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APPLICATION OF HOMOTOPY METHODS TO POWER SYSTEMS*[1)]

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Abstract

In this paper, the application of homotopy methods to the load flow multi-solution problems of power systems is introduced. By the generalized Bernshtein theorem, the combinatorial number C_{2n}^n is shown to be the BKK bound of the number of isolated solutions of the polynomial system transformed from load flow equations with generically chosen coefficients. As a result of the general Bezout number, the number of paths being followed is reduced significantly in the practical load flow computation. Finally, the complete P-V cures are obtained by tracking the load flow with homotopy methods.

Key words: Homotopy methods, Bezout number, Bernshtein-Khoranski-Kushnirenko (BKK) bound, Load flow computations.

1. Introduction

Load flow computations play an important role in the power system analysis([1], [2], [3]). In many cases, it comes down to solve a system of nonlinear algebraic equations with some constraint conditions. The algorithms available at present for this problem, such as Newton's methods and its variations , can be used to obtain individual solution only if the initial guess is in the near neighborhood of a solution. However, all solutions must be computed to explore the mechanism of voltage instability and collapse. Many power scientists and mathematicians have been very interested in this problem. It's well-known that there are few other general methods for determining all the solutions of nonlinear algebraic equations except homotopy methods ([4], [5], [6], [7]), which requires large amounts of computer time because of that many redundant paths must be followed during the computation.

This paper investigates homotopy methods and their numerical implementation for load flow multi-solution problems of power systems. The results significantly reduce the number of paths being followed. Since homotopy methods can calculate all isolated solutions of nonlinear algebraic equations in theory, they also are a benchmark for algorithmic development in this field.

2. Mathematical Model of Load Flow

Consider an $n+1$ bus (not including the earth bus) power system with r PQ buses, $n-r$ PV buses, and a slack bus. Since the computed results are independent of the ordering of the buses, without loss of generality, we assume that the previous r buses are PQ ones followed by

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PV buses and the last one is slack bus. From Kirchhoff law, we have a system of nonlinear complex-valued algebraic equations for the bus voltages $\dot{U}_1, \ldots, \dot{U}_n$,

$$P_i - \mathbf{j}\, Q_i = \overset{*}{\dot{U}_i} \sum_{j=1}^{n+1} Y_{ij} \dot{U}_j, \; i = 1, \ldots, n, \tag{1}$$

where both the slack bus voltage $\dot{U}_{n+1}$ and the admittance matrix $Y = (Y_{ij})_{(n+1)\times(n+1)}$ are known. The real and reactive bus powers $P_i, Q_i (i = 1, \ldots, r)$ on the PQ buses are also given, so that only the r bus voltages $\dot{U}_i (i = 1, \ldots, r)$ are unknown. On the other hand, the real bus powers and voltage peaks $P_i, V_i (i = r+1, \ldots, n)$ on the PV buses are given as well, only the bus reactive powers and bus angles $Q_i, \delta_i (i = r+1, \ldots, n)$ are expected. From the bus voltage peaks on the PV buses, we can also have $n - r$ additional equations

$$\dot{U}_i \overset{*}{\dot{U}_i} = V_i, \; i = r+1, \ldots, n. \tag{2}$$

Equations (1) and (2) determine all the $2n - r$ unknowns.

For the sake of symbolic simplicity, Equations (1) and (2) can be rewritten in the following form

$$\begin{cases} \bar{x}_i (b_{i1} x_1 + \ldots + b_{in} x_n + d_i) - w_i = 0, & i = 1, \ldots, n, \\ \bar{x}_i x_i = V_i^2, & i = r+1, \ldots, n. \end{cases} \tag{3}$$

where $x_i = \dot{U}_i \in \mathbf{C}$, "-" denotes the conjugate operation, $B = (b_{ij})_{n\times n} = (Y_{ij})_{n\times n}$, $d_i = Y_{i,n+1}\dot{U}_{n+1}$, $w_i = P_i - \mathbf{j}\, Q_i$ and B is a complex symmetrical matrix which is a highly sparse one when n is very large. The solutions of practical interests always have constraint conditions on the bus powers and voltages. However, the first phase of the algorithm dose not consider the constraint conditions.

