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The Explicit Inverses of CUPL-Toeplitz and CUPL-Hankel Matrices

Zhao-Lin Jiang[1], Xiao-Ting Chen[1,2,*] and Jian-Min Wang[1]

[1] *Department of Mathematics, Linyi University, Linyi 276005, P. R. China.*
[2] *School of Mathematical Sciences, Shandong Normal University, Jinan 250014, P. R. China.*

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Abstract. In this paper, we consider two innovative structured matrices, CUPL-Toeplitz matrix and CUPL-Hankel matrix. The inverses of CUPL-Toeplitz and CUPL-Hankel matrices can be expressed by the Gohberg-Heinig type formulas, and the stability of the inverse matrices is verified in terms of 1-, ∞- and 2-norms, respectively. In addition, two algorithms for the inverses of CUPL-Toeplitz and CUPL-Hankel matrices are given and examples are provided to verify the feasibility of these algorithms.

AMS subject classifications: 15A09, 15B05

Key words: CUPL-Toeplitz matrix, CUPL-Hankel matrix, inverse, stability, algorithm.

1. Introduction

As is well known, Toeplitz matrices family are also structured matrices family and have important applications in various disciplines including the elliptic Dirichlet-periodic boundary value problems [1], sinc discretizations of partial and ordinary differential equations [2–7], signal processing [8], numerical analysis [8], system theory [8], etc.

It is an ideal research area and hot issue to find the inverse of a Toeplitz matrix. In [9], Jiang and Wang firstly present an innovative structured matrix, RFPL-Toeplitz matrix, the group inverse of this new structured matrix can be represented as the sum of products of lower and upper triangular Toeplitz matrices, then the explicit expression and the decomposition of the group inverse is given. It turns out the inversion of Toeplitz matrix can be reconstructed by a low number of its columns and the entries of the original Toeplitz matrix. The result was first observed by Trench [10] and reconstructed by Gohberg and Semencul [11] from its first and last columns of T^{-1}, provided that the first component in the first column is not zero. The algorithm of Trench for the inversion of Toeplitz matrices is presented with a detailed proof in [12]. Gohberg and Krupnik [13] observed that

*Corresponding author. *Email addresses:* `jzh1208@sina.com` (Z.-L. Jiang), `chenxt0723@163.com` (X.-T. Chen), `wjm0818@163.com` (J.-M. Wang)

if the last component of the first column is not zero, then T^{-1} can be recovered from its first and second columns. Heinig and Rost [14] exhibited an inversion formula for every nonsingular Toeplitz matrix, which requires the solution of fundamental equations, where the right-hand side of one of them is a shifted column of the Toeplitz matrix T. In [15], the inverse was reconstructed through three columns of T^{-1}. Labahn and Ng modified this result in [16] and [17]. In [18], the inverse of the Toeplitz matrix was presented in the form of Toeplitz Bezoutian of two columns. Lv and Huang [19] gave a new Toeplitz matrix inversion formula in which the inverse can be denoted as a sum of products of circulant matrices and upper triangular Toeplitz matrices. Labahn [20] proposed formulas for the inverses of layered or striped Toeplitz matrices in terms of solutions of standard equations. The explicit inverses of nonsingular conjugate-Toeplitz and conjugate-Hankel matrices are provided in [21].

In [22] and [23], the stability of the algorithms emerging from Toeplitz matrix inversion formulas was considered. Xie and Wei [24] proposed a stability analysis of Gohberg-Semencul-Trench type formula for Moore-Penrose and group inverses of Toeplitz matrices.

In this paper, we present the explicit inverses of CUPL-Toeplitz and CUPL-Hankel matrices, which can be expressed as sum of products of circulant and upper triangular Toeplitz matrices, which is thought of a Gohberg-Heinig type formula for the inverse of an CUPL-Toeplitz matrix and CUPL-Hankel matrix. Moreover, the stability of the inverse formula is verified in terms of 1-, ∞- and 2-norms, respectively. Then the algorithms of the inverse formulas are provided. And in the final, examples are given to support the feasibility of the algorithms.

Definition 1.1. An $n\times n$ column upper-plus-lower Toeplitz matrix with the first row $(a_0, a_{-1}, a_{-2}, \cdots, a_{1-n})$ and the first column $(a_0, a_1, a_2, \cdots, a_{n-1})^T$ is meant a matrix of the form as

$$T_{CUPL} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{1-n} \\ a_1 & a_0 + a_1 & \ddots & \ddots & \vdots \\ a_2 & a_1 + a_2 & \ddots & \ddots & a_{-2} \\ \vdots & \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} + a_{n-1} & \cdots & a_1 + a_2 & a_0 + a_1 \end{pmatrix}, \tag{1.1}$$

denoted by $T_{CUPL}[fr(a_0, a_{-1}, a_{-2}, \cdots, a_{2-n}, a_{1-n});\ fc(a_0, a_1, a_2, \cdots, a_{n-2}, a_{n-1})^T]$, or by T_{CUPL} for short, where $a_0,\ a_{\pm 1},\ a_{\pm 2}, \cdots, a_{\pm(n-1)}$ are any complex numbers.

Obviously, the entries a_{ij} of the matrix in (1.1) are given by the following formulas:

$$a_{ij} = \begin{cases} a_{i-j}, & j = 1 \text{ or } j > i \\ a_{i-j} + a_{i-j+1}, & 2 \le j \le i. \end{cases} \tag{1.2}$$

Specially, if $a_{1-n} = a_1,\ a_{2-n} = a_2,\ \cdots,\ a_{-1} = a_{n-1}$, then T_{CUPL} is a row first-plus-last right circulant matrix, which is firstly defined in [25].

Definition 1.2. An $n\times n$ column upper-plus-lower Hankel matrix with the first row $(b_0, b_1, b_2, \cdots, b_{n-1})$ and the last column $(b_{n-1}, b_n, \cdots, b_{2n-2})^T$ is meant a matrix of the form as

$$H_{CUPL}=\begin{pmatrix} b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \\ b_1 & \iddots & \iddots & b_{n-1}+b_n & b_n \\ \vdots & \iddots & \iddots & b_n+b_{n+1} & \vdots \\ b_{n-2} & \iddots & \iddots & \vdots & b_{2n-3} \\ b_{n-1}+b_n & b_n+b_{n+1} & \cdots & b_{2n-3}+b_{2n-2} & b_{2n-2} \end{pmatrix}, \tag{1.3}$$

denoted by $H_{CUPL}[fr(b_0, b_1, \cdots, b_{n-2}, b_{n-1});\ lc(b_{n-1}, b_n, \cdots, b_{2n-3}, b_{2n-2})^T]$, or by H_{CUPL} for short, where $b_0,\ b_1,\ b_2, \cdots, b_{2n-3},\ b_{2n-2}$ are any complex numbers.

