criterion for the iteration is set to be

$$\frac{\| b-Ax^{(k)}\|_2}{\| b\|_2}\le 10^{-7}.$$

Preconditioner $V$ is an arbitrary symmetric definite matrix, for the convenience of operations, we set $V=W$ in the computation.

Denote $(\alpha_{exp},\beta_{exp})$ as the value of $(\alpha,\beta)$ that costs the minimum time of the GPMHSS method. To be fair, we adopt the same $(\alpha_{exp},\beta_{exp})$ as the parameters of our AGPMHSS

Table 1: The optimal parameters of the PMHSS, GPMHSS and AGPMHSS methods.

| m | PMHSS $\alpha_{exp}$ | GPMHSS $(\alpha_{exp}, \beta_{exp})$ | AGPMHSS $(\alpha, \beta, \delta)$ |
|---|---|---|---|
| 30 | 2.13 | (0.43,1.87) | (0.43,1.87,0.81) |
| 40 | 2.01 | (0.34,1.68) | (0.34,1.68,0.75) |
| 50 | 1.07 | (0.36,1.59) | (0.36,1.59,0.77) |

Table 2: The numerical results of the PMHSS, GPMHSS and AGPMHSS methods.

| m | CPU time | | | RES | | | IT | | |
|---|---|---|---|---|---|---|---|---|---|
| | PMHSS | GPMHSS | ADPMHSS | PMHSS | GPMHSS | ADPMHSS | PMHSS | GPMHSS | ADPMHSS |
| 30 | 0.1914 | 0.1682 | 0.1458 | 8.33e-07 | 6.12e-07 | 5.76e-07 | 136 | 97 | 53 |
| 40 | 0.3385 | 0.2943 | 0.1758 | 7.91e-07 | 5.55e-07 | 5.56e-07 | 212 | 115 | 88 |
| 50 | 0.5267 | 0.4927 | 0.3721 | 8.91e-07 | 7.66e-07 | 7.19e-07 | 294 | 201 | 132 |

method, and experimentally optimal parameters $\delta$. Similarly, for the PMHSS method, we adopt experimentally optimal parameters $\alpha$. Specific details can be obtained in Table 1.

In Table 2, numbers of iterations, which can reflect the rate of convergence, are denoted with IT, and we denote by "CPU" the CPU time used in seconds, by "RES" the relative error of the iterations.

It is obvious to see from Table 2 that the PMHSS method, GPMHSS method and AGPMHSS method all can solve equation $(W + iT)x = b$ efficiently. However, two parameters $\alpha$ and $\beta$ make the GPMHSS method converges faster and higher precision rate than the PMHSS method. Furthermore, the relaxation factor $\delta$ makes the AGPMHSS method have the best performing, which shows successive-overrelaxation acceleration is indeed effective.

## 6. Conclusions

In this paper we have introduced the AGPMHSS method, which is a successive overrelaxation acceleration scheme of the GPMHSS method for solving linear complex symmetric equations. We first established the fixed point equations of the GPMHSS method, then accelerate it, thus get our AGPMHSS method. And then we also have established the convergence of the new iteration. Finally, the numerical experiments indicate its efficiency. Since we just use the two optimal parameters in the GPMHSS method to implement the AGPMHSS method, our acceleration scheme should be able to obtain better results. For the normal/skew-Hermitian splitting (NSS) iteration method, Bai, Golub and Ng [9] ever considered an successive-overrelaxation (SOR) acceleration scheme and gave an optimal value of the SOR parameter. Theoretical analysis of optimal parameters for our AGPMHSS method with two parameters will be on the way.

## Acknowledgments

The authors are very much indebted to the referees for their valuable suggestions which greatly improved the original version of this paper. The paper is partly supported by the

National Natural Science Foundation of China (No. 11371145, No. 11471122), Science and Technology Commission of Shanghai Municipality (No. 13dz2260400).

## References


[1] S. R. ARRIDGE, *Optical Tomography in Medical Imaging*, Inverse Problems, 15 (1999), pp. 41–93.

[2] Z. Z. BAI, M. BENZI, AND F. CHEN, *Modified HSS iteration methods for a class of complex symmetric linear systems*, Computing, 87 (2010), pp. 93–111.

[3] Z. Z. BAI, M. BENZI, AND F. CHEN, *On preconditioned MHSS iteration methods for complex symmetric linear systems*, Numer. Algor., 56 (2011), pp. 297–317.