3. Homotopy Methods for Load Flow Problems

Equation (3) can not be directly solved by homotopy methods because it is not a traditional polynomial equation system in complex variables. Substituting $z_i = x_i, z_{n+i} = \bar{x}_i (i = 1, \ldots, n)$ into Eq. (3) and conjugating every equation and eliminating the unknown reactive powers at the PV buses give the following equation system

$$\begin{cases} z_{n+i}(b_{i1} z_1 + \ldots + b_{in} z_n + d_i) - w_i = 0, & i = 1, \ldots, r, \\ z_i (b_{i1} z_{n+1} + \ldots + b_{in} z_{2n} + d_i) - \bar{w}_i = 0, & i = 1, \ldots, r, \\ z_{n+r+i}(\sum_{j=1}^{n} b_{r+i,j} z_j + d_{r+i}) + z_{r+i}(\sum_{j=1}^{n} \bar{b}_{r+i,j} z_{n+j} + \bar{d}_{r+i}) & \\ -2P_{r+i} = 0, & i = 1, \ldots, n-r, \\ z_{r+i} z_{n+r+i} = V_{r+i}^2, & i = 1, \ldots, n-r. \end{cases} \tag{4}$$

This is a system of $2n$ quadratic polynomial equations for the complex variables $\{z_1, \ldots, z_{2n}\}$ with complex coefficients, which can be as denoted $P(z) = 0$ where $z \in \mathbf{C}^{2n}$. Equation (3) can be easily proved to be equivalent to

$$\begin{cases} P(z) = 0; \\ z_i = \bar{z}_{n+i}, \quad i = 1, \ldots, n. \end{cases} \tag{5}$$

for determining the bus voltages. Thus, $P(z) = 0$ can be firstly solved, then the result is checked to see if $z_i = \bar{z}_{n+i} (i = 1, \ldots, n)$, and finally all solutions of the bus voltages are obtained.

Obviously, it is unnatural to compute one individual root of Eq. (3) with Newton's methods in this way. However, in case of computing all solutions of the system (3) with homotopy methods, as we expected at present, Eq. (5) is more preferred. The reason is that a reasonable partition of the variables in $P(z)=0$ could be discovered easily when calculating the possibly optimal upper bound of Bezout number by multi-homogeneous structure, as a result, the number of paths being followed can be cut down.

The degree of $P(z)$ is 2^{2n}, so the computer cost is quite expensive if a homotopic mapping is constructed based on the degree which would follow a total of 2^{2n} paths. However, the Bezout number of $P(z)$ is reduced down to C_{2n}^n if a suitable grouping of variables $(\{z_1,\ldots,z_n\},\{z_{n+1},\ldots,z_{2n}\})$ is considered ([8]).

Theorem 1. *Let the coefficients* $b_{ij}, d_i (i,j=1,\ldots,n)$, $w_i (i=1,\ldots,r)$, $P_{r+i}, V_{r+i} (i=1,\ldots,n-r)$ *of the polynomial system* $P(z)$ *be generically chosen, then* C_{2n}^n *is the BKK bound of the number of all isolated solutions of* $P(z)=0$ *in* $\mathbf{C}^{2n}$.

Proof. Let $A=(A_1,\ldots,A_{2n})$ denote the support of $P(z)$. Expunging the constant terms of $P(z)$ and the terms with $\bar{b}_{ij}, \bar{d}_i (i=r+1,\ldots,n, j=1,\ldots,n)$ from the $(2r+1)$-th subexpression to the $(n+r)$-th subexpression of $P(z)$ gives the new polynomial system $\hat{P}(z)$ as below,

$$\begin{cases} z_{n+i}(b_{i1}z_1+\ldots+b_{in}z_n+d_i)=0, & i=1,\ldots,r, \\ z_i(b_{i1}z_{n+1}+\ldots+b_{in}z_{2n}+d_i)=0, & i=1,\ldots,r, \\ z_{n+r+i}(\sum_{j=1}^{n} b_{r+i,j}z_j+d_{r+i})=0, & i=1,\ldots,n-r, \\ z_{r+i}z_{n+r+i}=0, & i=1,\ldots,n-r. \end{cases} \tag{6}$$