Obviously, the entries b_{ij} of the matrix in (1.3) are given by the following formulas:

$$b_{ij}=\begin{cases} b_{i+j-2}, & j=n \text{ or } i+j\le n \\ b_{i+j-2}+b_{i+j-1}, & i+j>n \text{ and } j<n. \end{cases} \tag{1.4}$$

Specially, if $b_0=b_n,\ b_1=b_{n+1},\ \cdots, b_{n-2}=b_{2n-2}$, then H_{CUPL} is called a row first-plus-last left-circulant matrix, which is firstly defined in [26].

It should be mentioned that the CUPL-Toeplitz matrix is neither an extension of the Toeplitz matrix nor its special case and it is an innovative structured matrix. This is the same to CUPL-Hankel matrix as well.

2. The Inversion Formula of T_{CUPL}

In this section, we provide inversion formula for T_{CUPL} matrix as a sum of products of row first-plus-last right circulant matrices and upper triangular Toeplitz matrices.

Let T_{CUPL} be defined as in Definition 1.1 with the first row $(a_0, a_{-1}, \cdots, a_{1-n})$ and the first column $(a_0, a_1, \cdots, a_{n-1})^T$.

Theorem 2.1. *Let T_{CUPL} be an $n\times n$ column upper-plus-lower Toeplitz matrix and $f=(0, a_{1-n}-a_1, a_{2-n}-a_2, \cdots, a_{-1}-a_{n-1})^T,\ e_1=(1,0,\cdots,0)^T$. If each of the systems of equations $T_{CUPL}X=f,\ T_{CUPL}Y=e_1$ are solvable, where $X=(x_1, x_2, \cdots, x_n)^T,\ \ Y=(y_1, y_2, \cdots, y_n)^T$, then T_{CUPL} is invertible and*

$$T_{CUPL}^{-1}=V_1U_1+V_2U_2, \tag{2.1}$$

where

$$V_1 = \begin{pmatrix} y_1 & y_n & y_{n-1} & \cdots & y_2 \\ y_2 & y_1+y_2 & \ddots & \ddots & \vdots \\ y_3 & y_2+y_3 & \ddots & \ddots & y_{n-1} \\ \vdots & \vdots & \ddots & \ddots & y_n \\ y_n & y_{n-1}+y_n & \cdots & y_2+y_3 & y_1+y_2 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} x_1 & x_n & x_{n-1} & \cdots & x_2 \\ x_2 & x_1+x_2 & \ddots & \ddots & \vdots \\ x_3 & x_2+x_3 & \ddots & \ddots & x_{n-1} \\ \vdots & \vdots & \ddots & \ddots & x_n \\ x_n & x_{n-1}+x_n & \cdots & x_2+x_3 & x_1+x_2 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 1 & -x_n & -x_{n-1} & \cdots & -x_2 \\ & 1 & -x_n & \ddots & \vdots \\ & & \ddots & \ddots & -x_{n-1} \\ & & & 1 & -x_n \\ & & & & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & y_n & y_{n-1} & \cdots & y_2 \\ & 0 & y_n & \ddots & \vdots \\ & & \ddots & \ddots & y_{n-1} \\ & & & 0 & y_n \\ & & & & 0 \end{pmatrix},$$

V_1 and V_2 are both row first-plus-last right circulant matrices [25].

Proof. T_{CUPL} is a column upper-plus-lower Toeplitz matrix defined in Definition 1.1, which satisfies

$$KT_{CUPL} - T_{CUPL}K = fe_n^T - e_1 f^T \Sigma \tag{2.2}$$

where

$$K = \begin{pmatrix} 0 & & & & 1 \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} & & & & 1 \\ & & & 1 & 1 \\ & & \iddots & \iddots & \\ & 1 & 1 & & \\ 1 & 1 & & & \end{pmatrix},$$

and $e_n^T = (0, \cdots, 0, 1)$.

As $T_{CUPL}X = f,\ T_{CUPL}Y = e_1$, from (2.2) we can get

$$\begin{aligned} KT_{CUPL} &= T_{CUPL}K + fe_n^T - e_1 f^T \Sigma \\ &= T_{CUPL}(K + Xe_n^T - Y f^T \Sigma). \end{aligned}$$

From the above equation we can obtain

$$\begin{aligned} K^i T_{CUPL} &= K^{i-1} T_{CUPL}(K + Xe_n^T - Y f^T \Sigma) \\ &= T_{CUPL}(K + Xe_n^T - Y f^T \Sigma)^i. \end{aligned}$$

Therefore,

$$K^i e_1 = K^i T_{CUPL} Y = T_{CUPL}(K + Xe_n^T - Yf^T\Sigma)^i Y,$$

and use the mathematical induction method we can obtain

$$K^{i-1}e_1 = (0,1,C_{i-2}^1, C_{i-2}^2, \cdots, C_{i-2}^{i-3}, 1, 0, \cdots, 0)^T, \quad (i = 1,2,\cdots,n-1) \tag{2.3}$$

$$K^{n-1}e_1 = (0,1,C_{i-2}^1, C_{i-2}^2, \cdots, C_{i-2}^{i-3}, 1)^T, \tag{2.4}$$

where C_n^i is binomial coefficient $\left(\begin{smallmatrix} n \\ i \end{smallmatrix}\right)$.

If we denote

$$t_1 = Y,$$
$$t_i = (K + Xe_n^T - Yf^T\Sigma)\big[t_{i-1} - t_{i-2} + t_{i-3} + \cdots + (-1)^i t_1\big], \quad (i = 2,3,\cdots,n)$$

and

$$\hat{T}_{CUPL} = (t_1,\ t_2,\ t_3,\ \cdots, t_n),$$

then

$$T_{CUPL}t_1 = T_{CUPL}Y = e_1,$$
$$\begin{aligned} T_{CUPL}t_i &= T_{CUPL}(K + Xe_n^T - Yf^T\Sigma)\big[t_{i-1} - t_{i-2} + t_{i-3} + \cdots + (-1)^i t_1\big] \\ &= KT_{CUPL}\big[t_{i-1} - t_{i-2} + t_{i-2} + \cdots + (-1)^i t_1\big], \quad (i = 2,3,\cdots,n). \end{aligned}$$

We can obtain

$$\begin{aligned} & KT_{CUPL}\big[t_{i-1} - t_{i-2} + t_{i-3} + \cdots + (-1)^i t_1\big] \\ =&(-1)^0 C_{i-2}^0 K^{i-1}e_1 + (-1)^1 C_{i-2}^1 K^{i-2}e_1 + \cdots + (-1)^{i-2} C_{i-2}^{i-2} K e_1 \\ =&e_i, \end{aligned} \tag{2.5}$$

where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T$.