[4] Z. Z. BAI, M. BENZI, F. CHEN, AND Z. Q. WANG, *Preconditioned MHSS iteration methods for a class of block two-by-two linear systems with application to distributed control problems*, IMA J. Numer. Anal., 33 (2013), pp. 343–369.

[5] Z. Z. BAI AND G. H. GOLUB, *Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems*, IMA J. Numer. Anal., 27 (2007), pp. 1–23.

[6] Z. Z. BAI, G. H. GOLUB, AND C. K. LI, *Convergence properties of preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite matrices*, Math. Comput., 76 (2007), pp. 287–298.

[7] Z. Z. BAI, G. H. GOLUB, L. Z. LU, AND J. F. YIN, *Block triangular and skew-Hermitian splitting methods for positive-definite linear systems*, SIAM J. Sci. Comput., 26 (2005), pp. 844–863.

[8] Z. Z. BAI, G. H. GOLUB, AND M. K. NG, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl., 24 (2003), pp. 603–626.

[9] Z. Z. BAI, G. H. GOLUB, AND M. K. NG, *On successive-overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iterations*, Numer. Linear Algebra Appl., 14 (2007), pp. 319–335.

[10] Z. Z. BAI, G. H. GOLUB, AND J. Y. PAN, *Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems*, Numer. Math., 98 (2004), pp. 1–32.

[11] Z. Z. BAI AND X. P. GUO, *On Newton-HSS methods for systems of nonlinear equations with positive-definite Jacobian matrices*, J. Comput. Math., 28 (2010), pp. 235–260.

[12] D. BERTACCINI, *Efficient preconditioning for sequences of parametric complex symmetric linear systems*, Electr. Trans. Numer. Anal., 18 (2004), pp. 49–64.

[13] M. H. CHEN, R. F. LIN, AND Q. B. WU, *Convergence analysis of the modified Newton-HSS method under the hölder continuous condition*, J. Comput. Appl. Math., 264 (2014), pp. 115–130.

[14] M. DEHGHAN, M. DEHGHANI-MADISEH, AND M. HAJARIAN, *A generalized preconditioned MHSS method for a class of complex symmetric linear systems*, Math. Model. Anal., 18 (2013), pp. 561–576.

[15] A. FERIANI, F. PEROTTI, AND V. SIMONCINI, *Iterative system solvers for the frequency analysis of linear mechanical systems*, Comput. Methods Appl. Mech. Eng., 190 (2000), pp. 1719–1739.

[16] A. FROMMER, T. LIPPERT, B. MEDEKE, AND K. SCHILLING (EDS), *Numerical Challenges in Lattice Quantum Chromodynamics*, Springer, Berlin, 2000.

[17] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, John Hopkins University Press, Baltimore, 2012.

[18] X. P. GUO AND I. S. DUFF, *Semilocal and global convergence of the Newton-HSS method for systems of nonlinear equations*, Numer. Linear Algebra Appl., 18 (2011), pp. 299–315.

[19] C. T. KELLY, *Iterative Methods for Linear and Nonlinear Equations*, SIAM, Philadelphia, 1995.

[20] Y. LI AND X. P. GUO, *Multi-step modified Newton-HSS methods for systems of nonlinear equations with positive definite Jacobian matrices*, Numer. Algor., 10.1007/s11075-016-0196-6, 2016.

[21] B. POIRIER, *Efficient preconditioning scheme for block partitioned matrices with structured sparsity*, Numer. Linear Algebra Appl., 7 (2000), pp. 715–726.

[22] Y. SAAD, *Iterative Methods for Sparse Linear Systems*, SIAM, Philadelphia, 2003.

[23] D. W. VAN AND F. M. TOYAMA, *Accurate numerical solutions of the time-dependent Schrödinger equation*, Phys. Rev. E, 75 (2007), pp. 036707.

[24] Q. B. WU AND M. H. CHEN, *Convergence analysis of modified Newton-HSS method for solving systems of nonlinear equations*, Numer. Algor., 64 (2013), pp. 659–683.

[25] W. W. XU, *A generalization of PMHSS iteration method for complex symmetric indefinite linear systems*, Appl. Math. Comput., 219 (2013), pp. 10510–10517.

[26] A. L. YANG AND Y. J. WU, *Newton-MHSS methods for solving systems of nonlinear equations with complex symmetric Jacobian matrices*, Numer. Algebra, Control Optim., 2 (2012), pp. 839–853.

[27] H. X. ZHONG, G. L. CHEN, AND X. P. GUO, *On preconditioned modified Newton-MHSS method for systems of nonlinear equations with complex symmetric Jacobian matrices*, Numer. Algor., 69 (2015), pp. 553–567.