the support of which is denoted by $\hat{A}=(\hat{A}_1,\ldots,\hat{A}_{2n})$. Note that, the polynomial system $\hat{P}(z)$ with determinate support $\hat{A}$ has at most C_{2n}^n nonsingular solutions, and for generically chosen coefficients $b_{ij}, d_i (i,j=1,\ldots,n)$ it has C_{2n}^n nonsingular solutions. From Theorem 2.4 in [9], the BKK bound of the number of all isolated solutions of $\hat{P}(z)$ in $\mathbf{C}^{2n}$ is the mixed volume $M(\hat{A}\cup\{0\})$ where $\hat{A}\cup\{0\}$ denotes $(\hat{A}_1\cup\{0\},\ldots,\hat{A}_{2n}\cup\{0\})$, so $M(\hat{A}\cup\{0\})\geq C_{2n}^n$. Since $\hat{A}_i\cup\{0\}\subset A_i (i=1,\ldots,2n)$, from Theorem 2.4 and Theorem 1.1 in [9], the BKK bound of the number of all isolated solutions of $P(z)$ in $\mathbf{C}^{2n}$ is the mixed volume $M(A)$, which is bounded below by $M(\hat{A}\cup\{0\})$, namely

$$M(A)\geq M(\hat{A}\cup\{0\})\geq C_{2n}^n. \tag{7}$$

In addition, the Bezout number C_{2n}^n is just the BKK bound of the number of all isolated solutions of the trivial system determined by the partitioning $\{z_1,\ldots,z_n\}\{z_{n+1},\ldots,z_{2n}\}$, and the support of the original system $P(z)$ is included in the support of the trivial system. Hence, the Bezout number C_{2n}^n is an upper bound of the BKK bound $M(A)$, which means

$$M(A)\leq C_{2n}^n. \tag{8}$$

Finally, from (7) and (8), we can conclude $M(A)=C_{2n}^n$.

From Theorem 1, C_{2n}^n is the minimum number of paths to be followed when the coefficients of $P(z)$ are generically chosen. According to the multi-homogenous structure determined by the partitioning $\{z_1, \ldots, z_n\}\{z_{n+1}, \ldots, z_{2n}\}$, we construct the trivial system

$$\begin{cases} (e_{i1}z_1 + \ldots + e_{in}z_n + 1)(z_{n+i} + e_{i,n+1}) = 0, & i = 1, \ldots, r, \\ (z_i + \bar{e}_{i,n+1})(\bar{e}_{i1}z_{n+1} + \ldots + \bar{e}_{in}z_{2n}) = 0, & i = 1, \ldots, r, \\ (\sum_{j=1}^{n} e_{r+i,j}z_j + 1)(\sum_{j=1}^{n} \bar{e}_{r+i,j}z_{n+j} + 1) = 0, & i = 1, \ldots, n-r, \\ (z_{r+i} + e_{r+i,n+1})(z_{n+r+i} + \bar{e}_{r+i,n+1}) = 0, & i = 1, \ldots, n-r \end{cases} \quad (9)$$

which is simply denoted as $G(z) = 0$. For generically chosen $e_{ij}(i = 1, \ldots, n, j = 1, \ldots, n+1)$, the system (9) has C_{2n}^n nonsingular solutions which are easily searched. Define the homotopic mapping:

$$H(z, t) = (1-t)\gamma G(z) + tP(z), \ (t \in [0, 1], \gamma \in \mathbf{C}). \quad (10)$$

Theorem 2. *For almost all of $\gamma \in \mathbf{C}$, $H^{-1}(0)$ consists of smooth paths over $[0, 1)$ and each one gives an isolated solution of $P(z) = 0$ at $t = 1$.*

Proof. Since the trivial system $G(z)$ and the original system $P(z)$ have the same multi-homogenous structure determined by the partitioning $\{z_1, \ldots, z_n\}\{z_{n+1}, \ldots, z_{2n}\}$ except for the choice of the coefficients and $G(z)$ has the Bezout number C_{2n}^n nonsingular solutions, from the theorem proved in [5] the homotopic mapping (10) satisfies smoothness and accessibility. The result follows.

Using standard numerical techniques with a special step size strategy, the solution curves of $H^{-1}(0)$ (parameterized by $t \in [0, 1]$) are followed starting from the C_{2n}^n nonsingular solutions of $G(z) = 0$ to reach all isolated solutions of $G(z) = 0$ at $t = 1$. For load flow computations, matrix B is usually large and sparse, so that one can choose $e_{ij} = 0$ if $b_{ij} = 0$ and other coefficients are still generically chosen. In this way, the number of paths to be followed would be much less than C_{2n}^n. Moreover, the homotopic mapping (10) still has smoothness and accessibility ([6]). This option is referred to as comparative-generically chosen coefficients in the literature and the total number of all isolated solutions of the system (9) is called the general Bezout number of $P(z)$.