That is to say $T_{CUPL}t_i = e_i$, we can obtain $T_{CUPL}\hat{T}_{CUPL} = I_n$, where I_n is an identity matrix. From above, the matrix T_{CUPL} is invertible, and $T_{CUPL}^{-1} = \hat{T}_{CUPL}$.

Next, we can derive the representation of t_i. First of all, it is easy to see that

$$t_1 = Y, \qquad t_i = T_{CUPL}^{-1}e_i, \qquad \Sigma T_{CUPL}\Sigma^{-1} = T_{CUPL}^T, \qquad \Sigma^T = \Sigma$$
$$\Sigma e_1 = e_n, \qquad \Sigma e_i = e_{n-i+1} + e_{n-i+2},\ (i = 2,3,\cdots,n).$$

For $i > 1$,

$$\begin{aligned} t_i =&(K + Xe_n^T - Yf^T\Sigma)\big[t_{i-1} - t_{i-2} + t_{i-3} - \cdots + (-1)^i t_1\big] \\ =&\big[Kt_{i-1} - Kt_{i-2} + \cdots + (-1)^i K t_1\big] + (Xe_n^T - Yf^T\Sigma)t_{i-1} - (Xe_n^T - Yf^T\Sigma)t_{i-2} + \cdots \\ &+ (-1)^i (Xe_n^T - Yf^T\Sigma)t_1 \\ =&K\big[t_{i-1} - t_{i-2} + \cdots + (-1)^i t_1\big] + (Xy_{n-i+2} + Xy_{n-i+3} - Yx_{n-i+2} - Yx_{n-i+3}) \\ &- (Xy_{n-i+3} + Xy_{n-i+4} - Yx_{n-i+3} - Yx_{n-i+4}) + \cdots + (-1)^{i-1}(Xy_{n-1} + Xy_n - Yx_{n-1} \\ &- Yx_n) + (-1)^i (Xy_n - Yx_n) \\ =&K\big[t_{i-1} - t_{i-2} + \cdots + (-1)^i t_1\big] + Xy_{n-i+2} - Yx_{n-i+2}, \end{aligned}$$

in the middle of the above procedure, for $i > 2$,

$$\begin{aligned}
&(Xe_n^T - Yf^T\Sigma)t_{i-1}\\
=&Xe_n^T t_{i-1} - Yf^T\Sigma t_{i-1}\\
=&Xe_n^T T_{CUPL}^{-1}e_{i-1} - Yf^T\Sigma T_{CUPL}^{-1}e_{i-1}\\
=&Xe_n^T\Sigma^{-1}\Sigma T_{CUPL}^{-1}\Sigma^{-1}\Sigma e_{i-1} - Yf^T\Sigma T_{CUPL}^{-1}\Sigma^{-1}\Sigma e_{i-1}\\
=&Xe_n^T\Sigma^{-1}T_{CUPL}^{-T}\Sigma e_{i-1} - Yf^T T_{CUPL}^{-T}\Sigma e_{i-1}\\
=&Xe_1^T T_{CUPL}^{-T}(e_{n-i+2}+e_{n-i+3}) - Yf^T T_{CUPL}^{-T}(e_{n-i+2}+e_{n-i+3})\\
=&X(T_{CUPL}^{-1}e_1)^T(e_{n-i+2}+e_{n-i+3}) - Y(T_{CUPL}^{-1}f)^T(e_{n-i+2}+e_{n-i+3})\\
=&XY^T(e_{n-i+2}+e_{n-i+3}) - YX^T(e_{n-i+2}+e_{n-i+3})\\
=&Xy_{n-i+2}+Xy_{n-i+3}-Yx_{n-i+2}-Yx_{n-i+3}.
\end{aligned}$$

We have

$$\begin{aligned}
&t_1 = Y, \qquad t_2 = KY + Xy_n - Yx_n, \qquad \cdots\ ,\\
&t_n = \left[K^{n-1} + (-1)^1C_{n-2}^1K^{n-2} + (-1)^2C_{n-2}^2K^{n-3} + \cdots + (-1)^{n-2}C_{n-2}^{n-2}K^1\right]Y\\
&\quad + \left[K^{n-2} + (-1)^1C_{n-3}^1K^{n-3} + (-1)^2C_{n-3}^2K^{n-4} + \cdots + (-1)^{n-3}C_{n-3}^{n-3}K^1\right](Xy_n - Yx_n)\\
&\quad + \left[K^{n-3} + (-1)^1C_{n-4}^1K^{n-4} + (-1)^2C_{n-4}^2K^{n-5} + \cdots + (-1)^{n-4}C_{n-4}^{n-4}K^1\right](Xy_{n-1} - Yx_{n-1})\\
&\quad + \cdots + K(Xy_3 - Yx_3) + (Xy_2 - Yx_2),\\
&T_{CUPL}^{-1} = (t_1,\ t_2,\ t_3,\ \cdots, t_n) = V_1U_1 + V_2U_2,
\end{aligned}$$

where

$$\begin{aligned}
V_1 = \big(&Y, KY, (K^2-K)Y, \cdots,\\
&(K^{n-1} + (-1)^1C_{n-2}^1K^{n-2} + (-1)^2C_{n-2}^2K^{n-3} + \cdots + (-1)^{n-2}C_{n-2}^{n-2}K^1)Y\big),
\end{aligned}$$

$$U_1 = \begin{pmatrix} 1 & -x_n & -x_{n-1} & \cdots & -x_2 \\ & 1 & -x_n & \ddots & \vdots \\ & & \ddots & \ddots & -x_{n-1} \\ & & & 1 & -x_n \\ & & & & 1 \end{pmatrix},$$

$$\begin{aligned}
V_2 = \big(&X, KX, (K^2-K)X, \cdots,\\
&(K^{n-1} + (-1)^1C_{n-2}^1K^{n-2} + (-1)^2C_{n-2}^2K^{n-3} + \cdots + (-1)^{n-2}C_{n-2}^{n-2}K^1)X\big),
\end{aligned}$$

$$U_2 = \begin{pmatrix} 0 & y_n & y_{n-1} & \cdots & y_2 \\ & 0 & y_n & \ddots & \vdots \\ & & \ddots & \ddots & y_{n-1} \\ & & & 0 & y_n \\ & & & & 0 \end{pmatrix}.$$

We can write it in the formula

$$
\begin{aligned}
&T_{CUPL}^{-1}\\
&=\begin{pmatrix}
y_1 & y_n & y_{n-1} & \cdots & y_2\\
y_2 & y_1+y_2 & \ddots & \ddots & \vdots\\
y_3 & y_2+y_3 & \ddots & \ddots & y_{n-1}\\
\vdots & \vdots & \ddots & \ddots & y_n\\
y_n & y_{n-1}+y_n & \cdots & y_2+y_3 & y_1+y_2
\end{pmatrix}\times\begin{pmatrix}
1 & -x_n & -x_{n-1} & \cdots & -x_2\\
 & 1 & -x_n & \ddots & \vdots\\
 & & \ddots & \ddots & -x_{n-1}\\
 & & & 1 & -x_n\\
 & & & & 1
\end{pmatrix}\\
&\quad+\begin{pmatrix}
x_1 & x_n & x_{n-1} & \cdots & x_2\\
x_2 & x_1+x_2 & \ddots & \ddots & \vdots\\
x_3 & x_2+x_3 & \ddots & \ddots & x_{n-1}\\
\vdots & \vdots & \ddots & \ddots & x_n\\
x_n & x_{n-1}+x_n & \cdots & x_2+x_3 & x_1+x_2
\end{pmatrix}\times\begin{pmatrix}
0 & y_n & y_{n-1} & \cdots & y_2\\
 & 0 & y_n & \ddots & \vdots\\
 & & \ddots & \ddots & y_{n-1}\\
 & & & 0 & y_n\\
 & & & & 0
\end{pmatrix}.
\end{aligned}
$$

□

3. The Inversion Formula of H_{CUPL}

In this section, we provide the inversion formula of the H_{CUPL} matrix as a sum of products of row first-plus-last inverse right circulant matrices and upper triangular Toeplitz matrices.

Let H_{CUPL} be defined as in Definition 1.2 with the first row $(b_0, b_1, \cdots, b_{n-1})$ and the last column $(b_{n-1}, b_n, \cdots, b_{2n-2})^T$.

Theorem 3.1. *Let H_{CUPL} be an $n \times n$ column upper-plus-lower Hankel matrix and $r = (0, b_0 - b_n, b_1 - b_{n+1} \cdots, b_{n-2} - b_{2n-2})^T$, $e_1 = (1, 0, \cdots, 0)^T$. If each of the systems of equations $H_{CUPL}Z = r$, $H_{CUPL}W = e_1$ are solvable, where $Z = (z_1, z_2, \ldots, z_n)^T$, $W = (w_1, w_2, \ldots, w_n)^T$, then H_{CUPL} is invertible and*

$$H_{CUPL}^{-1} = S_1 T_1 + S_2 T_2, \tag{3.1}$$

where

$$
S_1 = \begin{pmatrix}
w_1 & w_2+w_1 & w_3+w_2 & \cdots & w_n+w_{n-1}\\
w_2 & w_3+w_2 & \iddots & \iddots & w_1\\
\vdots & \vdots & \iddots & \iddots & w_2\\
w_{n-1} & w_n+w_{n-1} & \iddots & \iddots & \vdots\\
w_n & w_1 & w_2 & \cdots & w_{n-1}
\end{pmatrix},
$$

$$S_2 = \begin{pmatrix} z_1 & z_2+z_1 & z_3+z_2 & \cdots & z_n+z_{n-1} \\ z_2 & z_3+z_2 & \iddots & \iddots & z_1 \\ \vdots & \vdots & \iddots & \iddots & z_2 \\ z_{n-1} & z_n+z_{n-1} & \iddots & \iddots & \vdots \\ z_n & z_1 & z_2 & \cdots & z_{n-1} \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 1 & -z_1 & -z_2 & \cdots & -z_{n-1} \\ & 1 & -z_1 & \ddots & \vdots \\ & & \ddots & \ddots & -z_2 \\ & & & 1 & -z_1 \\ & & & & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & w_1 & w_2 & \cdots & w_{n-1} \\ & 0 & w_1 & \ddots & \vdots \\ & & \ddots & \ddots & w_2 \\ & & & 0 & w_1 \\ & & & & 0 \end{pmatrix},$$

S_1 and S_2 are both row first-plus-last inverse right circulant matrices.

Proof. H_{CUPL} is a column upper-plus-lower Hankel matrix defined as Definition 1.2, which satisfies

$$KH_{CUPL} - H_{CUPL}\Gamma = re_1^T - e_1 r^T \Theta \tag{3.2}$$

where

$$K = \begin{pmatrix} 0 & & & & 1 \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & 1 & 1 & \\ & & & 1 & 1 \end{pmatrix}.$$

As $H_{CUPL}Z = r$, $H_{CUPL}W = e_1$, from (3.2) we can get

$$\begin{aligned} KH_{CUPL} &= H_{CUPL}\Gamma + re_1^T - e_1 r^T \Theta \\ &= H_{CUPL}(\Gamma + Ze_1^T - Wr^T\Theta), \end{aligned}$$

from the above equation we can obtain

$$\begin{aligned} K^i H_{CUPL} &= K^{i-1} H_{CUPL}(\Gamma + Ze_1^T - Wr^T\Theta) \\ &= H_{CUPL}(\Gamma + Ze_1^T - Wr^T\Theta)^i. \end{aligned}$$

Therefore,

$$K^i e_1 = K^i H_{CUPL} W = H_{CUPL}(\Gamma + Ze_1^T - Wr^T\Theta)^i W,$$

and we can obtain

$$K^{i-1}e_1 = (0, 1, C_{i-2}^1, C_{i-2}^2, \cdots, C_{i-2}^{i-3}, 1, 0, \cdots, 0)^T, \quad (i = 1, 2, \cdots, n-1) \tag{3.3}$$
$$K^{n-1}e_1 = (0, 1, C_{i-2}^1, C_{i-2}^2, \cdots, C_{i-2}^{i-3}, 1)^T. \tag{3.4}$$