Theorem 3. *For comparative-generically chosen coefficients of the trivial system $G(z)$ and almost all of $\gamma \in \mathbf{C}$, $H^{-1}(0)$ consists of smooth paths over $[0, 1)$ and each one gives an isolated solution of $P(z) = 0$ at $t = 1$.*

4. Numerical Results

In this section, homotopy methods is applied to three power systems with 3 buses, 4buses, and 5 buses respectively (see Fig.1-Fig.3). All the solutions of the three systems (see Table 1-Table 3) were obtained in the above way, and all the calculations were done in MATLAB 5.3.0.10183 (R11) on a PC Pentium 400MHz.

Table 4 presents the degree(DN), the Bezout number(BN), the general Bezout number(GBN) and the zero number(ZN) of the polynomial systems for the above three power systems. As we can see that, for practical problems the number of all isolated solutions of $P(z) = 0$ is much less than C_{2n}^n or the general Bezout number when n is large. This is because the matrix B has many zero elements and the support of the practical system is included by the support of $P(z)$ for generically or comparative-generically chosen coefficients. However, when constructing the homotopic mapping with the multi-homogenous structure, we must follow the C_{2n}^n or the general Bezout number paths in order to find out all the problem solutions.

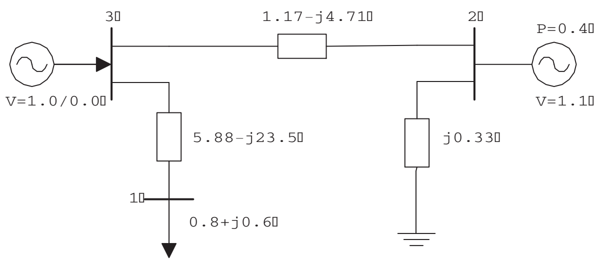

Figure 1: 3-bus power system

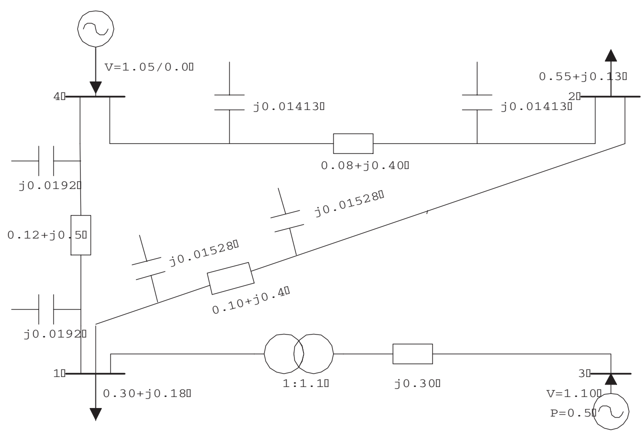

Figure 2: 4-bus power system

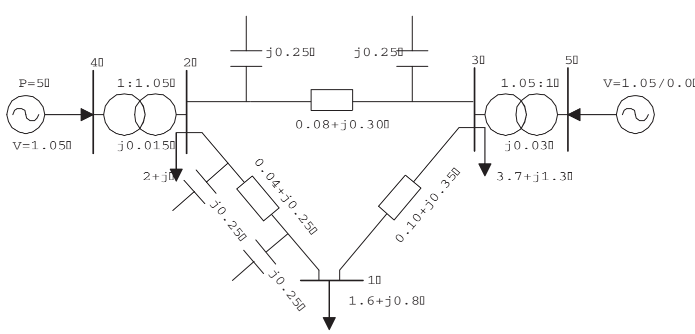

Figure 3: 5-bus power system

For the 4-bus system, a real parameter t was used to multiply the active power of the each bus, then for the different t the load flow was computed with homotopy methods. Fig. 4 and Fig. 5 show the t-V curves for the PQ bus 1 and 2 respectively, which are essentially the P-V curves. In the practical problems, the complete P-V cures can be obtained by tracking the load flow with homotopy methods.