If we denote

$$
\begin{aligned}
&h_1 = W,\\
&h_i = (\Gamma + Ze_1^T - Wr^T\Theta)\big[h_{i-1} - h_{i-2} + h_{i-3} + \cdots + (-1)^i h_1\big], \quad (i = 2, 3, \cdots, n)
\end{aligned}
$$

and

$$
\hat{H}_{CUPL} = (h_1,\ h_2,\ h_3,\ \cdots, h_n),
$$

then

$$
\begin{aligned}
H_{CUPL}h_1 &= H_{CUPL}W = e_1,\\
H_{CUPL}h_i &= H_{CUPL}(\Gamma + Ze_1^T - Wr^T\Theta)\big[h_{i-1} - h_{i-2} + h_{i-3} + \cdots + (-1)^i h_1\big]\\
&= KH_{CUPL}\big[h_{i-1} - h_{i-2} + h_{i-3} + \cdots + (-1)^i h_1\big], \quad (i = 2, 3, \cdots, n)
\end{aligned}
$$

and use the mathematical induction method we can obtain

$$
\begin{aligned}
&KH_{CUPL}\big[h_{i-1} - h_{i-2} + h_{i-3} + \cdots + (-1)^i h_1\big]\\
=&(-1)^0 C_{i-2}^0 K^{i-1}e_1 + (-1)^1 C_{i-2}^1 K^{i-2} e_1 + \cdots + (-1)^{i-2} C_{i-2}^{i-2} K e_1\\
=&e_i,
\end{aligned}
\tag{3.5}
$$

which means $H_{CUPL}h_i = e_i$, we can obtain $H_{CUPL}\hat{H}_{CUPL} = I_n$, where I_n is an identity matrix. From above, the matrix H_{CUPL} is invertible, and $H_{CUPL}^{-1} = \hat{H}_{CUPL}$.

Next we can obtain the representation of h_i. First of all, it is easy to see that

$$
\begin{aligned}
&h_1 = W, \qquad h_i = H_{CUPL}^{-1} e_i, \qquad H_{CUPL}^T = (\hat{I}_n\Sigma)H_{CUPL}(\hat{I}_n\Sigma^{-1}), \qquad \Sigma\hat{I}_n = \Theta,\\
&\Sigma^T = \Sigma, \qquad \Sigma e_1 = e_n, \qquad \Sigma e_i = e_{n-i+1} + e_{n-i+2}, \quad (i = 2, 3, \cdots, n),
\end{aligned}
$$

where $\hat{I}_n$ is an inverse identity matrix and Σ is of the form as (2.2).
Then for $i > 1$,

$$
\begin{aligned}
h_i =&(\Gamma + Ze_1^T - Wr^T\Theta)\big[h_{i-1} - h_{i-2} + h_{i-3} + \cdots + (-1)^i h_1\big]\\
=&\big[\Gamma h_{i-1} - \Gamma h_{i-2} + \cdots + (-1)^i \Gamma h_1\big] + (Ze_1^T - Wr^T\Theta)h_{i-1} - (Ze_1^T - Wr^T\Theta)h_{i-2} + \cdots\\
&+ (-1)^i (Ze_1^T - Wr^T\Theta)h_1\\
=&\Gamma\big[h_{i-1} - h_{i-2} + \cdots + (-1)^i h_1\big] + (Zw_{i-1} + Zw_{i-2} - Wz_{i-1} - Wz_{i-2}) - (Zw_{i-2} + Zw_{i-3}\\
&- Wz_{i-2} - Wz_{i-3}) + \cdots + (-1)^{i-1}(Zw_2 + Zw_1 - Wz_2 - Wz_1) + (-1)^i (Zw_1 - Wz_1)\\
=&\Gamma\big[h_{i-1} - h_{i-2} + \cdots + (-1)^i h_1\big] + Zw_{i-1} - Wz_{i-1},
\end{aligned}
$$

in the middle of the process, for $i>2$,

$$
\begin{aligned}
&(Ze_1^T-Wr^T\Theta)h_{i-1}\\
=&Ze_1^Th_{i-1}-Wr^T\Theta h_{i-1}\\
=&Ze_1^TH_{CUPL}^{-1}e_{i-1}-Wr^T\Theta H_{CUPL}^{-1}e_{i-1}\\
=&Ze_1^T(\hat{I}_n\Sigma^{-1})(\Sigma\hat{I}_n)H_{CUPL}^{-1}(\Sigma^{-1}\hat{I}_n)(\hat{I}_n\Sigma)e_{i-1}-Wr^T(\Sigma\hat{I}_n)H_{CUPL}^{-1}(\Sigma^{-1}\hat{I}_n)(\hat{I}_n\Sigma)e_{i-1}\\
=&Ze_1^T\hat{I}_n\Sigma^{-1}H_{CUPL}^{-T}\hat{I}_n\Sigma e_{i-1}-Wr^TH_{CUPL}^{-T}\hat{I}_n\Sigma e_{i-1}\\
=&Ze_n^T\Sigma^{-1}H_{CUPL}^{-T}\hat{I}_n(\Sigma e_{i-1})-Wr^TH_{CUPL}^{-T}\hat{I}_n(\Sigma e_{i-1})\\
=&Ze_1^TH_{CUPL}^{-T}\hat{I}_n(e_{n-i+2}+e_{n-i+3})-Wr^TH_{CUPL}^{-T}\hat{I}_n(e_{n-i+2}+e_{n-i+3})\\
=&Z(H_{CUPL}^{-1}e_1)^T(e_{i-1}+e_{i-2})-W(H_{CUPL}^{-1}r)^T(e_{i-1}+e_{i-2})\\
=&ZW^T(e_{i-1}+e_{i-2})-WZ^T(e_{i-1}+e_{i-2})\\
=&Zw_{i-1}+Zw_{i-2}-Wz_{i-1}-Wz_{i-2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&h_1=W,\qquad h_2=\Gamma h_1+Zw_1-Wz_1,\qquad \cdots,\\
&h_n=\left[\Gamma^{n-1}+(-1)^1C_{n-2}^1\Gamma^{n-2}+(-1)^2C_{n-2}^2\Gamma^{n-3}+\cdots+(-1)^{n-2}C_{n-2}^{n-2}\Gamma^1\right]W\\
&\quad+\left[\Gamma^{n-2}+(-1)^1C_{n-3}^1\Gamma^{n-3}+(-1)^2C_{n-3}^2\Gamma^{n-4}+\cdots+(-1)^{n-3}C_{n-3}^{n-3}\Gamma^1\right](Zw_1-Wz_1)\\
&\quad+\left[\Gamma^{n-3}+(-1)^1C_{n-4}^1\Gamma^{n-4}+(-1)^2C_{n-4}^2\Gamma^{n-5}+\cdots+(-1)^{n-4}C_{n-4}^{n-4}\Gamma^1\right](Zw_2-Wz_2)\\
&\quad+\cdots+\Gamma(Zw_{n-2}-Wz_{n-2})+(Zw_{n-1}-Wz_{n-1}),\\
&H_{CUPL}^{-1}=(h_1,h_2,\ h_3,\ \cdots,h_n)=S_1T_1+S_2T_2,
\end{aligned}
$$

where

$$
\begin{aligned}
S_1=&\big(W,\Gamma W,(\Gamma^2-\Gamma)W,\cdots,\\
&(\Gamma^{n-1}+(-1)^1C_{n-2}^1\Gamma^{n-2}+(-1)^2C_{n-2}^2\Gamma^{n-3}+\cdots+(-1)^{n-2}C_{n-2}^{n-2}\Gamma)W\big),\\
T_1=&\begin{pmatrix} 1 & -z_1 & -z_2 & \cdots & -z_{n-1}\\ & 1 & -z_1 & \ddots & \vdots\\ & & \ddots & \ddots & -z_2\\ & & & 1 & -z_1\\ & & & & 1\end{pmatrix},\\
S_2=&\big(Z,\Gamma Z,(\Gamma^2-\Gamma)Z,\cdots,\\
&(\Gamma^{n-1}+(-1)^1C_{n-2}^1\Gamma^{n-2}+(-1)^2C_{n-2}^2\Gamma^{n-3}+\cdots+(-1)^{n-2}C_{n-2}^{n-2}\Gamma)Z\big),\\
T_2=&\begin{pmatrix} 0 & w_1 & w_2 & \cdots & w_{n-1}\\ & 0 & w_1 & \ddots & \vdots\\ & & \ddots & \ddots & w_2\\ & & & 0 & w_1\\ & & & & 0\end{pmatrix}.
\end{aligned}
$$

We can write it in the formula

$$
\begin{aligned}
&H_{CUPL}^{-1}\\
&=\begin{pmatrix}
w_1 & w_1+w_2 & w_2+w_3 & \cdots & w_{n-1}+w_n\\
w_2 & w_2+w_3 & \iddots & \iddots & w_1\\
\vdots & \vdots & \iddots & \iddots & w_2\\
w_{n-1} & w_{n-1}+w_n & \iddots & \iddots & \vdots\\
w_n & w_1 & w_2 & \cdots & w_{n-1}
\end{pmatrix}
\times
\begin{pmatrix}
1 & -z_1 & -z_2 & \cdots & -z_{n-1}\\
 & 1 & -z_1 & \ddots & \vdots\\
 & & \ddots & \ddots & -z_2\\
 & & & 1 & -z_1\\
 & & & & 1
\end{pmatrix}\\
&\quad+\begin{pmatrix}
z_1 & z_1+z_2 & z_2+z_3 & \cdots & z_{n-1}+z_n\\
z_2 & z_2+z_3 & \iddots & \iddots & z_1\\
\vdots & \vdots & \iddots & \iddots & z_2\\
z_{n-1} & z_{n-1}+z_n & \iddots & \iddots & \vdots\\
z_n & z_1 & z_2 & \cdots & z_{n-1}
\end{pmatrix}
\times
\begin{pmatrix}
0 & w_1 & w_2 & \cdots & w_{n-1}\\
 & 0 & w_1 & \ddots & \vdots\\
 & & \ddots & \ddots & w_2\\
 & & & 0 & w_1\\
 & & & & 0
\end{pmatrix}.
\end{aligned}
$$

□

4. Stability Analysis

An algorithm is called forward stable if for all well conditioned problems, the computed solution $\hat{x}$ is closed to the true solution x that means the related error $\|x-\hat{x}\|/\|x\|$ is minimal. Now we present the error analysis of the inverse formula for nonsingular CUPL-Toeplitz matrix in terms of 1-norm, ∞-norm and 2-norm, respectively. Let

$$
\begin{aligned}
&g=(y_2,y_3,\cdots,y_{n-1},y_n,0)^T, && h=(0,y_2,y_3,\cdots,y_n)^T,\\
&p=(x_2,x_3,\cdots,x_{n-1},x_n,0)^T, && q=(0,x_2,x_3,\cdots,x_n)^T,\\
&s=(-x_2,-x_3,\cdots,-x_{n-1},-x_n,0)^T.
\end{aligned}
$$

From Theorem 2.1 we obtain the formula,

$$T_{CUPL}^{-1}=V_1U_1+V_2U_2,$$

which can be rewritten in the form as

$$
V_1=\begin{pmatrix}
0 & y_n & y_{n-1} & \cdots & y_2\\
 & & y_n & \ddots & \vdots\\
 & & & \ddots & y_{n-1}\\
 & & & & y_n\\
 & & & & 0
\end{pmatrix}
+\begin{pmatrix}
y_1 & & & & \\
y_2 & y_1 & & & \\
\vdots & \ddots & \ddots & & \\
y_{n-1} & \ddots & \ddots & y_1 & \\
y_n & y_{n-1} & \cdots & y_2 & y_1
\end{pmatrix}
+\begin{pmatrix}
0 & & & \\
y_2 & & & \\
y_3 & \ddots & & \\
\vdots & \ddots & y_2 & \\
y_n & \cdots & y_3 & y_2
\end{pmatrix},
$$

$$V_2 = \begin{pmatrix} 0 & x_n & x_{n-1} & \cdots & x_2 \\ & & x_n & \ddots & \vdots \\ & & & \ddots & x_{n-1} \\ & & & & x_n \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} x_1 & & & & \\ x_2 & x_1 & & & \\ \vdots & \ddots & \ddots & & \\ x_{n-1} & \ddots & \ddots & x_1 & \\ x_n & x_{n-1} & \cdots & x_2 & x_1 \end{pmatrix} + \begin{pmatrix} 0 & & & & \\ & x_2 & & & \\ & x_3 & \ddots & & \\ & \vdots & \ddots & x_2 & \\ & x_n & \cdots & x_3 & x_2 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 0 & -x_n & -x_{n-1} & \cdots & -x_2 \\ & 0 & -x_n & \ddots & \vdots \\ & & \ddots & \ddots & -x_{n-1} \\ & & & 0 & -x_n \\ & & & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 0 & y_n & y_{n-1} & \cdots & y_2 \\ & 0 & y_n & \ddots & \vdots \\ & & \ddots & \ddots & y_{n-1} \\ & & & 0 & y_n \\ & & & & 0 \end{pmatrix}.$$