Table 1. Four solutions of the 3-bus power system

Unkowns	Solutions			
(V, δ)	1	2	3	4
V_1	0.0427	0.0427	0.9665	0.9665
V_2	1.1000	1.1000	1.1000	1.1000
δ_1	-37.54	-37.54	-1.54	-1.54
δ_2	-155.08	2.98	-155.08	2.98

Table 2. Four solutions of the 4-bus power system

Unkowns	Solutions			
(V, δ)	1	2	3	4
V_1	0.1500	0.1514	0.6899	0.9818
V_2	0.2808	0.4208	0.1453	0.9633
V_3	1.1000	1.1000	1.1000	1.1000
δ_1	31.43	29.00	-5.04	-0.46
δ_2	-36.22	-22.26	-57.74	-6.45
δ_3	121.69	126.75	7.52	8.33

Table 3. Two solutions of the 5-bus power system

Unkowns	Solutions		Unkowns	Solutions	
(V)	1	2	(δ)	1	2
V_1	0.3499	0.8622	δ_1	-27.47	-4.78
V_2	1.0418	1.0779	δ_2	18.84	17.85
V_3	0.9908	1.0364	δ_3	-5.64	-4.28
V_4	1.0500	1.0500	δ_4	22.96	21.84

Tabel 4. DN, BN, GBN and ZN of the polynomial systems for the three power systems

Bus Number	n	DN	BN	GBN	ZN
3	2	16	6	4	4
4	3	64	20	18	10
5	4	256	70	60	22

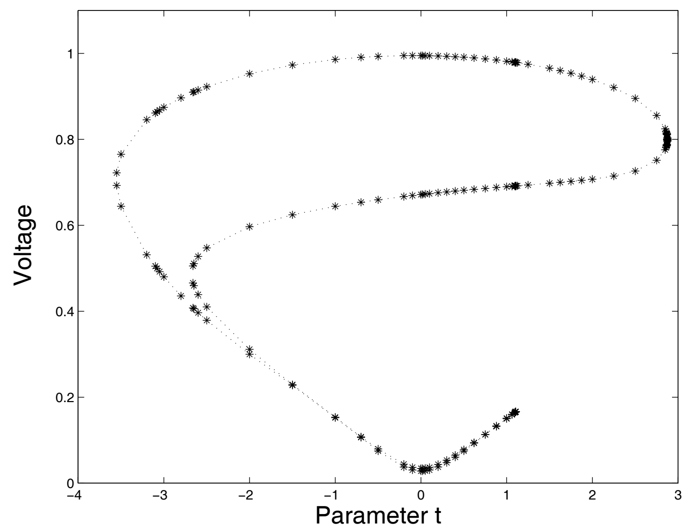

Figure 4: The t-V curve for the bus 1 of the 4-bus power system

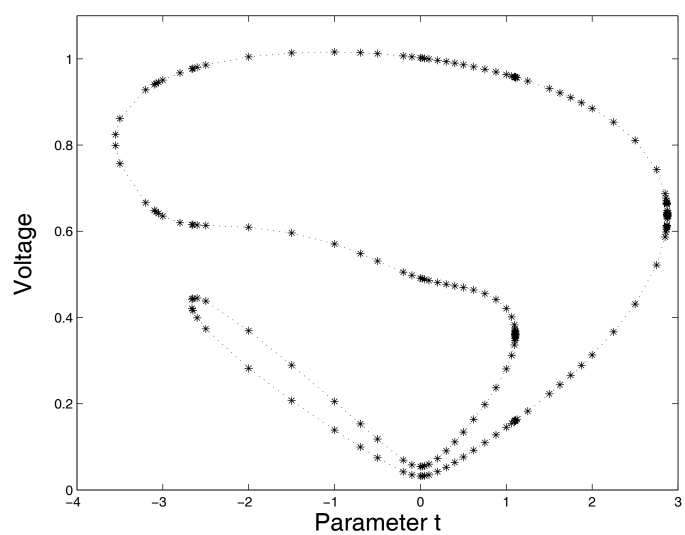

Figure 5: The t-V curve for the bus 2 of the 4-bus power system

5. Conclusions and Discussion

All of the solutions for power flow systems were computed with homotopy methods by transforming the initial problem into the problem of a system of polynomial equations with a proper variable partition, so that its multi-homogenous Bezout number is C_{2n}^{n}. By the general Bernshtein theorem, C_{2n}^{n} is the BKK bound of the number of isolated solutions of the polynomial system with generically chosen coefficients. As a result of the general Bezout number, the number of paths being followed is reduced significantly in the practical load flow computation. Lastly, the complete P-V cures are obtained by tracking the load flow with homotopy methods.

Since both the admittance matrix and the Jacobian of the homotopic mapping are highly sparse, the computer memory and the flops can be significantly decreased by using the disposal techniques for spare matrices. Note that the general Bezout number is still very large for a system of moderate size although it is much less than C_{2n}^{n}. Further work is needed to determine the optimal number of paths to be followed. Moreover, homotopy methods can be easily parallelized, which would further decrease the computer time.

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