Theorem 4.1. *Assume that T_{CUPL} to be invertible. Let $\epsilon > 0$ and $\hat{x}$, $\hat{y}$, $\hat{g}$, $\hat{h}$, $\hat{p}$, $\hat{q}$, $\hat{s}$, be the corresponding numerical least squares solutions of the linear systems for deriving the formula above. Denote by $\hat{T}_{CUPL}^{-1}$ the inverse of the perturbed matrix $\hat{T}_{CUPL}$. If $\frac{\|\hat{x}-x\|_1}{\|x\|_1} \le \epsilon$, $\frac{\|\hat{y}-y\|_1}{\|y\|_1} \le \epsilon$, $\frac{\|\hat{g}-g\|_1}{\|g\|_1} \le \epsilon$, $\frac{\|\hat{h}-h\|_1}{\|h\|_1} \le \epsilon$, $\frac{\|\hat{p}-p\|_1}{\|p\|_1} \le \epsilon$, $\frac{\|\hat{q}-q\|_1}{\|q\|_1} \le \epsilon$, $\frac{\|\hat{s}-s\|_1}{\|s\|_1} \le \epsilon$, then*

$$\|T_{CUPL}^{-1} - \hat{T}_{CUPL}^{-1}\|_1 \le 3\epsilon \|y\|_1 \big[2(2+\epsilon)\|x\|_1 + 1\big]. \tag{4.1}$$

Proof. From the above decomposition and for abbreviation, we write the inverse formula for T_{CUPL} given by (2.1) as

$$T_{CUPL}^{-1} = (R_g + L_y + L_h)(R_s + I) + (R_p + L_x + L_q)R_g.$$

Then we obtain

$$\begin{aligned} \|T_{CUPL}^{-1} - \hat{T}_{CUPL}^{-1}\|_1 \le & \|R_g R_s + L_y R_s + L_h R_s - \hat{R}_g \hat{R}_s - \hat{L}_y \hat{R}_s - \hat{L}_h \hat{R}_s\|_1 \\ & + \|R_g I + L_y I + L_h I - \hat{R}_g I - \hat{L}_y I - \hat{L}_h I\|_1 \\ & + \|R_p R_g + L_x R_g + L_q R_g - \hat{R}_p \hat{R}_g - \hat{L}_x \hat{R}_g - \hat{L}_q \hat{R}_g\|_1 \\ \triangleq & s_1 + s_2 + s_3. \end{aligned}$$

For the first part s_1, we present an upper bound,

$$
\begin{aligned}
s_1 =& \|R_gR_s + L_yR_s + L_hR_s - \hat{R}_g\hat{R}_s - \hat{L}_y\hat{R}_s - \hat{L}_h\hat{R}_s\|_1 \\
\le& \|R_gR_s - \hat{R}_g\hat{R}_s\|_1 + \|L_yR_s - \hat{L}_y\hat{R}_s\|_1 + \|L_hR_s - \hat{L}_h\hat{R}_s\|_1 \\
=& \|R_gR_s - R_g\hat{R}_s + R_g\hat{R}_s - \hat{R}_g\hat{R}_s\|_1 + \|L_yR_s - L_y\hat{R}_s + L_y\hat{R}_s - \hat{L}_y\hat{R}_s\|_1 \\
&+ \|L_hR_s - L_h\hat{R}_s + L_h\hat{R}_s - \hat{L}_h\hat{R}_s\|_1 \\
\le& \|R_g\|_1\|R_s - \hat{R}_s\|_1 + \|R_g - \hat{R}_g\|_1\|\hat{R}_s\|_1 + \|L_y\|_1\|R_s - \hat{R}_s\|_1 + \|L_y - \hat{L}_y\|_1\|\hat{R}_s\|_1 \\
&+ \|L_h\|_1\|R_s - \hat{R}_s\|_1 + \|L_h - \hat{L}_h\|_1\|\hat{R}_s\|_1 \\
=& \big(\|R_g\|_1 + \|L_y\|_1 + \|L_h\|_1\big)\|R_s - \hat{R}_s\|_1 + \big(\|R_g - \hat{R}_g\|_1 + \|L_y - \hat{L}_y\|_1 \\
&+ \|L_h - \hat{L}_h\|_1\big)\|\hat{R}_s\|_1 \\
=& \big(\|R_g\|_1 + \|L_y\|_1 + \|L_h\|_1\big)\|R_s - \hat{R}_s\|_1 + \big(\|R_g - \hat{R}_g\|_1 + \|L_y - \hat{L}_y\|_1 \\
&+ \|L_h - \hat{L}_h\|_1\big)\|\hat{R}_s - R_s + R_s\|_1 \\
\le& \big(\|R_g\|_1 + \|L_y\|_1 + \|L_h\|_1\big)\|R_s - \hat{R}_s\|_1 + \big(\|R_g - \hat{R}_g\|_1 + \|L_y - \hat{L}_y\|_1 + \|L_h - \hat{L}_h\|_1\big)\cdot \\
& \|\hat{R}_s - R_s\|_1 + \big(\|R_g - \hat{R}_g\|_1 + \|L_y - \hat{L}_y\|_1 + \|L_h - \hat{L}_h\|_1\big)\|R_s\|_1 \\
=& \big(\|g\|_1 + \|y\|_1 + \|h\|_1\big)\|s - \hat{s}\|_1 + \big(\|g - \hat{g}\|_1 + \|y - \hat{y}\|_1 + \|h - \hat{h}\|_1\big)\|s - \hat{s}\|_1 \\
&+ \big(\|g - \hat{g}\|_1 + \|y - \hat{y}\|_1 + \|h - \hat{h}\|_1\big)\|s\|_1 \\
\le& \big(\|g\|_1 + \|y\|_1 + \|h\|_1\big)\epsilon\|s\|_1 + \big(\epsilon\|g\|_1 + \epsilon\|y\|_1 + \epsilon\|h\|_1\big)\epsilon\|s\|_1 + \big(\epsilon\|g\|_1 + \\
& \epsilon\|y\|_1 + \epsilon\|h\|_1\big)\|s\|_1 \\
=& \|s\|_1\epsilon(2+\epsilon)\big(\|g\|_1 + \|y\|_1 + \|h\|_1\big)
\end{aligned}
$$

as $\|g\|_1 \le \|y\|_1$, $\|h\|_1 \le \|y\|_1$, $\|s\|_1 \le \|x\|_1$, we can obtain

$$
s_1 \le 3\epsilon(2+\epsilon)\|x\|_1\|y\|_1.
$$

Next we turn to consider s_2.

$$
\begin{aligned}
s_2 =& \|R_gI + L_yI + L_hI - \hat{R}_gI - \hat{L}_yI - \hat{L}_hI\|_1 \\
\le& \|R_g - \hat{R}_g\|_1\|I\|_1 + \|L_y - \hat{L}_y\|_1\|I\|_1 + \|L_h - \hat{L}_h\|_1\|I\|_1 \\
=& \|g - \hat{g}\|_1 + \|y - \hat{y}\|_1 + \|h - \hat{h}\|_1 \\
\le& \|g\|_1\epsilon + \|y\|_1\epsilon + \|h\|_1\epsilon = 3\epsilon\|y\|_1.
\end{aligned}
$$

Use the same method to obtain s_1, we can get s_3.

$$
s_3 \le 3\epsilon(2+\epsilon)\|y\|_1\|x\|_1.
$$

The conclusion is drawn by summing the above three inequalities. □

Theorem 4.2. *Under the assumption and notions of Theorem* 4.1*, we can obtain the same upper bound of* $\|T^{-1}_{CUPL} - \hat{T}^{-1}_{CUPL}\|_\infty$ *with the 1-norm in a similar way.*

$$
\|T^{-1}_{CUPL} - \hat{T}^{-1}_{CUPL}\|_\infty \le 3\epsilon\|y\|_1\big[2(2+\epsilon)\|x\|_1 + 1\big]. \tag{4.2}
$$

We can also present the upper bound with respect to 2-norm, since

$$\|T_{CUPL}^{-1} - \hat{T}_{CUPL}^{-1}\|_2^2 \le \|T_{CUPL}^{-1} - \hat{T}_{CUPL}^{-1}\|_1 \|T_{CUPL}^{-1} - \hat{T}_{CUPL}^{-1}\|_\infty,$$

from (4.1) and (4.2), we can get

$$\|T_{CUPL}^{-1} - \hat{T}_{CUPL}^{-1}\|_2 \le 3\epsilon\sqrt{n}\|y\|_2\big[2(2+\epsilon)\sqrt{n}\|x\|_2 + 1\big], \tag{4.3}$$

as $\|x\|_1 \le \sqrt{n}\|x\|_2$, $\|y\|_1 \le \sqrt{n}\|y\|_2$.

Therefore, the formula presented in Theorem 2.1 is forward stable. Use the same method, we can also learn that the formula presented in Theorem 3.1 is forward stable.

5. Two Algorithms for Finding T_{CUPL}^{-1} and H_{CUPL}^{-1}

In this section, two algorithms for finding T_{CUPL}^{-1} and H_{CUPL}^{-1} are given.

Algorithm 5.1. By using the Theorem 2.1, we proceed with

Step 1. Compute $f = (0, a_{1-n} - a_1, a_{2-n} - a_2, \cdots, a_{-1} - a_{n-1})^T$.

Step 2. Compute $X = (x_1, x_2, \cdots, x_n)^T$, $Y = (y_1, y_2, \cdots, y_n)^T$, by solving the systems of equations

$$T_{CUPL}X = f \;\; and \;\; T_{CUPL}Y = e_1,$$

where $e_1 = (1, 0, \cdots, 0)^T$.

Step 3. Compute T_{CUPL}^{-1} by the formula (2.1).

Algorithm 5.2. By using the Theorem 3.1, we proceed with

Step 1. Compute $r = (0, b_0 - b_n, b_1 - b_{n+1}, \cdots, b_{n-2} - b_{2n-2})^T$.

Step 2. Compute $Z = (z_1, z_2, \cdots, z_n)^T$, $W = (w_1, w_2, \cdots, w_n)^T$, by solving the systems of equations

$$H_{CUPL}Z = r \;\; and \;\; H_{CUPL}W = e_1,$$

where $e_1 = (1, 0, \cdots, 0)^T$.

Step 3. Compute H_{CUPL}^{-1} by the formula (3.1).

6. Numerical Examples

In this section, two examples are provided to verify the feasibility of the algorithms.

Example 6.1. Let T_{CUPL} be a 4×4 matrix as follows

$$T_{CUPL}=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Step 1. Compute

$$f=(0,a_{-3}-a_1,a_{-2}-a_2,a_{-1}-a_3)^T=(0,0,-1,0)^T.$$

Step 2. Compute X and Y by solving the equations: $T_{CUPL}X=f$ and $T_{CUPL}Y=e_1$, where $e_1=(1,0,0,0)^T$. We can get

$$X=\begin{pmatrix} 0 & 0 & -1 & 1 \end{pmatrix}^T,\qquad Y=\begin{pmatrix} 1 & 0 & -1 & 1 \end{pmatrix}^T.$$

Step 3. Compute T_{CUPL}^{-1} by (2.1):

$$T_{CUPL}^{-1}=V_1U_1+V_2U_2=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -1 & -1 & 1 & 0\\ 1 & 0 & -1 & 1 \end{pmatrix},$$

where

$$V_1=\begin{pmatrix} 1 & 1 & -1 & 0\\ 0 & 1 & 1 & -1\\ -1 & -1 & 1 & 1\\ 1 & 0 & -1 & 1 \end{pmatrix},\qquad U_1=\begin{pmatrix} 1 & -1 & 1 & 0\\ & 1 & -1 & 1\\ & & 1 & -1\\ & & & 1 \end{pmatrix},$$

$$V_2=\begin{pmatrix} 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\\ -1 & -1 & 0 & 1\\ 1 & 0 & -1 & 0 \end{pmatrix},\qquad U_2=\begin{pmatrix} 0 & 1 & -1 & 0\\ & 0 & 1 & -1\\ & & 0 & 1\\ & & & 0 \end{pmatrix}.$$

Example 6.2. Let H_{CUPL} be a 4×4 matrix as follows

$$H_{CUPL}=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Step 1. Compute

$$r=(0,b_0-b_4,b_1-b_5,b_2-b_6)^T=(0,0,-1,0)^T.$$

Step 2. Compute Z and W by solving the equations: $H_{CUPL}Z = r$ and $H_{CUPL}W = e_1$, where $e_1 = (1,0,0,0)^T$. We can obtain

$$Z = \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}^T, \qquad W = \begin{pmatrix} 1 & -1 & 0 & 1 \end{pmatrix}^T.$$

Step 3. Compute H_{CUPL}^{-1} by (3.1):

$$H_{CUPL}^{-1} = S_1T_1 + S_2T_2 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where

$$S_1 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ & 1 & -1 & 1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 1 & -1 & 0 \\ & 0 & 1 & -1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}.$$

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